NATURAL SCIENCES TRIPOS Part II

Wednesday 13 January 2016

THEORETICAL PHYSICS I

Answers

1 A bead with mass M slides, without friction, along an infinite fixed coil which constrains the bead's cylindrical coordinates, (r_B, θ_B, z_B), to be $(a, \theta_B, b\theta_B)$. A massless spring with zero natural length and spring constant k connects the bead to an unconstrained particle with mass m and cylindrical coordinates (r, θ, z).

(a) [**book work**] If the free particle is at coordinates (r, θ, z), and the bead is at θ_B, show that the Lagrangian for the system is:

$$\begin{aligned} L =& \frac{1}{2} m \left(\dot{r}^2 + \dot{z}^2 + r^2 \dot{\theta}^2 + \frac{M}{m}(a^2 + b^2)\dot{\theta}_B^2 \right) \\ &- \frac{1}{2} k \left(a^2 + r^2 - 2ar\cos(\theta - \theta_B) + (z - b\theta_B)^2 \right). \end{aligned}$$

The "free" particle has kinetic energy

$$KE_m = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 \right)$$

The bead has kinetic energy

$$KE_B = \frac{1}{2} M \left(a^2 \dot{\theta_B}^2 + \dot{z}_B^2 \right),$$

but the bead's z coordinate is simply $z_B = b\theta_B$ so this is

$$KE_B = \frac{1}{2} M \left(a^2 \dot{\theta_B}^2 + b^2 \dot{\theta}_B^2 \right).$$

The length of the spring, l, is given by Pythagoras,

$$l^2 = (r\cos(\theta) - a\cos(\theta_B))^2 + (r\sin(\theta) - a\sin(\theta_B))^2 + (z - z_B)^2.$$

Expanding out recalling $z_B = b\theta_B$ this gives the potential energy

$$\begin{aligned} V = \frac{1}{2} k l^2 &= \frac{1}{2} k (r^2 + a^2 - 2ar(\cos(\theta)\cos(\theta_B) + \sin(\theta)\sin(\theta_B)) + (z - b\theta_B)^2) \\ &= \frac{1}{2} k (r^2 + a^2 - 2ar\cos(\theta - \theta_B) + (z - b\theta_B)^2). \end{aligned}$$

The Lagrangian is thus

$$L = KE_m + KE_B - V$$
$$= \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2 + r^2\dot{\theta}^2 + \frac{M}{m}(a^2+b^2)\dot{\theta}_B^2\right) - \frac{1}{2}k\left(a^2 + r^2 - 2ar\cos(\theta-\theta_B) + (z-b\theta_B)^2\right)$$

however the a^2 term is an irrelevant constant, so this matches the given L.

(b) [**book work**] There are four coordinates, and hence four Euler-Lagrange equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \implies \frac{d}{dt}\left(mr^2\dot{\theta}\right) = -kar\sin(\theta-\theta_B)$$
$$\implies 2m\dot{r}\dot{\theta} + mr\ddot{\theta} = -ka\sin(\theta-\theta_B)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \implies m\ddot{r} = mr\dot{\theta}^2 - kr + ka\cos(\theta-\theta_B)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \implies m\ddot{z} = -k(z-b\theta_B)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_B} = \frac{\partial L}{\partial \theta_B} \implies M(a^2+b^2)\ddot{\theta}_B = kar\sin(\theta-\theta_B) + kb(z-b\theta_B).$$

(c) [**new**] The potential is of the form $V(r, \theta-\theta_B, z-b\theta_B)$. The helical symmetry reveals itself because $\theta \to \theta + c$, $\theta_B \to \theta_B + c$, $z \to z + cb$ leaves the length of the spring, and hence V, unchanged. It requires a simultaneous rotation and elevation of both masses.

The kinetic energy does not depend on θ, z or θ_B, so the Lagrangian is also in the form $L(\dot{r}, \dot{\theta}, \dot{z}, \dot{\theta}_B, r, \theta-\theta_B, z-b\theta_B)$. We thus have

$$\frac{\partial L}{\partial \theta_B} = -\frac{\partial L}{\partial \theta} - b\frac{\partial L}{\partial z}.$$

Applying the Euler=Lagrange equations this yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_B}\right) = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - b\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)$$

so the quantity

$$J = \frac{\partial L}{\partial \dot{\theta}_B} + \frac{\partial L}{\partial \dot{\theta}} + b\frac{\partial L}{\partial \dot{z}}$$

is conserved.

Evaluating this for the Lagrangian in this case, the conserved quantity is

$$J = mr^2\dot{\theta} + M(a^2+b^2)\dot{\theta}_B + bm\dot{z},$$

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and conservation means $\dot{J} = 0$.

(d) [**new**] Since the particle is released from rest and M is negligible, $J = 0$. We thus have

$$r^2\dot{\theta} = -b\dot{z}$$

Integrating both sides with respect to time, we get

$$\int_0^T r^2\dot{\theta}\,\mathrm{d}t = -b\int_0^T \dot{z}\,\mathrm{d}t = -b\Delta z.$$

Using the chain rule, the lhs can be transformed to

$$\int_{t=0}^{t=T} r^2\mathrm{d}\theta = -b\Delta z.$$

so we have

$$\Delta z = -\frac{2}{b}\left(\frac{1}{2}\int_{t=0}^{t=T} r^2\mathrm{d}\theta\right).$$

If the particle's trajectory is projected into the horizontal (r,θ) plane, it forms a closed loop, and $A = (1/2)\int r^2\mathrm{d}\theta$ is the area of this loop.

[Of the last three marks, one will be given if the candidate offers a correct but non-geometric interpretation relating Δz to the time integral of the angular momentum.]

2 (a) [**book work**] Given the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, one obtains two Maxwell's equations for a free electromagnetic field from the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = 0 = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}\right].$$

The derivative

$$\frac{\partial}{\partial(\partial_\mu A_\alpha)}F_{\delta\gamma}F^{\delta\gamma} = F^{\delta\gamma}\frac{\partial}{\partial(\partial_\mu A_\alpha)}F_{\delta\gamma} + F_{\delta\gamma}\frac{\partial}{\partial(\partial_\mu A_\alpha)}F^{\delta\gamma}.$$

The two terms are in fact equal, and by permuting indices each of these is equal to

$$2F^{\delta\gamma}\frac{\partial}{\partial(\partial_\mu A_\alpha)}\partial_\delta A_\gamma = 2F^{\mu\alpha}.$$

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The Euler-Lagrange equations therefore reduce to the 4-vector relation

$$\partial_\mu F^{\mu\alpha} = 0\,.$$

These are just the (inhomogeneous) Maxwell equations:

$$\nabla \cdot \boldsymbol{E} = 0\,, \qquad \nabla \times \boldsymbol{B} = \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}\,,$$

where we used $\partial_0 = \partial/\partial(ct)$,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

and the fact that $c^2 = 1/\mu_0\varepsilon_0$.
[**Not for credit**] The other two Maxwell's equations can be derived directly from the structure of the $F_{\mu\nu}$ tensor, using the so-called Bianchi identity

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0$$

which gives $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$.

(b) [**part book work, part new**] From the given form of the electromagnetic stress-energy tensor

$$T^{\mu\nu} = -F^{\mu}_{\ \lambda} F^{\nu\lambda} - g^{\mu\nu}\mathcal{L}$$

and from the form of the Lagrangian density given above, we obtain

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= -(\partial_\mu F^{\mu}_{\ \lambda})F^{\nu\lambda} - F^{\mu}_{\ \lambda}(\partial_\mu F^{\nu\lambda}) + \tfrac{1}{4} g^{\mu\nu}(\partial_\mu F_{\alpha\beta})F^{\alpha\beta} + \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta}(\partial_\mu F^{\alpha\beta}) \\ &= -(\partial_\mu F^{\mu\lambda})F^{\nu}_{\ \lambda} - F_{\mu\lambda}(\partial^\mu F^{\nu\lambda}) + \tfrac{1}{2} F_{\alpha\beta}(\partial^\nu F^{\alpha\beta})\,. \end{aligned}$$

The first term vanishes because of the Euler-Lagrange equation for a free electromagnetic field, $\partial_\mu F^{\mu\alpha} = 0$.

In the second term, we can relabel the mute indices $\mu \to \alpha$ and $\lambda \to \beta$ for simplicity and use Bianchi's identity

$$\partial^\nu F^{\alpha\beta} + \partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha} = 0$$

to arrive at

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= -F_{\alpha\beta}(\partial^\alpha F^{\nu\beta}) + \tfrac{1}{2} F_{\alpha\beta}(\partial^\alpha F^{\nu\beta} + \partial^\beta F^{\alpha\nu}) \\ &= \tfrac{1}{2} F_{\alpha\beta}(-\partial^\alpha F^{\nu\beta} + \partial^\beta F^{\alpha\nu}) \\ &= \tfrac{1}{2} F_{\alpha\beta}(\partial^\alpha F^{\beta\nu} + \partial^\beta F^{\alpha\nu}) = 0\,, \end{aligned}$$

where we used (repeatedly) the fact that the electromagnetic tensor is antisymmetric, and we finally noticed in the last line that the term in round brackets is instead symmetric in $\alpha \leftrightarrow \beta$.

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(c) **[part book work, part unseen]** The Lagrangian density for the interaction of a complex scalar field ϕ with the electromagnetic field A_μ is

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

where $D_\mu\phi = (\partial_\mu + iqA_\mu)\phi$ and $(D_\mu\phi)^* = (\partial_\mu - iqA_\mu)\phi^*$. Writing out the Lagrangian density in terms of ϕ, ϕ^* and A_μ explicitly, one obtains

$$\begin{aligned}\mathcal{L} &= (\partial_\mu\phi^* - \mathrm{i}qA_\mu\phi^*)(\partial^\mu\phi + \mathrm{i}qA^\mu\phi) - m^2\phi^*\phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= (\partial_\mu\phi^*)(\partial^\mu\phi) - \mathrm{i}qA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + q^2A_\mu A^\mu\phi^*\phi - m^2\phi^*\phi \\ &\quad -\tfrac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu).\end{aligned}$$

The Euler–Lagrange equations for A_μ give

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 \quad &\Rightarrow \quad -\mathrm{i}q(\phi^*\partial^\nu\phi - \phi\partial^\nu\phi^*) + 2q^2A^\nu\phi^*\phi + \partial_\mu F^{\mu\nu} = 0 \\ &\Rightarrow \quad \partial_\mu F^{\mu\nu} = \mathrm{i}q(\phi^*\partial^\nu\phi - \phi\partial^\nu\phi^*) - 2q^2A^\nu\phi^*\phi\end{aligned}$$

One can then verify that

$$\begin{aligned}J^\nu &\equiv \mathrm{i}q\left[\phi^*D^\nu\phi - \phi(D^\nu\phi)^*\right] \\ &= \mathrm{i}q[\phi^*(\partial^\nu\phi + \mathrm{i}qA^\nu\phi) - \phi(\partial^\nu\phi^* - \mathrm{i}qA^\nu\phi^*)] \\ &= \mathrm{i}q(\phi^*\partial^\nu\phi - \phi\partial^\nu\phi^*) - 2q^2A^\nu\phi^*\phi\,.\end{aligned}$$

(d) **[part book work, part unseen]** For the local transformations $\phi' = \mathrm{e}^{-\mathrm{i}q\alpha(x)}\phi$ and $A'_\mu = A_\mu + \partial_\mu\alpha$, one has $\delta\phi = -\mathrm{i}q\alpha\phi$, $\delta\phi^* = \mathrm{i}q\alpha\phi^*$ and $\delta A_\mu = \partial_\mu\alpha$. Thus, by Noether's theorem, $\partial_\mu j^\mu_{\mathrm{N}} = 0$, where

$$\begin{aligned}j^\mu_{\mathrm{N}} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\,\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\,\delta A_\nu \\ &= \mathrm{i}q\alpha\left[\phi^*D^\mu\phi - \phi(D^\mu\phi)^*\right] - F^{\mu\nu}\partial_\nu\alpha \\ &= \alpha J^\mu - F^{\mu\nu}\partial_\nu\alpha.\end{aligned}$$

Therefore, one has

$$\begin{aligned}\partial_\mu j^\mu_{\mathrm{N}} &= \alpha\partial_\mu J^\mu + (\partial_\mu\alpha)J^\mu - (\partial_\mu F^{\mu\nu})(\partial_\nu\alpha) - F^{\mu\nu}\partial_\mu\partial_\nu\alpha \\ &= \alpha\partial_\mu J^\mu\end{aligned}$$

since $\partial_\mu F^{\mu\nu} = J^\nu$ and $F^{\mu\nu} = -F^{\nu\mu}$. Therefore, Noether's theorem implies that $\partial_\mu J^\mu = 0$.

3 Consider the Klein-Gordon Lagrangian density for a complex scalar field in Minkowski space, coupled to an external vector potential A_μ and to a time-dependent driving force $f(t)$:

$$\mathcal{L} = (\partial_\mu\phi^*)\,(\partial^\mu\phi) - m^2\phi^*\phi + ieA_\mu\left[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right] + f(t)\,(\phi + \phi^*)$$

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where $A_\mu = (V(\boldsymbol{r}), 0, 0, 0)$ and $V(\boldsymbol{r})$ is a real function of the space coordinates $\boldsymbol{r}$ but is independent of time.

(a) [**book work**] In order to obtain the Euler-Lagrange equations, we need to compute

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \phi^*} &= -m^2\phi - ieA_\mu \partial^\mu \phi + f(t) \\ \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^*} &= \partial_\mu \left[\partial^\mu \phi + ieA^\mu \phi\right] = \partial_\mu \partial^\mu \phi + ie\left(\partial_\mu A^\mu\right)\phi + ieA^\mu \partial_\mu \phi \end{aligned}$$

For the given vector potential we readily see that $\partial_\mu A^\mu = 0$ and the corresponding Euler-Lagrange equation of motion can be written as

$$\partial_\mu \partial^\mu \phi + 2ieA^\mu(\boldsymbol{r})\partial_\mu \phi + m^2\phi = f(t),$$

and equivalently for ϕ^*.

(b) [**new**] The Green's function $\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t')$ is a solution of the above equation of motion when the right hand side is replaced by $\delta(t-t')\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')$. In order to find the corresponding equation in Fourier space, let us substitute the transform

$$\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} G(\boldsymbol{k}; \omega)\, e^{-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

in the equation

$$\left[\partial_\mu \partial^\mu + 2ieA^\mu(\boldsymbol{r})\partial_\mu + m^2\right] \mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \delta(t-t')\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')\ .$$

The left hand side becomes

$$\int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[-\omega^2 + k^2 + 2eV(\boldsymbol{r})\omega + m^2\right] G(\boldsymbol{k}; \omega)\, e^{-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

where we used $\partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$ and $A^\mu(\boldsymbol{r})\partial_\mu = A^0\partial_t$.
We then multiply both left and right hand side of the equation by $e^{i\omega_0(t-t') - i\boldsymbol{k}_0\cdot(\boldsymbol{r}-\boldsymbol{r}')}$, and integrate over t and $\boldsymbol{r}$. The right hand side gives straightforwardly 1. The left hand side has two contributions:

$$\begin{aligned} &\int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[-\omega^2 + k^2 + m^2\right] G(\boldsymbol{k}; \omega) \int dt \int d^3r\, e^{-i(\omega-\omega_0)(t-t') + i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')} \\ &\quad = \left[-\omega_0^2 + k_0^2 + m^2\right] G(\boldsymbol{k}_0; \omega_0) \\ &\int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[2e\omega \int d^3r\, V(\boldsymbol{r})\, e^{i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')}\right] G(\boldsymbol{k}; \omega) \int dt e^{-i(\omega-\omega_0)(t-t')} \\ &\quad = 2e\omega_0 \int \frac{d^3k}{(2\pi)^3} V(\boldsymbol{k}_0 - \boldsymbol{k}) G(\boldsymbol{k}; \omega_0) \end{aligned}$$

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where we used the fact that

$$\int dt\, e^{-i(\omega-\omega_0)(t-t')} = 2\pi\delta(\omega-\omega_0) \qquad \int d^3r\, e^{i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')} = (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0)$$

and

$$V(\boldsymbol{k}_0-\boldsymbol{k}) = \int d^3r\, V(\boldsymbol{r}) e^{i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

We can then combine these results (and change variables $\boldsymbol{k},\omega \to \boldsymbol{k}',\omega'$ and $\boldsymbol{k}_0,\omega_0 \to \boldsymbol{k},\omega$) to obtain the expression in the exam paper:

$$\left[-\omega^2+k^2+m^2\right] G(\boldsymbol{k};\omega) + 2e\omega \int \frac{d^3k'}{(2\pi)^3} V(\boldsymbol{k}-\boldsymbol{k}') G(\boldsymbol{k}';\omega) = 1\,.$$

(c) **[part book work, part new]** As instructed in the exam paper, we then consider the case where $V(\boldsymbol{k}-\boldsymbol{k}') = -(2\pi)^3\, i\gamma\, \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$:

$$\left[-\omega^2+k^2+m^2-2e\gamma i\omega\right] G(\boldsymbol{k};\omega) = 1\,.$$

It is straightforward to invert the equation and obtain $G(\boldsymbol{k};\omega)$, from which we get

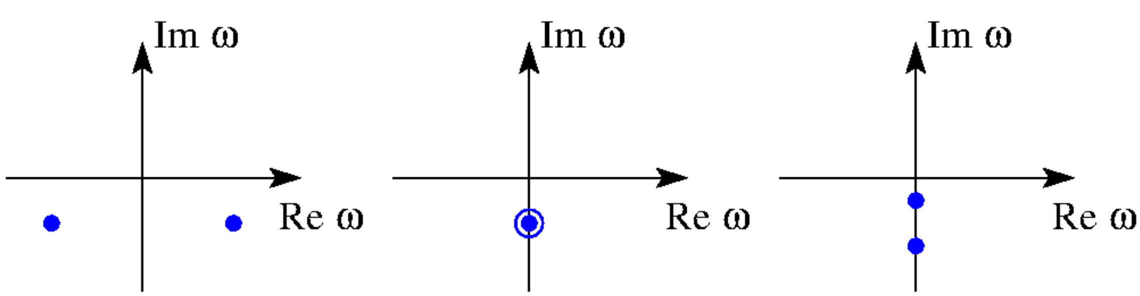
$$G(\boldsymbol{k};t,t') = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-\omega^2-2e\gamma i\omega+k^2+m^2}$$

The location of the poles can be obtained by solving

$$\omega^2+2e\gamma i\omega-k^2-m^2=0 \qquad \to \qquad \omega_{1,2} = -e\gamma i \pm \sqrt{k^2+m^2-e^2\gamma^2}$$

- If $k^2+m^2 > e^2\gamma^2$, the square root term is real and the two poles appear in the lower half of the complex ω plane, a distance $e\gamma$ below the real axis points $\pm\sqrt{k^2+m^2-e^2\gamma^2}$.
- If $k^2+m^2 < e^2\gamma^2$, the square root term is purely imaginary and the two poles sit on the imaginary axis of the complex ω plane. Since $\sqrt{e^2\gamma^2-k^2-m^2}$ is always smaller than $e\gamma$, the two poles lie again in the lower half plane, slightly above and slightly below the point $-ie\gamma$.
- Finally, if $k^2+m^2 = e^2\gamma^2$, the integral has a double pole at the point $-ie\gamma$ on the imaginary axis.

The location of the poles is illustrated schematically in the figure.



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(d) [**book work**] When $k^2 + m^2 > e^2\gamma^2$ (left panel in the figure above), the two poles are

$$\omega_{1,2} = -e\gamma i \pm \sqrt{k^2 + m^2 - e^2\gamma^2} \equiv -e\gamma i \pm \tilde{\omega}\,.$$

In order to compute

$$G(\boldsymbol{k}; t, t') = -\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{(\omega - \omega_1)(\omega - \omega_2)}$$

we use contour integration and Cauchy's theorem. For $t < t'$, we need to close the contour in the upper half plane (cf. the contribution $e^{\mathrm{Im}(\omega)(t-t')}$) and the integral vanishes trivially since the contour does not encircle any poles. For $t > t'$, we need to close the contour in the lower half plane thus encircling the two poles (in the clockwise direction!):

$$\begin{aligned} G(\boldsymbol{k}; t, t') &= -\frac{2\pi i}{2\pi} \left[-\frac{e^{-i\omega_1(t-t')}}{\omega_1 - \omega_2} - \frac{e^{-i\omega_2(t-t')}}{\omega_2 - \omega_1} \right] \\ &= i \left[\frac{e^{-i\tilde{\omega}(t-t')}}{2\tilde{\omega}} - \frac{e^{i\tilde{\omega}(t-t')}}{2\tilde{\omega}} \right] e^{-e\gamma(t-t')} = \frac{\sin\left[\tilde{\omega}(t-t')\right]}{\tilde{\omega}} e^{-e\gamma(t-t')}\,. \end{aligned}$$

4 A ferromagnet consists of a large number, N, of interacting vector spins, $\{\boldsymbol{s}_i\}$, which each have unit length but can point in any direction. Each spin interacts with many other spins via an interaction energy $E = -\boldsymbol{s}_i \cdot \boldsymbol{s}_j$, which favors alignment. We propose a Landau theory of the following form to study $\boldsymbol{m} \equiv \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{s}_i$, the average magnetization of the system:

$$f = am + bm^2 + cm^3 + dm^4$$

where $m = |\boldsymbol{m}|$.

(a) [**bookwork**] The energy has a rotational invariance: changing $\boldsymbol{m} \to R \cdot \boldsymbol{m}$ should not change the energy, for any rotation R, so the energy should be written in terms of tensor-invariants of $\boldsymbol{m}$. It should also be an analytic function. Thus we have

$$f = b\sum_i m_i m_i + d\sum_{i,j} m_i m_i m_j m_j$$

Thus $a = c = 0$ since they do not correspond to rotationally invariant analytic terms.

For $m \to \infty$ not to be the ground state, with divergent negative energy, we must have $d > 0$.

The parameter b controls the transition: $b > 0$ gives $m = 0$, the isotropic state, $b < 0$ gives $m \neq 0$, the aligned state.

(b) [**bookwork**] The observed $\boldsymbol{m}$ is that which minimizes f, which requires

$$\frac{\partial f}{\partial m} = 2bm + 4dm^3 = 0,$$

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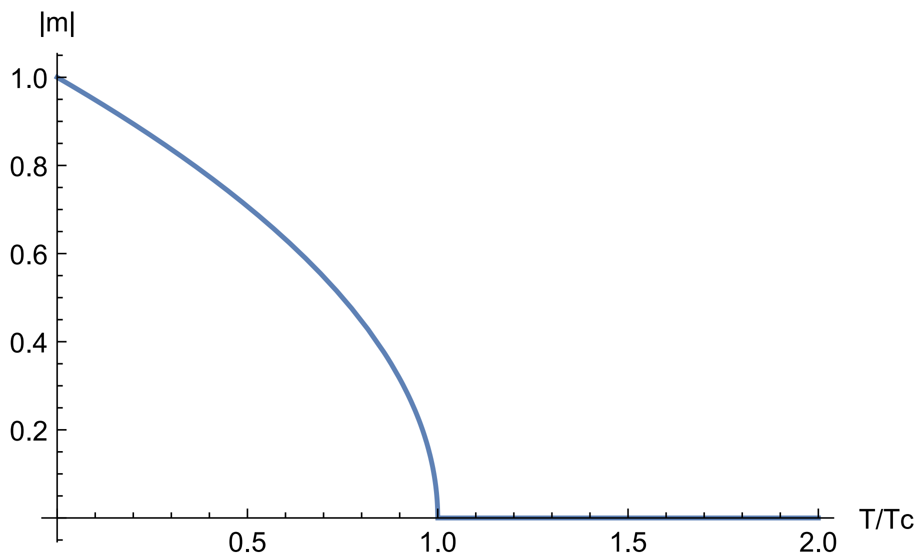
which is solved by $m = 0$, and by $m = \sqrt{-b/(2d)} = \sqrt{(T_c - T)/(2T_c d)}$. There is no $\pm$ as $|m|$ cannot be negative.

The $m = 0$ is the ground state for $b > 0$ (i.e. $T > T_c$) while $m = \sqrt{(T_c - T)/(2T_c d)}$ is the ground state for $b < 0$. Thus the plot looks like



The system breaks its spatial isotropy: the hamiltonian is isotropic, but the state with finite $|m|$ must "choose" a direction for $\boldsymbol{m}$.

Inspecting the plot above, the transition is continuous.

A nematic liquid crystal is similar to a ferromagnet, in that it consists of lots of rod shaped molecules each pointing along a vector $\boldsymbol{s}_i$, which has unit length but can point in any direction. However, in this case the molecules interact via an energy $E \propto -(\boldsymbol{s}_i \cdot \boldsymbol{s}_j)^2$ which equally favors alignment or anti-alignment.

(c)[**new**] The ferro-magnet energy, $E \propto -\boldsymbol{s}_i \cdot \boldsymbol{s}_j$ favors alignment, whereas the nematic energy, $E \propto -(\boldsymbol{s}_i \cdot \boldsymbol{s}_j)^2$, equally favors alignment or anti-alignment. Thus the ferro-magnet aligned state has the spins all pointing in the same direction, giving a finite $\langle \boldsymbol{s} \rangle$, whereas the nematic ground state can contain equal numbers of aligned and anti-aligned spins giving $\langle \boldsymbol{s} \rangle = 0$.

[bonus mark] The nematic energy does not actually distinguish between alignment and anti-alignment, so the fully aligned state is also a ground state. However, on combinatorial grounds there are many more ground states with $\langle \boldsymbol{s} \rangle = 0$, hence this is what is observed.

(d) [**new**] $S_{\alpha\alpha} = \frac{1}{N}\sum_{i=1}^{N}(3s_{i\alpha}s_{i\alpha} - \delta_{\alpha\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(3(1) - 3) = 0$

(e) [**new**] Substituting $S_{\alpha\beta} = Q(3n_\alpha n_\beta - \delta_{\alpha\beta})$ into the provided energy (easy if you remember it is symmetric, and hence diagonal in its principal frame) yields

$$f = 6aQ^2 + 6bQ^3 + 18cQ^4.$$

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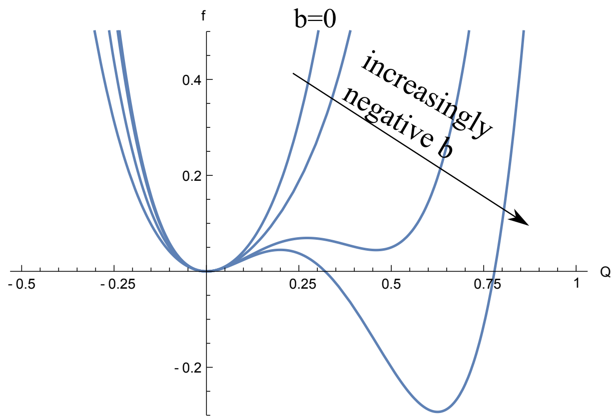
This is graphed for various values of $b < 0$ below:



If $b = 0$ this is a simple symmetric energy with a single minimum. As b is reduced below 0 the energy loses symmetry, and eventually a second minimum appears at positive Q. At some value of b it becomes the global minimum. The transition is thus discontinuous.

(f) [**new**] f always has one minimum with $f = 0$ at $Q = 0$. The second minimum will become the global minimum when it passes $f = 0$, so we examine

$$\begin{aligned} f &= 6aQ^2 + 6bQ^3 + 18cQ^4 = 0 \quad (1)\\ \implies &= aQ^2 + bQ^3 + 3cQ^4 = 0 \end{aligned}$$

which as three solutions, $Q = 0$ (as expected) and

$$Q = \frac{-b \pm \sqrt{b^2 - 12ca}}{6c}.$$

The point where we go from one to three solutions is the point when the second minimum cuts the x axis, which occurs when $b^2 = 12ca$.

So the transition happens when $b = -\sqrt{12ac}$, and the system jumps from $Q = 0$ to

$$Q = \frac{-b}{6c} = \sqrt{\frac{a}{3c}}.$$

END OF PAPER