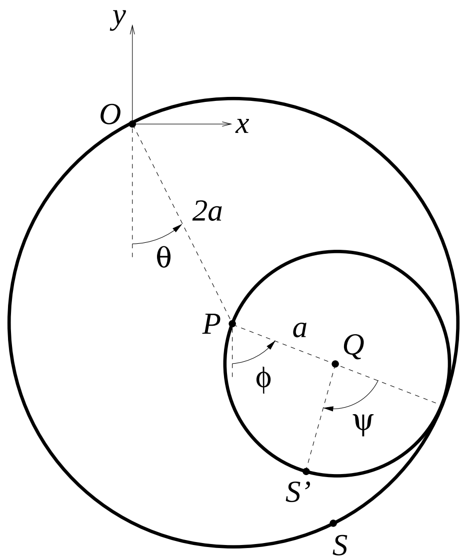
NATURAL SCIENCES TRIPOS Part II

Wednesday 15 January 2014

THEORETICAL PHYSICS I

Answers

1 Adopt the (x, y) coordinates as illustrated in the figure below. Also introduce the rotation angle ψ of the smaller cylinder, where points S and S' coincide in the equilibrium position $\theta = \phi = 0$.



(a) [**unseen, but similar to examples sheet problem**] The position vectors of the centres P and Q of the two cylinders are

$$\boldsymbol{r}_P = 2a(\sin\theta, -\cos\theta)$$
$$\boldsymbol{r}_Q = \boldsymbol{r}_P + a(\sin\phi, -\cos\phi) = a(2\sin\theta + \sin\phi, -2\cos\theta - \cos\phi).$$

Therefore, the velocities of the points P and Q are

$$\dot{\boldsymbol{r}}_P = 2a(\dot\theta\cos\theta, \dot\theta\sin\theta), \qquad \dot{\boldsymbol{r}}_Q = a(2\dot\theta\cos\theta + \dot\phi\cos\phi, 2\dot\theta\sin\theta + \dot\phi\sin\phi),$$

and hence

$$\dot{\boldsymbol{r}}_P \cdot \dot{\boldsymbol{r}}_P = 4a^2\dot\theta^2, \qquad \dot{\boldsymbol{r}}_Q \cdot \dot{\boldsymbol{r}}_Q = a^2[4\dot\theta^2 + \dot\phi^2 + 4\dot\theta\dot\phi\cos(\phi - \theta)],$$

where we have used the trigonometric relation $\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta - \phi)$. Therefore, the kinetic and potential energies of the system can be written as

$$\begin{aligned} T &= \tfrac{1}{2}Ma^2[8\dot{\theta}^2 + \dot{\phi}^2 + 4\dot{\theta}\dot{\phi}\cos(\phi - \theta)] + \tfrac{1}{2}I_1\dot{\theta}^2 + \tfrac{1}{2}I_2(\dot{\psi} - \dot{\phi})^2, \\ V &= -Mga(4\cos\theta + \cos\phi), \end{aligned}$$

where $I_1 = M(2a)^2 = 4Ma^2$ and $I_2 = Ma^2$ are the moments of inertia of the larger and smaller cylinder, respectively, about their centres.

If no friction acts between the cylinders and the angular velocity of the smaller cylinder about its own axis is fixed to zero throughout the motion, then $\psi = \phi$ (so that the line QS' remains vertical), and the kinetic energy termproportional to I_2 vanishes. Thus, one obtains

$$L = \tfrac{1}{2}Ma^2[12\dot{\theta}^2 + \dot{\phi}^2 + 4\dot{\theta}\dot{\phi}\cos(\phi - \theta)] + Mga(4\cos\theta + \cos\phi).$$

Expanding about the equilibrium position $\theta = \phi = 0$ to second-order in the angles and their time derivatives, one immediately finds (up to an irrelevant additive constant)

$$L = \tfrac{1}{2}Ma^2(12\dot{\theta}^2 + \dot{\phi}^2 + 4\dot{\theta}\dot{\phi}) - \tfrac{1}{2}Mga(4\theta^2 + \phi^2).$$

[9]

(b) [**unseen, but similar to examples sheet problem**] The Euler–Lagrange equations immediately give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} &= 0 \quad\Rightarrow\quad 12a\ddot{\theta} + 2a\ddot{\phi} + 4g\theta = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} &= 0 \quad\Rightarrow\quad 2a\ddot{\theta} + a\ddot{\phi} + g\phi = 0. \end{aligned}$$

Assuming that θ and ϕ have the same time dependence $\mathrm{e}^{\mathrm{i}\omega t}$ implies $\ddot{\theta} = -\omega^2\theta$ and $\ddot{\phi} = -\omega^2\phi$, so that

$$\begin{pmatrix} 12a\omega^2 - 4g & 2a\omega^2 \\ 2a\omega^2 & a\omega^2 - g \end{pmatrix}\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, the determinant on the LHS must vanish, which requires

$$2a^2\omega^4 - 4ag\omega^2 + g^2 = 0 \quad\Rightarrow\quad \omega^2 = \frac{4ag \pm \sqrt{16a^2g^2 - 8a^2g^2}}{4a^2} = \frac{(2 \pm \sqrt{2})g}{2a}.$$

Substituting these values for ω^2 back into the matrix equation above gives:
if $\omega = \omega_+ \Rightarrow \phi = -2(\sqrt{2} + 1)\theta \Rightarrow$ cylinders move in opposite directions,
if $\omega = \omega_- \Rightarrow \phi = \;\; 2(\sqrt{2} - 1)\theta \Rightarrow$ cylinders move in same direction. [10]

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(c) [**unseen**] If friction acts so that the smaller cylinder rolls without slipping, then $a\psi = 2a(\phi - \theta) \Rightarrow \psi = 2(\phi - \theta)$. Hence the total kinetic energy becomes (to second-order)

$$\begin{aligned} T &= \tfrac{1}{2}Ma^2(12\dot{\theta}^2 + \dot{\phi}^2 + 4\dot{\theta}\dot{\phi}) + \tfrac{1}{2}Ma^2[2(\dot{\phi} - \dot{\theta}) - \dot{\phi}]^2 \\ &= \tfrac{1}{2}Ma^2(12\dot{\theta}^2 + \dot{\phi}^2 + 4\dot{\theta}\dot{\phi}) + \tfrac{1}{2}Ma^2(\dot{\phi}^2 + 4\dot{\theta}^2 - 4\dot{\theta}\dot{\phi}) \\ &= \tfrac{1}{2}Ma^2(16\dot{\theta}^2 + 2\dot{\phi}^2), \end{aligned}$$

and hence the Lagrangian reads

$$L = \tfrac{1}{2}Ma^2(16\dot{\theta}^2 + 2\dot{\phi}^2) - \tfrac{1}{2}Mga(4\theta^2 + \phi^2).$$

[7]

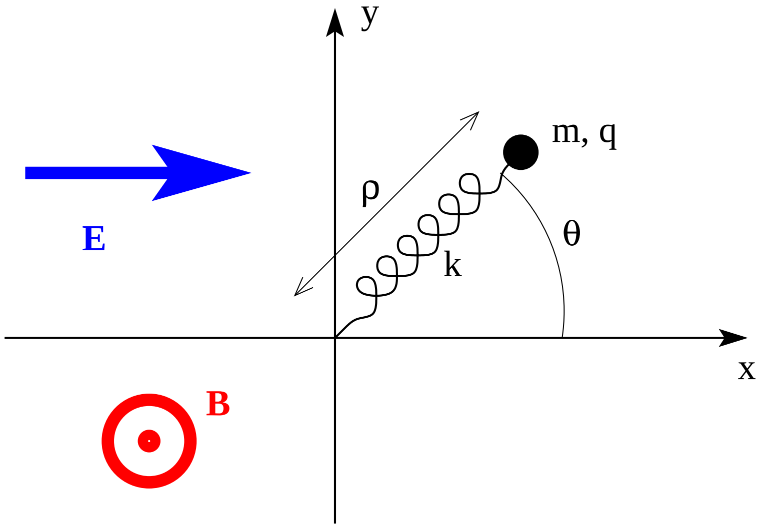
(d) [**unseen, but similar to examples sheet problem**] In this case, the Euler–Lagrange equations immediately give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad 4a\ddot{\theta} + g\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{4a}\theta = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad 2a\ddot{\phi} + g\phi = 0 \quad \Rightarrow \quad \ddot{\phi} + \frac{g}{2a}\phi = 0.$$

Thus, by inspection, the natural frequencies of oscillation are $\omega_1^2 = g/(4a)$ and $\omega_2^2 = g/(2a)$. Moreover, again by inspection:
if $\omega = \omega_1 \Rightarrow \phi = 0 \Rightarrow$ line PQ connecting centres of cylinders remains vertical,
if $\omega = \omega_2 \Rightarrow \theta = 0 \Rightarrow$ large cylinder remains stationary. [7]

2 Consider a particle of mass m and charge $q > 0$ moving in two dimensions, in presence of a uniform static electric field $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, and a uniform static magnetic field perpendicular to the plane, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ ($E > 0$, $B > 0$). The particle is attached to a fixed point on the plane (say, the origin of the reference frame) by an ideal spring of constant k (see figure.)



(a) [**book work**] Let us write the electrostatic potential as $\phi = -Ex$; following the hint in the exam paper, we further choose a vector potential of the form $\boldsymbol{A} = \frac{B}{2}\left(-\rho\sin\theta, \rho\cos\theta, 0\right)$. Using polar coordinates ρ, θ, the kinetic and potential energy of the particle can be written as:

$$T = \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\theta^2\right)$$

and

$$V = \frac{1}{2}k\rho^2 + q\left(\phi - \boldsymbol{v}\cdot\boldsymbol{A}\right) = \frac{1}{2}k\rho^2 - q\left(E\rho\cos\theta + \frac{B}{2}\rho^2\dot\theta\right)$$

which combine to give the Lagrangian in the exam paper: [9]

$$L = \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\theta^2\right) - \frac{1}{2}k\rho^2 + q\left(E\rho\cos\theta + \frac{B}{2}\rho^2\dot\theta\right)$$

(b) [**book work**] The conjugate momenta are

$$p_\rho = \frac{\partial L}{\partial\dot\rho} = m\dot\rho$$

and

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = m\rho^2\dot\theta + \frac{qB}{2}\rho^2$$

The Hamiltonian for the particle takes the form [5]

$$\begin{aligned} H &= p_\rho\dot\rho + p_\theta\dot\theta - L \\ &= m\dot\rho^2 + m\rho^2\dot\theta^2 + \frac{qB}{2}\rho^2\dot\theta - \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\theta^2\right) + \frac{1}{2}k\rho^2 - q\left(E\rho\cos\theta + \frac{B}{2}\rho^2\dot\theta\right) \\ &= \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\theta^2 + \frac{1}{2}k\rho^2 - qE\rho\cos\theta \end{aligned}$$

which as a function of the coordinates and conjugate momenta becomes

$$H = \frac{p_\rho^2}{2m} + \frac{1}{2m}\left(\frac{p_\theta}{\rho} - \frac{qB}{2}\rho\right)^2 + \frac{1}{2}k\rho^2 - qE\rho\cos\theta$$

Hamilton's equations of motion are [4]

$$\begin{aligned} \dot p_\rho &= -\frac{\partial H}{\partial\rho} = \frac{1}{m\rho}\left[\frac{p_\theta^2}{\rho^2} - \left(\frac{qB}{2}\right)^2\rho^2\right] - k\rho + qE\cos\theta \\ \dot\rho &= \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} \\ \dot p_\theta &= -\frac{\partial H}{\partial\theta} = -qE\rho\sin\theta \\ \dot\theta &= \frac{\partial H}{\partial p_\theta} = \frac{1}{m\rho}\left(\frac{p_\theta}{\rho} - \frac{qB}{2}\rho\right) \end{aligned}$$

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(c) [**book work**] Using the Lagrangian at point (a), the Euler-Lagrange equations of motion are

$$\begin{aligned}\frac{\partial L}{\partial \rho} - \frac{d}{dt}\frac{\partial L}{\partial \dot\rho} &= 0 = m\rho\dot\theta^2 - k\rho + qE\cos\theta + qB\rho\dot\theta - m\ddot\rho \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} &= 0 = -qE\rho\sin\theta - 2m\rho\dot\rho\dot\theta - m\rho^2\ddot\theta - qB\rho\dot\rho\end{aligned}$$

The equilibrium solution is obtained by setting to zero all the time derivatives in the above equations, leading to:

$$-k\rho_0 + qE\cos\theta_0 = 0 \qquad -qE\rho_0\sin\theta_0 = 0$$

which impose $\rho_0 = qE/k$ and $\theta_0 = 0$ (away from the singular point $\rho = 0$). [5]

(d) [**new**] Starting from the expanded Euler-Lagrange equations of motion given in the exam paper,

$$\begin{aligned}k\tilde\rho - qB\rho_0\frac{d\tilde\theta}{dt} + m\frac{d^2\tilde\rho}{dt^2} &= 0 \\ qE\tilde\theta + m\rho_0\frac{d^2\tilde\theta}{dt^2} + qB\frac{d\tilde\rho}{dt} &= 0,\end{aligned}$$

we substitute the suggested solution of the form $\tilde\rho = \varepsilon\cos\omega t$ and $\tilde\theta = (\varepsilon/\rho_0)\sin\omega t$, where $\varepsilon \ll 1$:

$$\begin{aligned}k\varepsilon\cos\omega t - qB\rho_0\omega\frac{\varepsilon}{\rho_0}\cos\omega t - m\omega^2\varepsilon\cos\omega t &= 0 = k - qB\omega - m\omega^2 \\ qE\frac{\varepsilon}{\rho_0}\sin\omega t - m\rho_0\omega^2\frac{\varepsilon}{\rho_0}\sin\omega t - qB\omega\varepsilon\sin\omega t &= 0 = qE\frac{1}{\rho_0} - m\omega^2 - qB\omega\end{aligned}$$

The two equations are identical once we recall that $\rho_0 = qE/k$, and they impose that the frequency ω satisfies the condition:

$$\omega_{1,2} = -\frac{qB}{2m}\left[1 \mp \sqrt{1 + \frac{4mk}{q^2B^2}}\right]$$

as in the exam paper. [7]

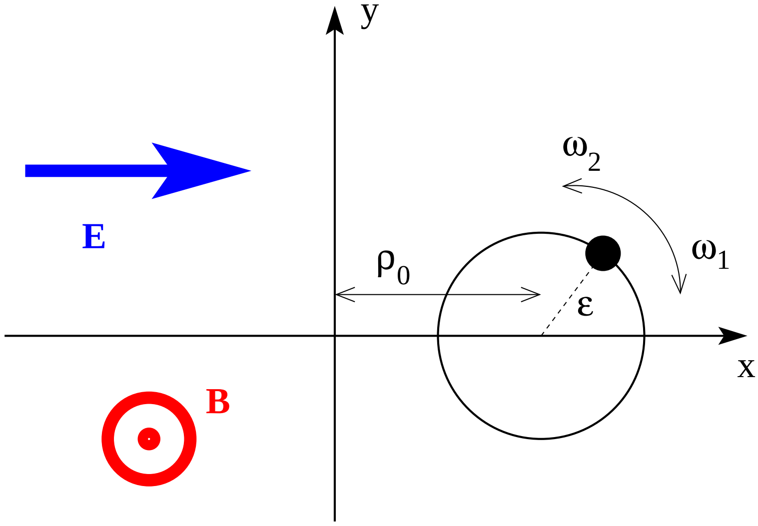
In order to draw the trajectory of the particle in the xy plane corresponding to this special solution, one can notice that

$$\rho = \rho_0 + \varepsilon\cos\omega t \qquad \theta = \theta_0 + \frac{\varepsilon}{\rho_0}\sin\omega t \qquad (\theta_0 = 0)$$

corresponds to the first order expansion in ε of the point $z = \rho_0 + \varepsilon e^{i\omega t}$ (in complex plane notation). Alternatively, one can work out the solution in cartesian coordinates up to first order in ε:

$$x = \rho\cos\theta = \rho_0 + \varepsilon\cos\omega t \qquad y = \rho\sin\theta = \varepsilon\sin\omega t$$

where we recognise the shape of a circle of radius ε centred at $(\rho_0, 0)$, illustrated in the figure.



The two solutions $\omega_{1,2}$ correspond to clockwise and counter-clockwise motion along the circle, respectively. The largest of the two frequencies corresponds to the clockwise motion, which is the direction expected for the motion of a charge in a magnetic field perpendicular to the plane (out of the page). The other solution surprisingly rotates in the opposite direction. [3]

3 (a) [**book work**] The Hamiltonian is defined by

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t), \qquad \text{where } p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Hence, its differential is given by

$$\begin{aligned} \mathrm{d}H &= \sum_i p_i \,\mathrm{d}\dot{q}_i + \sum_i \dot{q}_i \,\mathrm{d}p_i - \sum_i \frac{\partial L}{\partial q_i}\,\mathrm{d}q_i - \sum_i \frac{\partial L}{\partial \dot{q}_i}\,\mathrm{d}\dot{q}_i - \frac{\partial L}{\partial t}\,dt \\ &= \sum_i \dot{q}_i \,\mathrm{d}p_i - \sum_i \frac{\partial L}{\partial q_i}\,\mathrm{d}q_i - \frac{\partial L}{\partial t}\,dt \\ &= \sum_i \dot{q}_i \,\mathrm{d}p_i - \sum_i \dot{p}_i \,\mathrm{d}q_i - \frac{\partial L}{\partial t}\,dt, \qquad \text{since } \dot{p}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}. \end{aligned}$$

But, since $H = H(q_i, p_i, t)$,

$$dH = \sum_i \frac{\partial H}{\partial q_i}\,\mathrm{d}q_i + \sum_i \frac{\partial H}{\partial p_i}\,\mathrm{d}p_i + \frac{\partial H}{\partial t}\,\mathrm{d}t,$$

and thus by comparing terms one obtains Hamilton's equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

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[6]

(b) [**unseen, but canonical transformations are in the lectures**] Consider the transformation $Q_i = Q_i(q_j, p_j, t)$, $P_i = P_i(q_i, p_i, t)$ for $i = 1, \ldots, n$ and define

$$\mathcal{H}(Q_i, P_i, t) = \sum_i P_i \dot{Q}_i - \mathcal{L}(Q_i, \dot{Q}_i, t).$$

From part (a), one finds

$$\dot{Q}_i = \frac{\partial \mathcal{H}}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial \mathcal{H}}{\partial Q_i}, \qquad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \qquad \textbf{if} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}_i}\right) = \frac{\partial \mathcal{L}}{\partial Q_i},$$

and the new Euler–Lagrange equation will hold if

$$\mathcal{L}(Q_i, \dot{Q}_i, t) = L(q_i, \dot{q}_i, t) - \frac{\mathrm{d}G}{\mathrm{d}t},$$

since, from Hamilton's principle of least action, one has

$$\mathcal{S} \equiv \int_{t_1}^{t_2} \mathcal{L}\,\mathrm{d}t = \int_{t_1}^{t_2} \left(L - \frac{\mathrm{d}G}{\mathrm{d}t}\right) \mathrm{d}t = S - [G]_{t_1}^{t_2} \quad \Rightarrow \quad \delta\mathcal{S} = \delta S - \delta[G]_{t_1}^{t_2} = \delta S,$$

provided the variations δq_i and δQ_i vanish at the end-points, as assumed in the exam paper. [6]

(c) [**unseen**] In general,

$$\frac{\mathrm{d}G}{\mathrm{d}t} = L - \mathcal{L} = \sum_i p_i \dot{q}_i - \sum_i P_i \dot{Q}_i + (\mathcal{H} - H).$$

If $\mathcal{H}(Q_i, P_i, t) = H(q_i, p_i, t)$, then

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i p_i \dot{q}_i - \sum_i P_i \dot{Q}_i \quad \Rightarrow \quad dG = \sum_i p_i\,\mathrm{d}q_i - \sum_i P_i\,\mathrm{d}Q_i,$$

so the quantity on the RHS is an exact differential. [5]

(d) [**unseen**] In the general case, the relationship $\mathcal{H}(Q_i, P_i, t) = H(q_i, p_i, t)$ may not hold and so

$$dG = \sum p_i\,\mathrm{d}q_i - \sum_i P_i\,\mathrm{d}Q_i + (\mathcal{H} - H)\,\mathrm{d}t.$$

But, since $G = G(q_i, Q_i, t)$,

$$dG = \sum_i \frac{\partial G}{\partial q_i}\,\mathrm{d}q_i + \sum_i \frac{\partial G}{\partial Q_i}\,\mathrm{d}Q_i + \frac{\partial G}{\partial t}\,\mathrm{d}t,$$

and thus by comparing terms one obtains

$$p_i = \frac{\partial G}{\partial q_i}, \qquad P_i = -\frac{\partial G}{\partial Q_i}, \qquad \mathcal{H} - H = \frac{\partial G}{\partial t}.$$

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If the transformation is such that $\mathcal{H} \equiv 0$, then the equations of motions become $\dot{Q}_i = 0$ and $\dot{P}_i = 0$. This may be achieved by choosing G to satisfy

$$\frac{\partial G}{\partial t} + H(q_i, p_i, t) = 0 \quad \Rightarrow \quad \frac{\partial G}{\partial t} + H\left(q_i, \frac{\partial G}{\partial q_i}, t\right) = 0.$$

[6]

(e) **[unseen]** The Hamiltonian for a one-dimensional harmonic oscillator of mass m and natural frequency ω has the form

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.$$

Hence, in order to transform to new coordinates (Q, P) in which Hamilton's equations take the trivial form $\dot{Q} = 0$ and $\dot{P} = 0$, one requires G to satisfy

$$\frac{\partial G}{\partial t} + \frac{1}{2m}\left(\frac{\partial G}{\partial q}\right)^2 + \tfrac{1}{2}m\omega^2 q^2 = 0.$$

Assuming that $G = G_1(q) + G_2(t)$, one obtains

$$\frac{1}{2m}\left(\frac{\mathrm{d}G_1}{\mathrm{d}q}\right)^2 + \tfrac{1}{2}m\omega^2 q^2 = -\frac{\mathrm{d}G_2}{\mathrm{d}t}.$$

Since the LHS is a function of only q and the RHS is a function of only t, then both sides must be equal to some constant, say β. Therefore, omitting constants of integration

$$G_1(q) = \int \sqrt{2m(\beta - \tfrac{1}{2}m\omega^2 q^2)}\,\mathrm{dq}, \qquad G_2(t) = -\beta t,$$

and hence $G = G_1(q) + G_2(t)$ is given by

$$G = \int \sqrt{2m(\beta - \tfrac{1}{2}m\omega^2 q^2)}\,\mathrm{dq} - \beta t.$$

If one identifies the new coordinate Q with the constant β, thereby satisfying the first one of Hamilton's equation of motion $\dot{Q} = 0$, then from part (d) the new generalised momentum P is given by

$$P = -\frac{\partial G}{\partial Q} = -\frac{\partial G}{\partial \beta} = -\frac{\sqrt{2m}}{2}\int \frac{\mathrm{d}q}{\sqrt{\beta - \frac{1}{2}m\omega^2 q^2}} + t$$

But the second Hamilton's equation of motion is $\dot{P} = 0$, so $P = \text{constant} \equiv \gamma$, and thus

$$t - \gamma = \frac{\sqrt{2m}}{2}\int \frac{\mathrm{d}q}{\sqrt{\beta - \frac{1}{2}m\omega^2 q^2}} = \frac{1}{\omega}\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2\beta}}q\right).$$

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Hence one finally obtains the expected result

$$q = \sqrt{\frac{2\beta}{m\omega^2}} \sin \omega(t-\gamma).$$

[10]

4 (a) **[book work]** The Lagrangian density for a free complex scalar field ϕ of mass m is

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi,$$

where ϕ and ϕ^* are considered as independent fields.

The Euler–Lagrange equations for ϕ and ϕ^*, respectively, give

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \Rightarrow \quad -m^2 \phi^* - \partial_\mu \partial^\mu \phi^* = 0 \;\Rightarrow\; (\partial_\mu \partial^\mu + m^2)\phi^* = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) = 0 \;\Rightarrow\; -m^2 \phi - \partial_\mu \partial^\mu \phi = 0 \quad \Rightarrow \; (\partial_\mu \partial^\mu + m^2)\phi = 0.$$

If $j^\mu = \mathrm{i}q(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$, then

$$\begin{aligned} \partial_\mu j^\mu &= \mathrm{i}q[(\partial_\mu \phi^*)(\partial^\mu \phi) + \phi^* \partial_\mu \partial^\mu \phi - (\partial_\mu \phi)(\partial^\mu \phi^*) - \phi \partial_\mu \partial^\mu \phi^*], \\ &= \mathrm{i}q[\phi^* \partial_\mu \partial^\mu \phi - \phi \partial_\mu \partial^\mu \phi^*], \\ &= \mathrm{i}q[\phi^*(-m^2\phi) - \phi(-m^2\phi^*)] = 0, \end{aligned}$$

where in the last line we use the equations of motions for ϕ and ϕ^*. [4]

(b) **[book work]** For the global transformation $\phi' = \mathrm{e}^{-\mathrm{i}q\alpha}\phi$ we have $\delta\phi = -\mathrm{i}q\alpha\phi$ and $\delta\phi^* = \mathrm{i}q\alpha\phi^*$. Thus, by Noether's theorem, $\partial_\mu j^\mu = 0$ where

$$j^\mu \propto \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^* = \mathrm{i}q\alpha(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*).$$

Hence the required result holds immediately, since $\alpha =$ constant. [4]

(c) **[part book work, part unseen]** The Lagrangian density for the interaction of a complex scalar field ϕ with the electromagnetic field A_μ is

$$\hat{\mathcal{L}} = (\bar{D}_\mu \phi^*)(D^\mu \phi) - m^2 \phi^* \phi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where $D_\mu \phi = (\partial_\mu + iqA_\mu)\phi$, $\bar{D}_\mu \phi^* = (\partial_\mu - iqA_\mu)\phi^*$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Writing out the Lagrangian density in terms of ϕ, ϕ^* and A_μ explicitly, one obtains

$$\begin{aligned} \hat{\mathcal{L}} &= (\partial_\mu \phi^* - \mathrm{i}qA_\mu \phi^*)(\partial^\mu \phi + \mathrm{i}qA^\mu \phi) - m^2 \phi^* \phi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &= (\partial_\mu \phi^*)(\partial^\mu \phi) - \mathrm{i}qA_\mu(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) + q^2 A_\mu A^\mu \phi^* \phi - m^2 \phi^* \phi \\ &\quad -\tfrac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu). \end{aligned}$$

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The Euler–Lagrange equations for A_μ give

$$\frac{\partial \hat{\mathcal{L}}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\mu A_\nu)} \right) = 0 \quad \Rightarrow \quad -\mathrm{i}q(\phi^* \partial^\nu \phi - \phi \partial^\nu \phi^*) + 2q^2 A^\nu \phi^* \phi + \partial_\mu F^{\mu\nu} = 0$$
$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = \mathrm{i}q(\phi^* \partial^\nu \phi - \phi \partial^\nu \phi^*) - 2q^2 A^\nu \phi^* \phi$$
$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = j^\nu - 2q^2 A^\nu \phi^* \phi$$

Now consider

$$\begin{aligned} J^\nu &\equiv \mathrm{i}q(\phi^* D^\nu \phi - \phi \bar{D}^\nu \phi^*) \\ &= \mathrm{i}q[\phi^*(\partial^\nu \phi + \mathrm{i}qA^\nu \phi) - \phi(\partial^\nu \phi^* - \mathrm{i}qA^\nu \phi^*)] \\ &= \mathrm{i}q(\phi^* \partial^\nu \phi - \phi \partial^\nu \phi^*) - 2q^2 A^\nu \phi^* \phi = j^\nu - 2q^2 A^\nu \phi^* \phi. \end{aligned}$$

Therefore the Euler–Lagrange equation can be written $\partial_\mu F^{\mu\nu} = J^\nu$, from which one obtains

$$\begin{aligned} \partial_\nu J^\nu &= \partial_\nu \partial_\mu F^{\mu\nu} \\ &= -\partial_\nu \partial_\mu F^{\nu\mu} \quad \text{since } F^{\nu\mu} = -F^{\mu\nu} \\ &= -\partial_\mu \partial_\nu F^{\nu\mu} \quad \text{swapping order of partial derivatives} \\ &= -\partial_\nu \partial_\mu F^{\mu\nu} \quad \text{swapping indices } \mu \text{ and } \nu \end{aligned}$$

Thus, since it is equal to minus itself, $\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu J^\nu = 0$. [10]

(d) [**part book work, part unseen**] For the local transformation $\phi' = \mathrm{e}^{-\mathrm{i}q\alpha(x)}\phi$, the derivative $D_\mu \phi$ transforms as

$$\begin{aligned} (D_\mu \phi)' &= (\partial_\mu + iqA'_\mu)\phi \mathrm{e}^{-\mathrm{i}q\alpha} \\ &= \mathrm{e}^{-\mathrm{i}q\alpha} \partial_\mu \phi - \mathrm{i}q(\partial_\mu \alpha)\phi \mathrm{e}^{-\mathrm{i}q\alpha} + \mathrm{i}qA'_\mu \phi \mathrm{e}^{-\mathrm{i}q\alpha} \\ &= \mathrm{e}^{-\mathrm{i}q\alpha}(\partial_\mu + \mathrm{i}qA_\mu)\phi \quad \text{since } A'_\mu = A_\mu + \partial_\mu \alpha, \\ &= \mathrm{e}^{-\mathrm{i}q\alpha} D_\mu \phi. \end{aligned}$$

Similarly, since $\phi'^* = \mathrm{e}^{\mathrm{i}q\alpha(x)}\phi^*$ one has $(\bar{D}_\mu \phi^*)' = \mathrm{e}^{\mathrm{i}q\alpha} \bar{D}_\mu \phi^*$.

Combining these two results, one sees that the first term in $\hat{\mathcal{L}}$ is invariant under the local transformation. Moreover, $\phi'^* \phi' = \phi^* \phi$ immediately and

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha - \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{since} \quad \partial_\mu \partial_\nu \alpha = \partial_\nu \partial_\mu \alpha \\ &= F_{\mu\nu}, \end{aligned}$$

and so the full Lagrangian density $\hat{\mathcal{L}}$ is invariant under the local transformation. [7]

(e) [**part book work, part unseen**] For the local transformations $\phi' = \mathrm{e}^{-\mathrm{i}q\alpha(x)}\phi$ and $A'_\mu = A_\mu + \partial_\mu \alpha$, one has $\delta\phi = -\mathrm{i}q\alpha\phi$, $\delta\phi^* = \mathrm{i}q\alpha\phi^*$ and $\delta A_\mu = \partial_\mu \alpha$. Thus, by Noether's theorem, $\partial_\mu j^\mu_{\mathrm{N}} = 0$, where

$$\begin{aligned} j^\mu_{\mathrm{N}} &= \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\mu \phi^*)} \delta\phi^* + \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_\mu A_\nu)} \delta A_\nu \\ &= \mathrm{i}q\alpha(\phi^* D^\mu \phi - \phi \bar{D}^\mu \phi^*) - F^{\mu\nu} \partial_\nu \alpha \\ &= \alpha J^\mu - F^{\mu\nu} \partial_\nu \alpha. \end{aligned}$$

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Therefore, one has

$$\begin{aligned} \partial_\mu j^\mu_{\mathrm{N}} &= \alpha \partial_\mu J^\mu + (\partial_\mu \alpha) J^\mu - (\partial_\mu F^{\mu\nu})(\partial_\nu \alpha) - F^{\mu\nu} \partial_\mu \partial_\nu \alpha \\ &= \alpha \partial_\mu J^\mu \quad \text{since } \partial_\mu F^{\mu\nu} = J^\nu \text{ and } F^{\mu\nu} = -F^{\nu\mu} \end{aligned}$$

Thus, Noether's theorem implies that $\partial_\mu J^\mu = 0$. [8]

5 Consider the Klein-Gordon Lagrangian density for a complex scalar field in Minkowski space, coupled to an external (static) electromagnetic field and to a time-dependent driving force

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi + ieA_\mu \left[\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi\right] + f(t)(\phi + \phi^*)$$

where A_μ is a function of the space coordinates $\boldsymbol{r}$ but it is independent of time.

(a) **[part book work, part new]** In order to obtain the Euler-Lagrange equations, we need to compute

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \phi^*} &= -m^2 \phi - ieA_\mu \partial^\mu \phi + f(t) \\ \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^*} &= \partial_\mu \left[\partial^\mu \phi + ieA^\mu \phi\right] = \partial_\mu \partial^\mu \phi + ie\left(\partial_\mu A^\mu\right)\phi + ieA^\mu \partial_\mu \phi \end{aligned}$$

In the Lorenz gauge $\partial_\mu A^\mu = 0$, the corresponding Euler-Lagrange equation of motion can be written as

$$\partial_\mu \partial^\mu \phi + 2ieA^\mu(\boldsymbol{r})\partial_\mu \phi + m^2 \phi = f(t),$$

and equivalently for ϕ^*. [8]

(b) **[new]** The Green's function $\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t')$ is a solution of the above equation of motion when the right hand side is replaced by $\delta(t-t')\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$. In order to find the corresponding equation in Fourier space, let us substitute the transform

$$\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} G(\boldsymbol{k}; \omega)\, e^{-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

in the equation

$$\left[\partial_\mu \partial^\mu + 2ieA^\mu(\boldsymbol{r})\partial_\mu + m^2\right] \mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \delta(t-t')\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\ .$$

The left hand side becomes

$$\int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[-\omega^2 + k^2 + 2eA^0(\boldsymbol{r})\omega - 2e\boldsymbol{A}(\boldsymbol{r})\cdot \boldsymbol{k} + m^2\right] G(\boldsymbol{k}; \omega)\, e^{-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

where we used $\partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$ and $A^\mu(\boldsymbol{r})\partial_\mu = A^0 \partial_t + \boldsymbol{A}\cdot \boldsymbol{\nabla}$.

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We then multiply both left and right hand side of the equation by $e^{i\omega_0(t-t')-i\boldsymbol{k}_0\cdot(\boldsymbol{r}-\boldsymbol{r}')}$, and integrate over t and $\boldsymbol{r}$. The right hand side gives straightforwardly 1. The left hand side has two contributions:

$$\int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[-\omega^2 + k^2 + m^2\right] G(\boldsymbol{k};\omega) \int dt \int d^3r \, e^{-i(\omega-\omega_0)(t-t')+i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$
$$= \left[-\omega_0^2 + k_0^2 + m^2\right] G(\boldsymbol{k}_0;\omega_0)$$
$$\int dt \int d^3r \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[2eA^0(\boldsymbol{r})\omega - 2e\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{k}\right] G(\boldsymbol{k};\omega) \, e^{-i(\omega-\omega_0)(t-t')+i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$
$$= 2e \int \frac{d^3k}{(2\pi)^3} \left[A^0(\boldsymbol{k}_0-\boldsymbol{k})\omega_0 - \boldsymbol{A}(\boldsymbol{k}_0-\boldsymbol{k})\cdot\boldsymbol{k}\right] G(\boldsymbol{k};\omega_0)$$

where we used the fact that

$$\int dt \, e^{-i(\omega-\omega_0)(t-t')} = 2\pi\delta(\omega-\omega_0) \qquad \int d^3r \, e^{i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')} = (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0)$$

and

$$A^0(\boldsymbol{k}_0-\boldsymbol{k}) = \int d^3r \, A^0(\boldsymbol{r}) e^{i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')} \qquad \boldsymbol{A}(\boldsymbol{k}_0-\boldsymbol{k}) = \int d^3r \, \boldsymbol{A}(\boldsymbol{r}) e^{i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

We can then combine these results (and change variables $\boldsymbol{k}_0, \omega_0 \to \boldsymbol{k}, \omega$) to obtain the expression in the exam paper: [10]

$$\left[-\omega^2 + k^2 + m^2\right] G(\boldsymbol{k};\omega) + 2e \int \frac{d^3k'}{(2\pi)^3} \left[A^0(\boldsymbol{k}-\boldsymbol{k}')\omega - \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{k}'\right] G(\boldsymbol{k}';\omega) = 1$$

(c) [**part book work, part new**] As instructed in the exam paper, we then consider the case where $\boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}') = 0$ and $A^0(\boldsymbol{k}-\boldsymbol{k}') = (2\pi)^3 \, i\gamma \, \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$:

$$\left[-\omega^2 + k^2 + m^2 + 2e\gamma i\omega\right] G(\boldsymbol{k};\omega) = 1$$

It is straightforward to invert the equation and obtain $G(\boldsymbol{k};\omega)$, from which we get [3]

$$G(\boldsymbol{k};t,t') = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-\omega^2 + 2e\gamma i\omega + k^2 + m^2}$$

The location of the poles can be obtained by solving

$$\omega^2 - 2e\gamma i\omega - k^2 - m^2 = 0 \qquad \to \qquad \omega_{1,2} = e\gamma i \pm \sqrt{k^2 + m^2 - e^2\gamma^2}$$

- If $k^2 + m^2 > e^2\gamma^2$, the square root term is real and the two poles appear in the upper half of the complex ω plane, a distance $e\gamma$ above the real axis points $\pm\sqrt{k^2 + m^2 - e^2\gamma^2}$.

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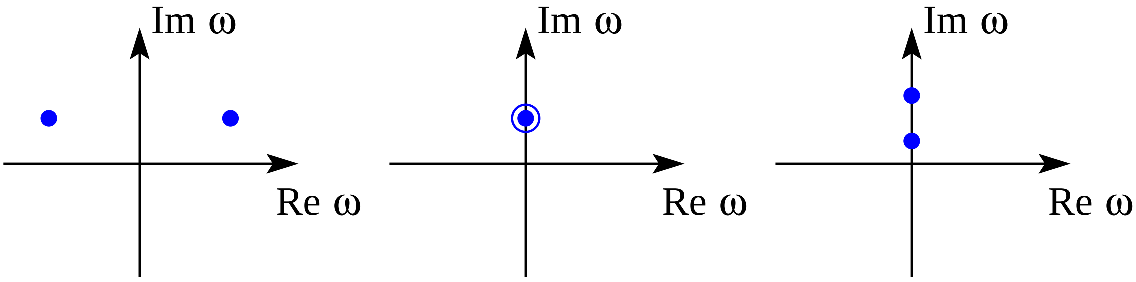
- If $k^2+m^2<e^2\gamma^2$, the square root term is purely imaginary and the two poles sit on the imaginary axis of the complex ω plane. Since $\sqrt{e^2\gamma^2-k^2+m^2}$ is always smaller than $e\gamma$, the two poles lie again in the upper half plane, slightly above and slightly below the point $ie\gamma$.
- Finally, if $k^2+m^2=e^2\gamma^2$, the integral has a double pole at the point $ie\gamma$ on the imaginary axis.

The location of the poles is illustrated schematically in the figure. [8]



(d) [**book work**] When $k^2+m^2>e^2\gamma^2$ (left panel in the figure above), the two poles are

$$\omega_{1,2}=e\gamma i\pm\sqrt{k^2+m^2-e^2\gamma^2}\equiv e\gamma i\pm\tilde{\omega}$$

In order to compute

$$G(\boldsymbol{k};t,t')=-\int\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{(\omega-\omega_1)(\omega-\omega_2)}$$

we use contour integration and Cauchy's theorem. For $t>t'$, we need to close the contour in the lower half plane (cf. the contribution $e^{\mathrm{Im}(\omega)(t-t')}$) and the integral vanishes trivially since the contour does not encircle any poles. For $t<t'$, we need to close the contour in the upper half plane thus encircling the two poles: [4]

$$\begin{aligned}G(\boldsymbol{k};t,t') &= -i\left[\frac{e^{-i\omega_1(t-t')}}{\omega_1-\omega_2}+\frac{e^{-i\omega_2(t-t')}}{\omega_2-\omega_1}\right]\\ &= -i\left[\frac{e^{-i\tilde{\omega}(t-t')}}{2\tilde{\omega}}-\frac{e^{i\tilde{\omega}(t-t')}}{2\tilde{\omega}}\right]e^{e\gamma(t-t')}=-\frac{\sin\left[\tilde{\omega}(t-t')\right]}{\tilde{\omega}}e^{e\gamma(t-t')}\end{aligned}$$

6 Consider the Landau free energy expansion of a system with complex order parameter $\phi(x)$ in 1D:

$$\beta H=\int f\,dx=\int\left[a\phi^*\phi+\frac{1}{2}\left(\phi^*\phi\right)^2+c\left(\partial_x\phi^*\right)\left(\partial_x\phi\right)+d\left(\partial_x^2\phi^*\right)\left(\partial_x^2\phi\right)\right]\,dx$$

with the coefficients a,c,d real.

(a) [**book work**] The physical state of the system is obtained by minimizing the free energy. When $c>0$ and $d=0$, this is equivalent to the free energy expected

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for an Ising ferromagnet discussed in the lecture notes, which is minimized by a uniformly constant order parameter $\phi(x) = \overline{\phi}$, except that the order parameter is now complex. We thus need to minimize the function

$$f\Big|_{\phi(x)=\overline{\phi}} = a\overline{\phi}^*\overline{\phi} + \frac{1}{2}\left(\overline{\phi}^*\overline{\phi}\right)^2 = a|\overline{\phi}|^2 + \frac{1}{2}|\overline{\phi}|^4$$

This is the mexican hat potential encountered when studying spontaneous symmetry breaking in the context of the relativistic scalar complex ϕ^4 field theory. For $a > 0$, the solution is $\overline{\phi} = 0$, whereas for $a < 0$ a continuous line of solutions equally minimize the free energy, $|\overline{\phi}| = \sqrt{-a}$. [6]

The order parameter develops continuously across the transition at $a = 0$, with a discontinuity in its first derivative. This is therefore a second order phase transition, which spontaneously breaks a global phase change symmetry. [6]

(b) [**book work**] Let us consider a uniform magnetic field B coupled to the order parameter $\phi(x)$, where B points along the real axis in the complex ϕ plane. The free energy should then include a further term $-B\left(\phi + \phi^*\right)/2$.

The response of the system to such field will be in the real component of ϕ, and we can therefore simplify the expression for the free energy by considering ϕ to be a real scalar field. Once again, for $c > 0$ and $d = 0$ the free energy is minimised by a uniform constant field $\phi(x) = m$, with m real in this case:

$$f\Big|_{\phi(x)=m} = am^2 + \frac{1}{2}m^4 - Bm\ , \qquad \frac{\partial f}{\partial m} = 0 = 2am + 2m^3 - B$$

This is equivalent to the case of the Ising ferromagnet considered in the lecture notes. The zero-field susceptibility is obtained by taking the derivative with respect to B of the equation that determines the physical value of m (above, right hand side),

$$2a\chi + 6m^2\chi = 1$$

where $\chi = \partial m/\partial B|_{B=0}$. Above the transition ($a > 0$), $m(B = 0) = 0$ and $\chi = 1/2a$. Below the transition ($a < 0$), $m(B = 0) = \pm\sqrt{-a}$ and $\chi = -1/4a$. [6]

(c) [**new**] We then consider $d = 1$, where both a and c are allowed to take on negative values and assume, as suggested in the exam paper, an order parameter of the form $\phi(x) = \phi_0 e^{i(kx+\delta)}$, where $\phi_0 > 0$, k and δ are real constants. A straightforward substitution into the free energy of the system gives:

$$f\Big|_{\phi(x)=\phi_0 e^{i(kx+\delta)}} = a\phi_0^2 + \frac{1}{2}\phi_0^4 + ck^2\phi_0^2 + k^4\phi_0^2 = \left(a + ck^2 + k^4\right)\phi_0^2 + \frac{1}{2}\phi_0^4$$

In order to minimize the free energy, we need to solve the system of differential equations

$$\begin{cases} \frac{\partial f}{\partial \phi_0} = 2\left(a + ck^2 + k^4\right)\phi_0 + 2\phi_0^3 = 0 \\ \frac{\partial f}{\partial k} = 2k\left(c + 2k^2\right)\phi_0^2 = 0 \end{cases}$$

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- If $a > 0$ and $c > 0$, then the only solution is $\phi_0 = 0$ (for any k). This corresponds to the disordered "paramagnetic" phase where the order parameter vanishes.
- If $a < 0$ and $c > 0$, then an additional solution appears, for $k = 0$ and $\phi_0^2 = -a$. In order to see which of the two solutions is the global minimum (and thus corresponds to the physical state), we need to compare the free energies: (i) $f|_{\phi_0=0} = 0$; and (ii) $f|_{k=0,\, \phi_0^2=-a} = -a^2/2$. Clearly the free energy (ii) is lowest, and this region of parameter space corresponds to an ordered phase with uniform ($k = 0$) order parameter $\phi_0^2 = -a$.
- If $c < 0$, a new solution to the second differential equation appears for $k^2 = -c/2$. With this choice for k, the first differential equation gives $\phi_0^2 = c^2/4 - a$ (in addition to $\phi_0 = 0$, which is always a solution). Notice that this new finite value of the order parameter for finite k is allowed only if $a < c^2/4$. The free energy of this new solution

$$f\Big|_{k^2=-c/2,\, \phi_0^2=c^2/4-a} = -\frac{1}{2}\left(\frac{c^2}{4} - a\right)^2 \,,$$

is always negative, thus lower than that for $\phi_0 = 0$, $f = 0$.
In the region of parameter space where $a < 0$ and $c < 0$, this new solution should also be compared to the other one: $k = 0$, $\phi_0^2 = -a$, $f = -a^2/2$:

$$-\frac{1}{2}\left(\frac{c^2}{4} - a\right)^2 < -\frac{a^2}{2} \qquad \to \qquad \frac{c^2}{2}\left(\frac{c^2}{8} - a\right) > 0$$

which is always satisfied for $a < 0$. Therefore, whenever the new finite-k solution is allowed, it is indeed the global minimum of the free energy and thus the physical state of the system. Notice that $k^2 = -c/2$, $\phi_0^2 = c^2/4 - a$ corresponds to a modulated rather than uniform order parameter, with wave vector $k = \pm\sqrt{-c/2}$.

[12]

There is no dependence on δ in the free energy. This term represents a global phase in the order parameter, and the free energy is symmetric upon changes in the global phase; therefore it cannot depend on δ.
As discussed in part (b) above, the system undergoes a spontaneous breaking of the global phase symmetry. In the ordered phase, the system will spontaneously choose a value for δ, but this choice is arbitrary rather than being dictated by a free energy minimisation principle. [3]

END OF PAPER