NATURAL SCIENCES TRIPOS Part II

Wednesday 18 January 2017

THEORETICAL PHYSICS I

Answers

1 (a) [**bookwork**] The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\pi a^2 \rho \dot{\psi}^2 - \frac{1}{2}\pi a^2 K \psi'^2$$

The momentum conjugate to ψ is

$$\phi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \pi a^2 \rho \dot{\psi}.$$ [1]

The Hamiltonian density is

$$\mathcal{H} = \phi\dot{\psi} - \mathcal{L} = \frac{\phi^2}{2\pi a^2 \rho} + \frac{1}{2}\pi a^2 K \psi'^2$$ [2]

The Lagrangian equation of motion is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{d}{dz}\frac{\partial \mathcal{L}}{\partial \psi'} = \frac{\partial L}{\partial \psi}$$ [1]

$$\pi a^2 \rho \ddot{\psi} - \pi a^2 K \psi'' = 0$$ [1]

Substituting in $\psi = f(z - ct)$, we get

$$c^2 \pi a^2 \rho f'' - \pi a^2 K f'' = 0$$

which is a solution provided $c = \sqrt{\frac{K}{\rho}}$. [1]

(b) [**part bookwork part new calculation, particularly case 2, though scale symmetry is on the example sheet.**]

(1) The operation is a displacement of the whole fluid, $\psi \to \psi + b$. [1]

This does not change ψ' or $\dot{\psi}$, so it does not change $\mathcal{L}$ or S. [1]

The corresponding conservation law is simply the equation of motion for ψ

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{d}{dz}\frac{\partial \mathcal{L}}{\partial \psi'} = \frac{\partial L}{\partial \psi} = 0$$ [1]

which is a conservation law for the the total momentum:

$$Q_1 = \int \frac{\partial \mathcal{L}}{\partial \dot{\psi}} dz = \int \pi a^2 \rho \dot{\psi} dz.$$ [1]

(2) The operation is a scaling of the wave-packet, $\psi(z,t) \to \psi(bz, bt)$, which for an infinitesimal transformation, $b = 1 + \epsilon$, gives $\delta\psi = \epsilon(t\dot{\psi} + z\psi')$. [1]

Because of the second derivatives in each term, this shifts $\mathcal{L}(z,t) \to b^2 \mathcal{L}(bz, bt)$ but, by a simple change of dummy variables, $Z = bz$ $T = bt$, does not change $S = \int\int \mathcal{L} dz dt$. [1]

Applying Noether's theorem, we have

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta\psi\right) + \frac{d}{dz}\left(\frac{\partial \mathcal{L}}{\partial \psi'} \delta\psi\right) = \delta L$$ [1]

$$= \epsilon(t\dot{L} + zL' + 2L)$$ [1]

$$= \frac{d}{dt}(t\mathcal{L}) + \frac{d}{dz}(z\mathcal{L}).$$ [1]

which is a conservation law for the quantity

$$Q_2 = \int \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta\psi - t\mathcal{L}\right) dz = \int \frac{\pi a^2}{2} \left(t(\rho\dot{\psi}^2 + K\psi'^2) + 2\rho z\dot{\psi}\psi'\right) dz.$$ [1]

(c) [**calculation**]

$$Q_1 = \int_{-\infty}^{\infty} \pi a^2 \rho \dot{\psi} dz = -\int_{-\infty}^{\infty} c\pi a^2 \rho f'(z - ct) dz = -c\pi a^2 \rho \left[f(z - ct)\right]_{z=-\infty}^{z=\infty} = 0$$ [1]

$$\begin{aligned} Q_2 &= \int_{-\infty}^{\infty} \frac{\pi a^2}{2} \left(t(2c^2) - 2zc\right) \rho f'(z - ct)^2 dz \\ &= \int_{-\infty}^{\infty} \frac{\pi a^2}{2} (tc - z)\, 2c\rho f'(z - ct)^2 dz \\ &= \int_{\infty}^{\infty} \pi a^2 z c \rho f'(z)^2 dz \end{aligned}$$

[2]

Which are both manifestly conserved.

(d) [**New**] $\mathcal{L}_1$ is not invariant under $\psi(z,t) \to \psi(bz, bt)$, so Q_2 is not conserved. [1]

It is invariant under $\psi \to \psi + b$, leading to conservation of a slightly modified

$$Q_1 = \int \frac{\partial \mathcal{L}}{\partial \dot{\psi}} dz = \int \pi a^2 \rho(z) \dot{\psi} dz.$$ [1]

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$\mathcal{L}_2$ is invariant under $\psi \to \psi + b$, so Q_1 is still conserved, and since $\frac{\partial \mathcal{L}}{\partial \dot{\psi}}$ is unchanged, its form is also unchanged. [1]

The ψ'^4 term transforms differently under $\psi(z,t) \to \psi(bz, bt)$ so Q_2 is no longer conserved. [1]

$\mathcal{L}_3$ is no longer invariant under $\psi \to \psi + b$, so Q_1 is not conserved. [1]

The new term still transforms as $\mathcal{L}(z,t) \to b^2\mathcal{L}(bz, b, t)$ under $\psi(z,t) \to \psi(bz, bt)$ so Q_3 is conserved but in a modified form since $\mathcal{L}$ has changed. [1]

2 (a) [**bookwork**] The general form for the Lagrangian of a relativistic particle moving in and E-M field is

$$L = -mc^2/\gamma - q(\phi - \boldsymbol{v} \cdot \boldsymbol{A}). \qquad [1]$$

For the particle in the question, we have $\boldsymbol{v} = \dot{z}\hat{\boldsymbol{z}} + \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\theta}\hat{\boldsymbol{\theta}}$. [1]

For the cylindrically symmetric magnetic field in the question, which has no θ component, we can take $\boldsymbol{A} = A(\rho, z)\hat{\theta}$ and $\phi = 0$. [1]

The Lagrangian thus becomes

$$L = -mc^2\sqrt{1 - \frac{\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\theta}^2}{c^2}} + q\rho\dot{\theta}A(\rho, z)\,. \qquad [1]$$

Outside the solenoid we have $A = 0$, inside we need $A = B\rho/2$ so that $\nabla \times \boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A)}{\partial \rho}\hat{\boldsymbol{z}} = B\hat{\boldsymbol{z}}$. [1]

(b) [**part bookwork part calcuation**] Time invariance leads to conservation of the Hamiltonian. [1]

In this case, the canonical momenta are

$$p_z = \gamma m\dot{z}, \qquad p_\theta = \gamma m\rho^2\dot{\theta} + q\rho A, \qquad p_\rho = \gamma m\dot{\rho} \qquad [1]$$

so the Hamiltonian is

$$H = p_z\dot{z} + p_\rho\dot{\rho} + p_\theta\dot{\theta} - L = \gamma mc^2,$$

i.e. the γ factor, or the particle speed, is a constant of the motion. [1]

Cylindrical symmetry leads to the conservation of $p_\theta = \gamma m\rho^2\dot{\theta} + q\rho A$. [1]

The particle starts at infinity with $\dot{\theta} = 0$ and therefore $p_\theta = 0$, so it has $\dot{\theta} = 0$ outside and $\dot{\theta} = -\frac{qB}{2\gamma m}$ inside. [1]

(c) [**calculation**] The z equation is

$$\frac{d}{dt}p_z = \frac{\partial L}{\partial z} \implies \gamma m\ddot{z} = 0 \qquad [1]$$

both inside and outside.

The ρ equation is

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$$\frac{d}{dt}p_\rho = \frac{\partial L}{\partial \rho} \implies \gamma m \ddot{\rho} = \gamma m \rho \dot{\theta}^2 + \begin{cases} 0 & \text{outside} \\ q\rho\dot{\theta}B & \text{inside.} \end{cases} \qquad [1]$$

Inside the solenoid we have $\dot{\theta} = -\frac{qB}{2\gamma m}$, so this equation becomes

$$\gamma m \ddot{\rho} = -\frac{q^2B^2}{4\gamma m}\rho. \qquad [1]$$

Despite appearances $\dot{z}$ is not a constant of the motion, since the potential V is a function of z, changing when the particle enters the solenoid. Thus, although $\dot{z}$ is constant outside and inside the solenoid, it changes when the particle enters, as we already know since γ is conserved but $\dot{\theta}$ changes. [1]

(d) [**new form of familiar motion**] Introducing $\omega = \frac{qB}{2\gamma m}$, we can solve the θ motion trivially to get

$$\theta = -\omega t\,. \qquad [1]$$

The ρ equation inside the cylinder is an SHM equation with the solution

$$\rho = \rho_0 \cos(\omega t)\;. \qquad [1]$$

The z equation can be trivially integrated to get

$$z = v_i t \qquad [1]$$

for some velocity v_i, which we find from conservation of γ as the particle enters the solenoid, which requires,

$$v^2 = v_i^2 + \rho_0^2\omega^2 \implies v_i = \sqrt{v^2 - \rho_0^2\omega^2}. \qquad [1]$$

Translating the ρ and θ results into Cartesians, we have $y = -\rho\sin(\theta) = \rho_0\sin(2\omega t)/2$ and $x = \rho\cos(\theta) = \rho_0(\cos(2\omega t)+1)/2$. As we expect for a charged particle in a uniform magnetic field, this motion is a helix with the cyclotron frequency (2ω) that points along $\hat{\boldsymbol{z}}$ and has radius $\rho_0/2$ and center at $\rho_0/2$. [3]

(e) [**new**] The particles are in the solenoid a time $t = l/v$, so they exit with

$$\rho_f = \rho_0\cos(\omega l/v) \qquad [1]$$

$$\dot{\rho} = -\rho_0\omega\sin(\omega l/v) \qquad [1]$$

Outside the solenoid, the particles move in a straight line, so they reach the z axis after a further time $T = \rho_f/\dot{\rho}$. [1]

In this time the particle travels a further distance along the z axis

$$f = vT = \frac{v}{\omega}\cot\left(\frac{\omega l}{v}\right). \qquad [1]$$

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3 The Lagrangian density for a self-interacting, complex scalar field in 3+1 dimensions $\phi(\boldsymbol{r}, t)$ is given by:

$$\mathcal{L} = (\partial_\mu \phi^*)\,(\partial^\mu \phi) - V(\phi)\,,$$

where $\partial_\mu = \partial/\partial x^\mu$ and $V(\phi) = -\phi^*\phi + \exp(\lambda\,\phi^*\phi)$.

(a) **[book work]** The Hamiltonian density of the system is

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}\partial_0\phi + \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^*}\partial_0\phi^* - \mathcal{L} = (\partial_0\phi^*)(\partial_0\phi) + (\nabla\phi^*)\cdot(\nabla\phi) - \phi^*\phi + \exp(\lambda\,\phi^*\phi)\ .$$

If $\lambda \leq 0$, the energy diverges to negative infinity for uniform fields with $\phi^*\phi \to \infty$ and the Lagrangian theory becomes intractable. Therefore one must choose $\lambda > 0$. [2]

Since the derivative terms in the Hamiltonian are always non-negative, they can be minimised separately from $V(\phi)$ by choosing a field $\phi = \phi_0 e^{i\theta}$ that is uniform throughout space time. We are then left with the task of finding the minima of $V(\phi) = -\phi_0^2 + \exp(\lambda\phi_0^2)$. Considering $dV/d\phi_0 = -\phi_0 + \lambda\phi_0\exp(\lambda\phi_0^2) = 0$, we see that $\phi_0 = 0$ is the minimum for $\lambda \geq 1$, whereas for $\lambda < 1$ the minima are given by

$$\exp(\lambda\phi_0^2) = 1/\lambda\,,$$

i.e., $\phi_0 = \sqrt{-\ln(\lambda)/\lambda}$. The Lagrangian density is symmetric upon global phase changes, and so are the minima of the energy. Therefore, when $\lambda \in (0,1)$ we have an infinitely degenerate set of minima $\phi = \phi_0 e^{i\theta}$ with $\phi_0 = \sqrt{-\ln(\lambda)/\lambda}$ and $\theta \in [0, 2\pi)$. [4]

(b) **[part book work, part new]** Expanding about the chosen minimum, $\phi = \sqrt{-\ln(\lambda)/\lambda} + \chi$ and $\phi^* = \sqrt{-\ln(\lambda)/\lambda} + \chi^*$, the derivative terms in the Lagrangian density give straightforwardly $(\partial_\mu\chi^*)\,(\partial^\mu\chi)$. We then expand the potential to second order,

$$\begin{aligned}
-\,(\phi_0+\chi^*)\,(\phi_0+\chi) &= -\phi_0^2 - \phi_0\,(\chi+\chi^*) - \chi^*\chi \\
\exp\left[\lambda\,(\phi_0+\chi^*)\,(\phi_0+\chi)\right] &= \exp\left[\lambda\phi_0^2\right]\exp\left[\lambda\phi_0\,(\chi+\chi^*)\right]\exp\left[\lambda\chi^*\chi\right] \\
&\simeq \exp\left[\lambda\phi_0^2\right]\left[1+\lambda\phi_0\,(\chi+\chi^*) + \frac{\lambda^2\phi_0^2}{2}\,(\chi+\chi^*)^2\right]\left[1+\lambda\chi^*\chi\right] \\
&\simeq \exp\left[\lambda\phi_0^2\right]\left[1+\lambda\phi_0\,(\chi+\chi^*) + \frac{\lambda^2\phi_0^2}{2}\,(\chi+\chi^*)^2 + \lambda\chi^*\chi\right] + \mathcal{O}(\chi^3)
\end{aligned}$$

Using the fact that $\phi_0^2 = -\ln\lambda/\lambda$, $\lambda\phi_0^2 = -\ln\lambda$, and $\exp\left[\lambda\phi_0^2\right] = 1/\lambda$, summing the two terms above gives

$$V(\phi) \;\simeq\; \frac{\ln\lambda + 1}{\lambda} - \frac{\ln\lambda}{2}\,(\chi+\chi^*)^2 + \mathcal{O}(\chi^3)\,.$$

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Finally, the Lagrangian density to second order takes the form

$$\mathcal{L} = (\partial_\mu \chi^*)(\partial^\mu \chi) + \frac{\ln \lambda}{2} (\chi + \chi^*)^2 \,,$$

up to an irrelevant constant. [6]

Upon spontaneously breaking a continuous symmetry, we find that the fluctuations of a complex scalar field about one of the minima depend only on the real part of the fluctuating field, $\chi^* + \chi$. Upon writing $\chi = \chi_1 + i\chi_2$, we see that the real scalar field χ_1 is massive (note that $\lambda < 1$ and therefore $\ln \lambda < 0$), whereas the field χ_2 is massless. The appearance of such *soft mode* is typical of the breaking of a continuous symmetry and takes the name of Goldstone mode. [2]

(c) [**book work**] When the complex scalar field is coupled to an electromagnetic gauge field as described in the question, the massless Goldstone mode is absorbed by the electromagnetic field and disappears. In return, the electromagnetic field acquires a mass. This phenomenon is called the Higgs mechanism. [3]

(d) [**new**] The new term in the potential:

$$V(\phi) = -\frac{1}{2}\left(\phi^{*2} + \phi^2\right) - \phi^*\phi + \exp\left(\lambda\, \phi^*\phi\right) \,,$$

breaks the continuous symmetry of a global phase change, but retains the discrete symmetry $\phi \leftrightarrow \phi^*$ (complex conjugation) and $(\phi, \phi^*) \to (-\phi, -\phi^*)$. [2]

Following the hint, we rewrite the first two terms in the potential as $-(\phi^* + \phi)^2/2$, and substitute $\phi = \phi_0 e^{i\theta}$ to obtain:

$$V(\phi) = -2\phi_0^2 \cos^2\theta + \exp\left(\lambda\, \phi_0^2\right) \,.$$

Taking derivatives with respect to θ and ϕ_0 we find:

$$\begin{cases} -4\phi_0 \cos^2\theta + 2\lambda\phi_0 \exp\left(\lambda\, \phi_0^2\right) = 0 \\ -4\phi_0^2 \cos\theta \sin\theta = 0 \end{cases}$$

where $\phi_0 = 0$ is always a solution for all θ. In addition, we see that when $\sin(2\theta) = 0$, other solutions may be present if $2\cos^2\theta = \lambda \exp(\lambda\phi_0^2)$. One readily verifies that $\theta = \pi/2,\ 3\pi/2$ can be discarded, and that $\theta = 0,\ \pi$ are indeed the minima of $V(\phi)$ for $\phi_0 = \sqrt{\ln(2/\lambda)/\lambda}$, provided that $\lambda \in (0, 2)$. [2]

When $\lambda < 2$, the system chooses one of the minima at low energies and breaks the complex conjugation symmetry spontaneously. Expanding about the minimum $\phi_0 = \sqrt{\ln(2/\lambda)/\lambda}$ for convenience, and using the earlier results for the expansion of the exponential term, we obtain:

$$\begin{aligned} V(\phi) &\simeq \text{const.} - 2\phi_0 (\chi^* + \chi) - \frac{1}{2}(\chi^* + \chi)^2 \\ &+ e^{\lambda\phi_0^2}\lambda\phi_0 (\chi^* + \chi) + e^{\lambda\phi_0^2}\frac{\lambda^2\phi_0^2}{2}(\chi^* + \chi)^2 + e^{\lambda\phi_0^2}\lambda\, \chi^*\chi \\ &\simeq \left(\ln\frac{2}{\lambda} - \frac{1}{2}\right)(\chi^* + \chi)^2 + 2\,\chi^*\chi\,, \end{aligned}$$

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where we used the fact that $\lambda\phi_0^2 = \ln(2/\lambda)$ and $\exp(\lambda\phi_0^2) = 2/\lambda$. [3]

Substituting $\chi = \chi_1 + i\chi_2$, we see that $V(\phi) = 4\ln(2/\lambda)\,\chi_1^2 + 2\,\chi_2^2$. In the broken symmetry phase ($\lambda < 2$) both modes have a finite positive mass. Indeed, no Goldstone mode is expected since the symmetry that has been broken spontaneously is discrete and not continuous. [1]

4 A charged particle radiates energy at a rate proportional to the square of its acceleration, $\ddot{x}^2$. For *periodic* motion, this is equivalent to the action of a force $\dddot{x}$ on the particle. If we consider simple harmonic oscillations in one dimension, the equation of motion can then be written as

$$\ddot{x} + x - \varepsilon\dddot{x} = f(t) \qquad\qquad \varepsilon \in \mathbb{R}\,,$$

where $f(t)$ is an external driving force.

(a) **[book work]** The equation of motion can be solved introducing the real time Green's function $G(t-t')$, [1]

$$\left[\frac{d^2}{dt^2} + 1 - \varepsilon\frac{d^3}{dt^3}\right] G(t-t') = \delta(t-t')\,,$$

such that $x(t) = \int G(t-t')f(t')\,dt'$. Given the Fourier transform convention

$$G(t-t') = \int G(\omega)\,e^{-i\omega(t-t')}\,\frac{d\omega}{2\pi}\,,$$

one obtains $x(\omega) = G(\omega)f(\omega)$ and $[-\omega^2 + 1 - i\varepsilon\omega^3]\,G(\omega) = 1$, whereby [3]

$$G(\omega) = \frac{1}{-\omega^2 + 1 - i\varepsilon\omega^3}\,.$$

$G(t-t')$ is the solution to the equation of motion (namely, the response of the system) given a driving force that acts at a single instant in time, $t = t'$. Causality therefore demands that $G(t-t') = 0$ for $t < t'$. [2]

The integral

$$G(t-t') = \int G(\omega)\,e^{-i\omega(t-t')}\,\frac{d\omega}{2\pi}$$

can be computed via Cauchy integration. The contour ought to be closed in the *upper* half complex ω plane for $t < t'$ and in the *lower* half plane for $t > t'$. Since the Cauchy integral vanishes whenever no poles are enclosed, causality demands that $G(\omega)$ has no poles in the upper half plane. [2]

(b) **[1998-99 exam]** The two poles for $\varepsilon = 0$ are $\omega_{1,2}^{(0)} = \pm 1$ (solutions of $-\omega^2 + 1 = 0$). [2]

Following the hint, we substitute the general solution $\omega_{1,2} \simeq \pm 1 + \alpha_{1,2}$ into the equation and we expand to leading order in $|\alpha_{1,2}| \sim |\varepsilon|$:

$$-\omega_{1,2}^2 + 1 - i\varepsilon\omega_{1,2}^3 \simeq \mp 2\alpha_{1,2} \mp i\varepsilon = 0\,.$$

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Therefore, $\alpha_{1,2} = -i\varepsilon/2$ and the two poles for small $|\varepsilon|$ are $\omega_{1,2} \simeq \pm 1 - i\varepsilon/2$. [5]

[part 1998-99 exam, part textbook] Firstly, we note that both poles shift either in the upper or lower half plane depending on the sign of ε. Causality requires them to move in the lower half plane, and therefore $\varepsilon > 0$. (Notice – not for marking – that $\varepsilon < 0$ in the equation of motion corresponds instead to a dipole absorbing energy rather than radiating it.) [3]

The case of the damped simple harmonic oscillator discussed in the lecture notes has a damping term $\gamma\dot{x}$ in the equation of motion, with $\gamma > 0$ corresponding to the physical case of damping (rather than pumping energy into the system, $\gamma < 0$). First order dissipative terms ($\gamma\dot{x}$) have therefore the opposite sign with respect to third order dissipative terms ($-\varepsilon\dddot{x}$). [2]

(c) **[1998-99 exam]** Substituting $\omega_3 \simeq \alpha_3/\varepsilon$ into the equation as suggested,

$$-\omega_3^2 + 1 - i\varepsilon\omega_3^3 \simeq -\frac{\alpha^2 + i\alpha^3}{\varepsilon^2} = 0\,,$$

we see that $\alpha_3 = i$ is a solution, to leading order (the constant term is negligible with respect to the diverging contributions $\propto 1/\varepsilon$). [3]

The third pole $\omega_3 \simeq i/\varepsilon$ appears in the upper half plane for $\varepsilon > 0$, thence violating causality. [2]

(Notice – not for marking – that radiative energy emission from an accelerating charged particle is a relativistic effect, and we have already seen in the lecture notes how relativistic terms in the equation of motion can be incompatible with non-relativistic causality conditions. See for instance the case of a relativistic vs. non-relativistic quantum particle.)

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