NATURAL SCIENCES TRIPOS Part II

Wednesday 17 January 2018

THEORETICAL PHYSICS I

Answers

1 The action for a system consisting of a relativistic charged particle moving in an electromagnetic field is given by

$$S = -\int mc^2 d\tau - \int eA_\mu dx^\mu \, ,$$

where $x^\mu = (ct, \boldsymbol{x})$, $A^\mu = (\phi/c, \boldsymbol{A})$, and τ is the proper time.

(a) [**book work**] Derive the equations of motion in terms of the electric and magnetic fields, given by $\boldsymbol{E} = -\nabla\phi - \frac{\partial}{\partial t}\boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, respectively. [8]

We start from $dt = \gamma d\tau$, where $\gamma^{-2} = 1 - v^2/c^2$. We have that $dx^\mu = \frac{dx^\mu}{dt}dt$ so that the lagrangian may be written as

$$L = -\frac{mc^2}{\gamma} - e(\phi - \boldsymbol{A}\cdot\boldsymbol{v}) \, .$$

The Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) = \frac{\partial L}{\partial \boldsymbol{x}}.$$

Using

$$\frac{\partial L}{\partial \boldsymbol{v}} = \gamma m\boldsymbol{v} + e\boldsymbol{A}$$

and

$$\frac{\partial L}{\partial \boldsymbol{x}} = -e\nabla\phi + e\boldsymbol{v}\nabla\cdot\boldsymbol{A} \, ,$$

we get the Euler-Lagrange equation

$$\frac{d}{dt}(\gamma m\boldsymbol{v} + e\boldsymbol{A}) = -e\nabla\phi + e\boldsymbol{v}\nabla\cdot\boldsymbol{A} \, .$$

Now, by the chain rule, $\frac{d}{dt}\boldsymbol{A}(\boldsymbol{x},t) = \frac{\partial}{\partial t}\boldsymbol{A} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$, such that this reduces to

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = -e\nabla\phi - e\frac{\partial}{\partial t}\boldsymbol{A} + e\boldsymbol{v}\nabla\cdot\boldsymbol{A} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$$

or

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = -e\nabla\phi - e\frac{\partial}{\partial t}\boldsymbol{A} + e\boldsymbol{v}\times(\nabla\times\boldsymbol{A}).$$

Using the definitions of electric and magnetic fields, we obtain

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}).$$

(b) [**unseen calculation**] Suppose that $\boldsymbol{B} = 0$, that $\boldsymbol{E}$ is constant and that at $t = 0$ the particle has velocity $\boldsymbol{v}_0$. Find the subsequent velocity of the particle. [5]

When $\boldsymbol{B} = 0$, we may integrate this equation directly to obtain

$$\gamma m\boldsymbol{v} = e\boldsymbol{E}t + m\gamma_0\boldsymbol{v}_0\,,$$

where $\boldsymbol{v}_0$ is the initial velocity and γ_0 the corresponding value of γ. Taking the dot product of this relation with itself, we find that

$$\gamma^2v^2/c^2(=\gamma^2-1) = \frac{|e\boldsymbol{E}t + m\gamma_0\boldsymbol{v}_0|^2}{m^2c^2}$$

such that

$$\gamma = \sqrt{1 + \frac{|e\boldsymbol{E}t + m\gamma_0\boldsymbol{v}_0|^2}{m^2c^2}}$$

and so

$$\boldsymbol{v} = \frac{e\boldsymbol{E}t/m + \gamma_0\boldsymbol{v}_0}{\sqrt{1 + \frac{|e\boldsymbol{E}t + m\gamma_0\boldsymbol{v}_0|^2}{m^2c^2}}}.$$

(c) [**unseen calculation**] Find the limiting velocity of the particle as $t \to \infty$. [3]

As $t \to \infty$, we find that $\boldsymbol{v} \to \frac{e\boldsymbol{E}}{|e\boldsymbol{E}|}c$. No matter what velocity we start with (provided its magnitude is less than c), the ultimate velocity is aligned with the electric field, has magnitude c, and is aligned either parallel or anti-parallel to the field, depending on whether the charge is positive or negative, respectively.

Note: an answer that simply states that c is the limiting velocity of any particle subject to a constant force will receive 2 marks out of 3 because it does not discuss the direction of the velocity.

(d) [**unseen calculation**] Suppose that instead $\boldsymbol{E} = 0$ (and generically $\boldsymbol{B} \neq 0$). Show that γ, and hence the total speed, are constant. [4]

In this case, we must solve the equation

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = e\boldsymbol{v}\times\boldsymbol{B}.$$

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If we first take the dot product with the velocity, then we find that

$$\boldsymbol{v}\cdot\frac{d}{dt}(\gamma m\boldsymbol{v}) = mc^2\frac{d\gamma}{dt} = 0\,.$$

Hence γ and the speed are both constant. We thus may write the equation of motion as $\frac{d\boldsymbol{v}}{dt} = \frac{e}{m\gamma}\boldsymbol{v}\times\boldsymbol{B}$.

(e) [**unseen calculation, similar to non-relativistic case**] Suppose now that $\boldsymbol{E}=0$ and $\boldsymbol{B}$ is constant. Show that the time dependence of the perpendicular velocity vector $\boldsymbol{v}_\perp = \boldsymbol{v} - \boldsymbol{B}\frac{(\boldsymbol{v}\cdot\boldsymbol{B})}{B^2}$ is periodic and find the period. [5]

Now differentiate with respect to time again, to get that

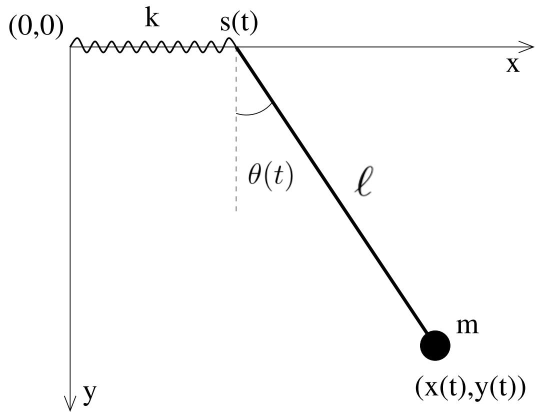
$$\frac{d^2\boldsymbol{v}}{dt^2} = \frac{e}{m\gamma}\frac{d\boldsymbol{v}}{dt}\times\boldsymbol{B} = \left(\frac{e}{m\gamma}\right)^2(\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B} = -\left(\frac{e}{m\gamma}\right)^2(\boldsymbol{v}B^2 - \boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{B}))\,.$$

In terms of the perpendicular component $\boldsymbol{v}_\perp = \boldsymbol{v} - \boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{B}/B^2)$, we get, by resolving components, that

$$\frac{d^2\boldsymbol{v}_\perp}{dt^2} = -\left(\frac{eB}{m\gamma}\right)^2\boldsymbol{v}_\perp$$

which represents periodic motion with period $T = \frac{2\pi m\gamma}{eB}$.

2 A massless rod of length ℓ makes an angle $\theta(t)$ with the vertical, has a point mass m at one end, and is in a constant gravitational field $\boldsymbol{g} = g\hat{\boldsymbol{y}}$. The other end of the rod is attached to a horizontal line with a frictionless hinge, and connected to a point along the line by a massless spring of constant k and zero rest length, as illustrated in the figure.



Let us call $s(t)$ the instantaneous horizontal displacement of the hinge from the origin (i.e., the fixed point of the spring).

(a) [**book work**] Introducing $\eta(t) = s(t)/\ell$, the coordinates of the mass can be written as [2]

$$x = \ell\eta + \ell\sin\theta \qquad\qquad y = \ell\cos\theta\,.$$

Correspondingly, the kinetic energy is given by [3]

$$T = \frac{1}{2}m\left[\left(\ell\dot\eta + \ell\dot\theta\cos\theta\right)^2 + \left(\ell\dot\theta\sin\theta\right)^2\right] = \frac{1}{2}m\ell^2\left[\dot\eta^2 + \dot\theta^2 + 2\dot\theta\dot\eta\cos\theta\right]$$

and the potential energy by [3]

$$V = -mg\ell\cos\theta + \frac{1}{2}k\ell^2\eta^2\,.$$

We can then obtain the Lagrangian $L = T - V$, more conveniently rescaled by a factor $(m\ell^2)^{-1}$: [2]

$$L = \frac{1}{2}\left[\dot\eta^2 + 2\dot\eta\dot\theta\cos\theta + \dot\theta^2\right] + \frac{g}{\ell}\cos\theta - \frac{1}{2}\frac{k}{m}\eta^2\,,$$

(b) [**part book work, part new**] To obtain the equations of motion of this system we need to compute: [4]

$$\frac{\partial L}{\partial\eta} = -\frac{k}{m}\eta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\eta} = \ddot\eta + \frac{d}{dt}(\dot\theta\cos\theta) = \ddot\eta + \ddot\theta\cos\theta - \dot\theta^2\sin\theta$$

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and

$$\frac{\partial L}{\partial \theta} = -\dot{\eta}\dot{\theta}\sin\theta - \frac{g}{\ell}\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \ddot{\theta} + \frac{d}{dt}(\dot{\eta}\cos\theta) = \ddot{\theta} + \ddot{\eta}\cos\theta - \dot{\eta}\dot{\theta}\sin\theta$$

Finally, the generic equations of motion are: [1]

$$\begin{cases} \ddot{\eta} + \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta + \frac{k}{m}\eta = 0 \\ \ddot{\theta} + \ddot{\eta}\cos\theta + \frac{g}{\ell}\sin\theta = 0\,. \end{cases}$$

If we assume that the dynamical variables and their derivatives are small, the equations of motion expanded to linear order can be written as

$$\begin{cases} \ddot{\theta} + \ddot{\eta} + \omega_0^2\theta = 0 \\ \ddot{\theta} + \ddot{\eta} + \omega_1^2\eta = 0\,, \end{cases}$$

where $\omega_0^2 = g/\ell$ and $\omega_1^2 = k/m$. [3]

(c) [**new**] The expanded equations of motion imply a proportionality relation between η and θ: $\eta = (\omega_0^2/\omega_1^2)\theta$. [2]

Assuming a solution of the form $\theta(t) = \theta_0\sin(\omega t)$, we have to require the form $\eta(t) = (\omega_0^2/\omega_1^2)\theta_0\sin(\omega t)$. The two equations above are then linearly dependent on one another and they are satisfied only if

$$-\omega^2\frac{\omega_0^2}{\omega_1^2} - \omega^2 + \omega_0^2 = 0\,,$$

which gives $\omega^2 = \omega_0^2\omega_1^2/(\omega_0^2 + \omega_1^2)$. [3]

In the limit $k \to \infty$, $\omega_1^2 \to \infty$ and $\eta \to 0$, which in turn gives $\omega^2 = \omega_0^2$. This is consistent with the expectation for a pendulum where the top hinge is fixed (infinite spring stiffness), in the approximation of small oscillations. [2]

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3 (a) **[bookwork]** Explain why a total derivative term in the Lagrangian (or Lagrangian density) of a dynamical system does not affect the equations of motion and may be discarded. [4]

A total derivative in the Lagrangian or Lagrangian density can be integrated to give a contribution on the boundary on spacetime, so does not affect the variations used to derive the equations of motion, which are taken to vanish on the boundary.

(b) **[unseen calculation]** A system is described by a real scalar field $h(\boldsymbol{x}, t)$ with a Lagrangian density containing spacetime derivatives of $h(\boldsymbol{x}, t)$ up to and including second order. Derive the corresponding Euler-Lagrange equations of motion. [10]

The variation of the action may be written in terms of the Lagrangian $\mathcal{L}(h, \partial_\mu h, \partial_\mu \partial_\nu h)$ as

$$0 = \delta S = \int dx^\mu \delta\mathcal{L} = \int dx^\mu \left[\frac{\delta\mathcal{L}}{\delta h}\delta h + \frac{\delta\mathcal{L}}{\delta\partial_\mu h}\delta\partial_\mu h + \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu h}\delta\partial_\mu\partial_\nu h \right] .$$

Integrating by parts and neglecting boundary contributions, we get,

$$0 = \int dx^\mu \left[\frac{\delta\mathcal{L}}{\delta h} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu h} + \partial_\mu\partial_\nu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu h} \right] \delta h \, .$$

For this to vanish for arbitrary δh, we require that

$$0 = \frac{\delta\mathcal{L}}{\delta h} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu h} + \partial_\mu\partial_\nu \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu h} \, .$$

(c) **[unseen calculation]** The height $h(\boldsymbol{x}, t)$ of a surface grown over the $\boldsymbol{x} = (x^1, x^2)$ plane by random deposition of atoms is described by the action

$$S = \int d^2\boldsymbol{x}\, dt \left(\frac{\partial h}{\partial t} - \nu\nabla^2 h \right)^2 ,$$

where ν is a positive constant. Find the Euler-Lagrange equation of motion governing the dynamics of $h(\boldsymbol{x}, t)$. [7]

It helps to first expand out the quadratic terms and to notice that (after integration by parts and neglecting a trivial boundary term) the cross-term $-2\nu\dot{h}\nabla^2 h = +2\nu\nabla\dot{h}\nabla h = \frac{d}{dt}\left(2\nu(\nabla h)^2\right)$ is a total derivative and may be discarded. Next, one may either use the formula formula derived in the previous part, or, more simply, just use the usual Euler-Lagrange equations, integrating by parts where necessary in order that only first-order derivatives appear. Doing so, we find

$$\frac{\delta\mathcal{L}}{\delta\dot{h}} = 2\dot{h}$$

and

$$\frac{\delta \mathcal{L}}{\delta \nabla h} = -2\nu^2 \nabla \nabla^2 h$$

(where we have freely integrated by parts) in order to arrive at the equation of motion

$$\ddot{h} - \nu^2 \nabla^4 h = 0.$$

(d) [**unseen**] What symmetries does the system possess? [4]

The system is invariant under the discrete symmetry $h \to -h$,spacetime translations, and under rotations of $\boldsymbol{x}$.

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4 Consider the Lagrangian density of 1-dimensional elastic rod with density $\rho = 1$ and elastic constant $\kappa = 1$, namely

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 ,$$

where $\phi(x,t)$ is the local displacement field.

(a) [**book work**] The Euler-Lagrange equation of motion for the field $\phi(x,t)$ are given by [3]

$$-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\phi'} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0 \qquad\qquad \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = 0\,.$$

(b) [**book work**] The total angular momentum tensor of the system is given by

$$J^{\mu\nu} = \int dx\, M^{0\mu\nu} = \int dx\, \left[x^\mu T^{0\nu} - x^\nu T^{0\mu}\right] ,$$

where $M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$ and $T^{\mu\nu}$ is the stress energy tensor. [3]

In order to evaluate the stress energy tensor for the elastic rod described above, we need the terms

$$\frac{\partial\mathcal{L}}{\partial\partial_0\phi} = \dot{\phi} \qquad \frac{\partial\mathcal{L}}{\partial\partial_1\phi} = -\phi' \qquad \partial^0\phi = \dot{\phi} \qquad \partial^1\phi = -\phi' ,$$

from which we obtain [2]

$$T^{00} = \dot{\phi}^2 - \mathcal{L} = \mathcal{H} \qquad T^{01} = -\dot{\phi}\phi' \qquad T^{10} = -\dot{\phi}\phi' \qquad T^{11} = \phi'^2 + \mathcal{L} = \mathcal{H}\,.$$

By construction $J^{\mu\nu} = J^{\nu\mu}$, and therefore we only need to compute J^{01} since $J^{00} = J^{11} = 0$ and $J^{10} = -J^{01}$. For the rod we obtain [1]

$$J^{01} = \int dx\, \left[-t\phi'\dot{\phi} - x\mathcal{H}\right] .$$

The stress-energy tensor is symmetric upon exchanging the indices μ and ν because, for the choice of density and elastic constant equal to one another, the system is relativistic invariant, which is a higher symmetry than just space-time translations. As a result, $\partial_\lambda M^{\lambda\mu\nu} = 0$ and the total angular momentum tensor is the corresponding conserved charge. [1]

(c) [**new**] Consider adding a viscous damping term to the equation of motion of the rod, $\gamma\partial_t\partial_x^2\phi$ where γ is a positive constant. Substituting the Fourier transform

$$G(x,t) = \iint G(k,\omega)\, e^{-ikx-i\omega t}\, \frac{dk\, d\omega}{(2\pi)^2}$$

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into the equation

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\gamma\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\right)G(x,t)=\delta(x)\delta(t)$$

we obtain

$$\iint\left[\omega^2-k^2+i\gamma k^2\omega\right]G(k,\omega)\,e^{-ikx-i\omega t}\,\frac{dk\,d\omega}{(2\pi)^2}=\delta(x)\delta(t)\,,$$

which in turn gives [4]

$$G(k,\omega)=\frac{1}{\omega^2-k^2+i\gamma k^2\omega}\,.$$

The denominator has roots [1]

$$\omega_{1,2}=-i\gamma k^2/2\pm\sqrt{k^2-k^4\gamma^2/4}\,.$$

(d) **[new]** Assuming that $k^2<4/\gamma^2$, the square root term in the roots is real and both $\omega_{1,2}$ lie below the real ω axis, to the right and left of the imaginary ω axis, respectively. [1]

To compute

$$G(k,t)=\int G(k,\omega)\,e^{-i\omega t}\,\frac{d\omega}{2\pi}=\int\frac{e^{-i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}\,\frac{d\omega}{2\pi}$$

we can use Cauchy integration provided we close the contour in the upper half complex ω plane for $t<0$, and in the lower half plane for $t>0$ (indeed, the exponential at the numerator is proportional to $e^{\mathrm{Im}(\omega)t}$). Both poles are in the lower half plane, which is consistent with causality: $G(t<0)=0$. [1]

For $t>0$ we obtain [3]

$$G(k,t)=-i\left[\frac{e^{-i\omega_1 t}}{\omega_1-\omega_2}+\frac{e^{-i\omega_2 t}}{\omega_2-\omega_1}\right]=\frac{2}{\omega_1-\omega_2}\frac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2i}$$

$$=-\frac{e^{-\gamma k^2 t/2}}{\sqrt{k^2-k^4\gamma^2/4}}\sin\left(\sqrt{k^2-k^4\gamma^2/4}\,t\right)\,.$$

We can finally take the limit $\gamma\to 0$,

$$G(k,t)=-\frac{\sin(kt)}{k}\,,$$

and compute

$$G(x,t)=\int G(k,t)\,e^{-ikx}\,\frac{dk}{2\pi}=-\int\frac{\sin(kt)}{k}\,e^{-ikx}\,\frac{dk}{2\pi}\,.$$

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(Note that we were able to replace $\sin(|k|t)/|k|$ with $\sin(kt)/k$ by taking advantage of the fact that $t > 0$ and sin is an odd function of its argument.) [1]

Using the definition of the top hat function

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-isx}\, ds = \mathrm{TH}(x)\,,$$

we arrive at the result [2]

$$G(x,t) = -\frac{1}{2\pi}\int \frac{\sin s}{s}\, e^{-isx/t}\, ds = -\frac{1}{2}\mathrm{TH}\left(\frac{x}{t}\right)\,.$$

This is consistent with the choice of initial conditions $\delta(x)\delta(t)$: for $t = 0$, $G(x,t)$ does not vanish only at $x = 0$. Moreover, the edges of the support of $G(x,t)$ are at $x/t = \pm 1$, propagating in space as $x(t) = \pm t$, namely with velocity 1 as expected for an elastic rod that satisfies the condition $\rho = \kappa$. [2]

END OF PAPER