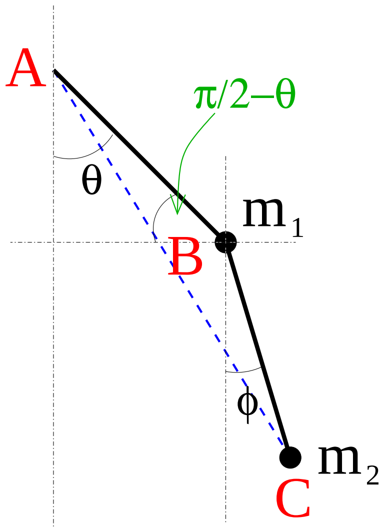
NATURAL SCIENCES TRIPOS   Part II

Wednesday 16 January 2013

THEORETICAL PHYSICS I

Answers

1 Consider a double pendulum composed of two masses m_1 and m_2 attached to two rigid massless rods of equal length ℓ as illustrated in the figure. The two rods are connected by a frictionless hinge at point B and the other end of the first rod is pinned by a frictionless hinge to rotate about point A. A massless spring of elastic constant κ connects the ends points A and C.



(a) [**book work**] Consider the case $m_1 = m_2 = m$. The angle between the two rods is $\pi/2 - \theta + \pi/2 + \phi = \pi + \phi - \theta$ and thus the distance between points A and C is $2\ell \sin[\pi/2 + (\phi - \theta)/2] = 2\ell \cos[(\phi - \theta)/2]$. The positions of mass m_1 and m_2 are given by

$$\boldsymbol{r}_1 = \ell(\sin\theta, -\cos\theta) \qquad \boldsymbol{r}_2 = \boldsymbol{r}_1 + \ell(\sin\phi, -\cos\phi).$$

Therefore, the kinetic and potential energies of the system can be written as

$$T = m\ell^2\dot{\theta}^2 + m\ell^2\dot{\theta}\dot{\phi}\cos(\theta - \phi) + \tfrac{1}{2}m\ell^2\dot{\phi}^2$$
$$V = -2mg\ell\cos\theta - mg\ell\cos\phi + 2\kappa\ell^2\cos^2\left(\tfrac{\theta-\phi}{2}\right),$$

where we used the trigonometric relation $\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) = \cos(\theta - \phi)$. Expanding to second order in θ, ϕ, $\dot\theta$, and $\dot\phi$, after a few lines of algebra, we obtain

$$T = m\ell^2 \left[\dot\theta^2 + \dot\theta\dot\phi + \tfrac{1}{2}\dot\phi^2\right]$$
$$V = mg\ell\theta^2 + \tfrac{1}{2}mg\ell\phi^2 - \tfrac{1}{2}\kappa\ell^2 (\phi - \theta)^2 ,$$

where we ignored irrelevant constants in $V(\theta, \phi)$. These in turn give the expected Lagrangian [9]

$$L = m\ell^2 \left(\dot\theta^2 + \dot\theta\dot\phi + \frac{1}{2}\dot\phi^2\right) - mg\ell\left(\theta^2 + \frac{1}{2}\phi^2\right) + \frac{1}{2}\kappa\ell^2 (\theta - \phi)^2 .$$

(b) [**part book work, part new**] In order to write the Euler-Lagrange equations, we compute

$$\frac{\partial L}{\partial \theta} = -2mg\ell\theta + \kappa\ell^2(\theta - \phi)$$
$$\frac{\partial L}{\partial \phi} = -mg\ell\phi - \kappa\ell^2(\theta - \phi)$$
$$\frac{\partial L}{\partial \dot\theta} = m\ell(2\dot\theta + \dot\phi)$$
$$\frac{\partial L}{\partial \dot\phi} = m\ell(\dot\phi + \dot\theta),$$

to obtain

$$\begin{cases} m\ell^2(2\ddot\theta + \ddot\phi) = -2mg\ell\theta + \kappa\ell^2(\theta - \phi) \\ m\ell^2(\ddot\phi + \ddot\theta) = -mg\ell\phi - \kappa\ell^2(\theta - \phi). \end{cases}$$

For convenience, we introduce the two frequencies $\omega_0^2 = g/\ell$ and $\omega_1^2 = \kappa/m$ and re-write the equations of motion as

$$\begin{cases} 2\ddot\theta + \ddot\phi + 2\omega_0^2\theta - \omega_1^2(\theta - \phi) = 0 \\ \ddot\phi + \ddot\theta + \omega_0^2\phi + \omega_1^2(\theta - \phi) = 0. \end{cases}$$

An oscillatory solution where θ and ϕ have the same frequency and satisfy the initial conditions $\theta(0) = -\phi(0) = \xi$ and $\dot\theta(0) = \dot\phi = 0$ takes the form

$$\theta(t) = \xi\cos(\omega t)$$
$$\phi(t) = -\xi\cos(\omega t).$$

Substituting into the equations of motion and cancelling out the common factor of $\cos(\omega t)$,

$$\begin{cases} -\omega^2 + 2\omega_0^2 - 2\omega_1^2 = 0 \\ -\omega_0^2 + 2\omega_1^2 = 0 \end{cases}$$

we immediately see that a solution exists only if $\omega_0^2 = 2\omega_1^2$ (i.e., if $g/\ell = 2\kappa/m$), in which case $\omega = 2\omega_1^2 = \omega_0^2$. The initial conditions correspond to the two masses at rest with the first rod at an angle ξ, say, to the right of the vertical and the second rod at an opposite angle. Since the two rods have the same length, this means that the mass m_2 lies exactly underneath the hinge at A. The latter condition holds

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throughout the motion of the pendulum, where the mass m_1 swings from side to side whereas the mass m_2 only moves up and down. [7]

(c) [**book work**] If $m_1 = 0, m_2 = m$, the Lagrangian of the system can be obtained in a similar manner,

$$\begin{aligned} T &= m\ell^2\left(\tfrac{1}{2}\dot\theta^2 + \dot\theta\dot\phi + \tfrac{1}{2}\dot\phi^2\right) \\ V &= \tfrac{1}{2}mg\ell\theta^2 + \tfrac{1}{2}mg\ell\phi^2 - \tfrac{1}{2}\kappa\ell^2(\phi-\theta)^2 \\ L &= \tfrac{1}{2}m\ell^2\left(\dot\theta+\dot\phi\right)^2 - \tfrac{1}{2}mg\ell\left(\theta^2+\phi^2\right) + \tfrac{1}{2}\kappa\ell^2(\theta-\phi)^2 \,. \end{aligned}$$

The Euler-Lagrange equations of motion are

$$\begin{cases} m\ell^2(\ddot\theta+\ddot\phi) + mg\ell\theta - \kappa\ell^2(\theta-\phi) = 0 \\ m\ell^2(\ddot\theta+\ddot\phi) + mg\ell\phi + \kappa\ell^2(\theta-\phi) = 0 \end{cases},$$

which can be readily re-written in terms of $\eta = \theta + \phi$ and $\nu = \theta - \phi$ by taking the sum and difference of the two equations,

$$\begin{cases} \ddot\eta + \omega_0^2\eta = 0 \\ \left(\omega_0^2 - \omega_1^2\right)\nu = 0 \end{cases},$$

where this time we defined $\omega_0^2 = g/2\ell$ and $\omega_1^2 = \kappa/m$. [Note the factor of 2 in the new definition of ω_0 with respect to the definition used in part (b) above]. In the new variables, the two equations are decoupled since each variable appears in one and only one of them. The variable η follows simple harmonic motion whereas $\nu = 0$.

If we set $\omega_0 = \omega_1$, the variable ν is undetermined by the equations of motion. Had we made the change of variables $\theta, \phi \to \eta, \nu$ at the Lagrangian level, we would have indeed noticed that L is then independent of ν. This is of course an artefact of the second order expansion. [5]

(d) [**part book work, part new**] Let us consider the Fourier transform of the equation in the paper,

$$\ddot\eta + \gamma\dot\eta + \omega_0^2\eta = \begin{cases} 0 & t < 0 \\ Ae^{-\alpha t} & t \geq 0 \end{cases},$$

where A is a constant of dimensions (time)$^{-2}$:

$$-\omega^2\hat\eta + i\omega\gamma\hat\eta + \omega_0^2\hat\eta = \int dt A e^{-\alpha t} e^{-i\omega t}\,\Theta(t) = \frac{-iA}{\omega - i\alpha},$$

(using the chosen definition of Fourier transform). The Green's function of the left hand side is nothing but

$$\frac{-1}{\omega^2 - i\omega\gamma - \omega_0^2} = \frac{-1}{\left(\omega - i\gamma/2 - \varpi\right)\left(\omega - i\gamma/2 + \varpi\right)},$$

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where $\overline{\omega} = \sqrt{\omega_0^2 - \gamma^2/4}$ (which is a real number as we are working under the assumption that $\omega_0 > \gamma/2$).

A solution to the equation of motion is thus given in reciprocal space by

$$\hat{\eta}(k) = \frac{iA}{(\omega - i\gamma/2 - \overline{\omega})\,(\omega - i\gamma/2 + \overline{\omega})\,(\omega - i\alpha)}.$$

In order to obtain $\eta(t)$ we need to compute the inverse Fourier transform

$$\eta(t) = \int \frac{d\omega}{2\pi} \frac{iAe^{i\omega t}}{(\omega - i\gamma/2 - \overline{\omega})\,(\omega - i\gamma/2 + \overline{\omega})\,(\omega - i\alpha)},$$

which can be done via Cauchy integration of the three simple poles at

$$\begin{aligned} \omega_1 &= i\gamma/2 + \overline{\omega} \\ \omega_2 &= i\gamma/2 - \overline{\omega} \\ \omega_3 &= i\alpha. \end{aligned}$$

Notice that all the poles are above the real axis. For $t < 0$ we ought to close the contour in the lower half plane; the path encircles no poles and the integral vanishes: $\eta(t) = 0$ for $t < 0$. For $t > 0$, the path is in the upper half plane and it encloses all three poles, leading to the solution:

$$\eta(t) = A\left\{\frac{-e^{i\overline{\omega}t-\gamma t/2}}{2\overline{\omega}(\overline{\omega} + i\gamma/2 - i\alpha)} + \frac{e^{-i\overline{\omega}t-\gamma t/2}}{2\overline{\omega}(-\overline{\omega} + i\gamma/2 - i\alpha)} - \frac{e^{-\alpha t}}{(\overline{\omega} + i\gamma/2 - i\alpha)(-\overline{\omega} + i\gamma/2 - i\alpha)}\right\}$$

$$= A\frac{(-\overline{\omega} + i\gamma/2 - i\alpha)e^{i\overline{\omega}t-\gamma t/2} - (\overline{\omega} + i\gamma/2 - i\alpha)e^{-i\overline{\omega}t-\gamma t/2} + 2\overline{\omega}e^{-\alpha t}}{2\overline{\omega}\left[\overline{\omega}^2 + (\alpha - \gamma/2)^2\right]}$$

$$= A\frac{\overline{\omega}\cos(\overline{\omega}t)e^{-\gamma t/2} - (\gamma/2 - \alpha)\sin(\overline{\omega}t)e^{-\gamma t/2} + \overline{\omega}e^{-\alpha t}}{\overline{\omega}\left[\overline{\omega}^2 + (\alpha - \gamma/2)^2\right]}$$

A general solution of the differential equation we are interested in can alternatively be produced by summing the solution of the relative homogeneous problem ($\ddot{\eta} + \gamma\dot{\eta} + \omega_0^2\eta = 0$) to a specific solution of the full equation (which can be guessed of the form $Ce^{-\alpha t}$ and $C = A/(\alpha^2 - \gamma\alpha + \omega_0^2)$ is readily obtained):

$$\eta(t) = C_1\cos(\overline{\omega}t)e^{-\gamma t/2} + C_2\sin(\overline{\omega}t)e^{-\gamma t/2} + \frac{Ae^{-\alpha t}}{\alpha^2 - \gamma\alpha + \omega_0^2}.$$

Observing that $\overline{\omega}^2 + (\alpha - \gamma/2)^2 = \alpha^2 - \gamma\alpha + \omega_0^2$, the last term in the solution obtained via the Green's function is clearly the same as the one in the general solution. The remaining two terms are of the same form in both solutions. A quick calculation shows that the solution obtained via Green's function satisfies $\eta(0) = 0$ and $\dot{\eta}(0) = 0$. [12]

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2 Consider two charged particles of mass m_1 and m_2 and of charge $e_1 = -e_2 = e$ constrained to move in the $x - y$ plane in presence of a magnetic field perpendicular to the plane, $\boldsymbol{B} = B\hat{z}$. The two particles interact via the Coulomb potential $V(r) = -e^2/r$, $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.

(a) [**part book work, part new**] Using the inverse relations $\boldsymbol{r}_{1,2} = \boldsymbol{R} \mp m_{2,1}\boldsymbol{r}/M$, the kinetic energy of the system can be written as

$$T = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 = \frac{M}{2}\dot{\boldsymbol{R}}^2 + \frac{\mu}{2}\dot{\boldsymbol{r}}^2,$$

where $\boldsymbol{R} = (m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2)/M$, $M = m_1 + m_2$, $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, and $\mu = m_1m_2/M$.

With the suggested gauge choice $\boldsymbol{A}(\boldsymbol{r}) = (\boldsymbol{B} \times \boldsymbol{r})/2$, the potential energy of the system is

$$V = -\frac{e^2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} - e\dot{\boldsymbol{r}}_1 \cdot \boldsymbol{A}(\boldsymbol{r}_1) + e\dot{\boldsymbol{r}}_2 \cdot \boldsymbol{A}(\boldsymbol{r}_2)$$

$$= -\frac{e^2}{r} - \frac{e}{2}\left(\dot{\boldsymbol{R}} - \frac{m_2}{M}\dot{\boldsymbol{r}}\right) \cdot \left[\boldsymbol{B} \times \left(\boldsymbol{R} - \frac{m_2}{M}\boldsymbol{r}\right)\right] + \frac{e}{2}\left(\dot{\boldsymbol{R}} + \frac{m_1}{M}\dot{\boldsymbol{r}}\right) \cdot \left[\boldsymbol{B} \times \left(\boldsymbol{R} + \frac{m_1}{M}\boldsymbol{r}\right)\right]$$

$$= -\frac{e^2}{r} + \frac{e}{2}\frac{m_1 - m_2}{M}\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) + \frac{e}{2}\dot{\boldsymbol{R}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) + \frac{e}{2}\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{R})$$

$$= -\frac{e^2}{r} + \frac{e}{2}\frac{m_1 - m_2}{M}\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) + \frac{e}{2}\dot{\boldsymbol{R}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) - \frac{e}{2}\boldsymbol{R} \cdot (\boldsymbol{B} \times \dot{\boldsymbol{r}})$$

$$= -\frac{e^2}{r} + \frac{e}{2}\frac{m_1 - m_2}{M}\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) + e\dot{\boldsymbol{R}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) - \frac{e}{2}\dot{\boldsymbol{R}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) - \frac{e}{2}\boldsymbol{R} \cdot (\boldsymbol{B} \times \dot{\boldsymbol{r}}),$$

where we used the cyclic property of $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. Notice that the last two terms can be neglected as they are proportional to the total time derivative of $\boldsymbol{R} \cdot (\boldsymbol{B} \times \boldsymbol{r})$ and thus contribute only vanishing boundary terms. The Lagrangian can be written as

$$L = \frac{M}{2}\dot{\boldsymbol{R}}^2 + \frac{\mu}{2}\dot{\boldsymbol{r}}^2 + \frac{e^2}{r} - \frac{e}{2}\frac{m_1 - m_2}{M}\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) - e\dot{\boldsymbol{R}} \cdot (\boldsymbol{B} \times \boldsymbol{r}),$$

as expected. [10]

(b) [**book work**] In order to obtain the Hamiltonian of the system we need to evaluate

$$H = \boldsymbol{p} \cdot \dot{\boldsymbol{r}} + \boldsymbol{P} \cdot \dot{\boldsymbol{R}} - L$$

as a function of $\boldsymbol{p}$, $\boldsymbol{P}$, $\boldsymbol{r}$, and $\boldsymbol{R}$, using the relations

$$\boldsymbol{p} = \frac{dL}{d\dot{\boldsymbol{r}}} = \mu\dot{\boldsymbol{r}} - e^*\boldsymbol{B} \times \boldsymbol{r}$$

and

$$\boldsymbol{P} = \frac{dL}{d\dot{\boldsymbol{R}}} = M\dot{\boldsymbol{R}} - e\boldsymbol{B} \times \boldsymbol{r}.$$

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Substituting into the expression for H one gets

$$H = \frac{\boldsymbol{p}}{\mu}\cdot(\boldsymbol{p} + e^*\boldsymbol{B}\times\boldsymbol{r}) + \frac{\boldsymbol{P}}{M}\cdot(\boldsymbol{P} + e\boldsymbol{B}\times\boldsymbol{r}) - \frac{1}{2M}(\boldsymbol{P} + e\boldsymbol{B}\times\boldsymbol{r})^2 - \frac{1}{2\mu}(\boldsymbol{p} + e^*\boldsymbol{B}\times\boldsymbol{r})^2$$

$$-\frac{e^2}{r} + \frac{e^*}{\mu}(\boldsymbol{p} + e^*\boldsymbol{B}\times\boldsymbol{r})\cdot(\boldsymbol{B}\times\boldsymbol{r}) + \frac{e}{M}(\boldsymbol{P} + e\boldsymbol{B}\times\boldsymbol{r})\cdot(\boldsymbol{B}\times\boldsymbol{r}),$$

which gives, after some algebra,

$$H = \frac{[\boldsymbol{P} + e(\boldsymbol{B}\times\boldsymbol{r})]^2}{2M} + \frac{[\boldsymbol{p} + e^*(\boldsymbol{B}\times\boldsymbol{r})]^2}{2\mu} - \frac{e^2}{r}.$$

The Hamiltonian does not depend directly on time and therefore the energy of the system is conserved. Moreover, the Hamiltonian does not depend on $\boldsymbol{R}$ and thus $\boldsymbol{P}$ is conserved. [6]

(c) **[part book work, part new]** If we set $\boldsymbol{P} = 0$ and observe that $|\boldsymbol{B}\times\boldsymbol{r}| = Br$, the Hamiltonian reduces to

$$H = \frac{p^2}{2\mu} + \frac{e^*}{\mu}\boldsymbol{p}\cdot(\boldsymbol{B}\times\boldsymbol{r}) - \frac{e^2}{r} + \frac{e^2B^2r^2}{2M} + \frac{(e^*)^2B^2r^2}{2\mu}$$

$$= \frac{p^2}{2\mu} + \frac{e^*}{\mu}\boldsymbol{p}\cdot(\boldsymbol{B}\times\boldsymbol{r}) - \frac{e^2}{r} + \frac{e^2B^2r^2}{8\mu}.$$

Hamilton's equations of motion are therefore

$$\dot{p}_x = -\frac{dH}{dx} = -\frac{e^*}{\mu}p_yB - \frac{e^2}{r^3}x - \frac{e^2B^2x}{4\mu}$$

$$\dot{p}_y = -\frac{dH}{dy} = \frac{e^*}{\mu}p_xB - \frac{e^2}{r^3}y - \frac{e^2B^2y}{4\mu}$$

$$\dot{x} = \frac{dH}{dp_x} = \frac{p_x}{\mu} - \frac{e^*}{\mu}By \qquad \dot{y} = \frac{dH}{dp_y} = \frac{p_y}{\mu} + \frac{e^*}{\mu}Bx,$$

where $\boldsymbol{r} = (x, y)$. [6]

(d) **[part book work, part new]** From the last two equations of motion we can obtain an expression for p_x and p_y, respectively, as a function of x, y and their time derivatives. Taking the time derivative of the same expressions, we obtain similarly the dependence of $\dot{p}_x$ and $\dot{p}_y$. Substituting these expressions into the first two equations of motion, we obtain

$$\begin{cases} \mu\ddot{x} + e^*B\dot{y} = -\frac{e^2B^2}{4\mu}x - \frac{e^2}{r^3}x - \frac{e^*B}{\mu}(\mu\dot{y} - e^*Bx) \\ \mu\ddot{y} - e^*B\dot{x} = -\frac{e^2B^2}{4\mu}y - \frac{e^2}{r^3}y + \frac{e^*B}{\mu}(\mu\dot{x} + e^*By), \end{cases}$$

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and with a few simplifications,

$$\begin{cases} \mu\ddot{x} + 2e^*B\dot{y} = -\frac{e^2B^2}{M}x - \frac{e^2}{r^3}x \\ \mu\ddot{y} - 2e^*B\dot{x} = -\frac{e^2B^2}{M}y - \frac{e^2}{r^3}y. \end{cases}$$

Using the solutions for x and y suggested in the paper,

$$\begin{cases} x = R\cos(\omega t) \\ y = R\sin(\omega t), \end{cases}$$

with R, ω constant, the two equations above reduce to the same equation

$$\mu\omega^2 - 2e^*B\omega - \frac{e^2B^2}{M} - \frac{e^2}{R^3} = 0$$

which always admits real-valued solutions for ω (recall that M, R, and μ are positive). This demonstrates that the suggested time dependence of $\boldsymbol{r}$, whereby one particle moves in a perfect circle at constant angular velocity in the reference frame of the other particle, is indeed a solution of the equations of motion. This solution is consistent with the expectation that charged particles moving in the $x - y$ plane subject to a magnetic field perpendicular to the plane follow circular trajectories. It is also consistent with the centrosymmetric Coulomb potential between two charged particles, which admits solutions in the form of circular orbits.

The values of ω for which the suggested time dependence is a solution of the equations of motion are given by

$$\omega = \frac{e^*B \pm \sqrt{(e^*)^2B^2 + \mu\left(\frac{e^2B^2}{M} + \frac{e^2}{R^3}\right)}}{\mu}.$$

[11]

3 A dynamical system with Hamiltonian $\mathcal{H}$ is described by independent coordinates q_i $(i = 1, ..., n)$ and corresponding generalised (canonical) momenta p_i.

(a) [**book work**] The *Poisson Bracket* (PB) $\{f, g\}$ of two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ that depend on the generalised coordinates and on time is defined by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right),$$

where in the partial derivatives all the other coordinates, canonical momenta and time are held fixed. From now on we use the summation convention and introduce a shorthand notation $\partial_{qi} f = \partial f / \partial q_i$ etc.

Suppose that $g = q_j$, then $\partial_{qi} g = \delta_{ij}$ and $\partial_{pi} g = 0$ and similarly for $g = p_j$. Hence

$$\{f, q_j\} = -\frac{\partial f}{\partial p_j}, \qquad \{f, p_j\} = +\frac{\partial f}{\partial q_j},$$

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and using the second of these relationships with $f = q_i$ gives $\{q_i, p_j\} = \delta_{ij}$, as expected. This can also of course be obtained directly from the PB definition.

Hamilton's equations of motion are

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p_i}.$$

Hence

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial f}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}$$

and substituting for the coordinate time derivatives immediately gives the required result:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\}.$$

Substituting $h = \mathcal{H}$ in the Jacobi Identity:

$$\{\mathcal{H}, \{f, g\}\} = -\{f, \{g, \mathcal{H}\}\} - \{g, \{\mathcal{H}, f\} = \{f, \frac{\partial g}{\partial t}\} - \{g, \frac{\partial f}{\partial t}\} = \frac{\partial}{\partial t}\{f, g\}$$

and therefore $h = \{f, g\}$ satisfies the same equation as f and g.

(b) A new set of coordinates and momenta (Q_i, P_i) is defined by

$$Q_i = Q_i(q_j, p_j), \qquad P_i = P_i(q_j, p_j), \qquad i = 1, ..., n.$$

A necessary and sufficient condition for this transformation to be *canonical* is $\{Q_i, P_j\} = \delta_{ij}$, $\{Q_i, Q_j\} = 0$, $\{P_i, P_j\} = 0$, where as before the partial derivatives are with respect to the q_i and p_i.

For a system with two degrees of freedom, two new coordinates are defined by

$$Q_1 = q_1^2, \qquad Q_2 = q_1 + q_2.$$

Then

$$\begin{aligned}
0 &= \{Q_1, P_2\} = 2q_1\partial_{p1}P_2 \Rightarrow \partial_{p1}P_2 = 0 \\
0 &= \{Q_2, P_1\} = \partial_{p1}P_1 + \partial_{p2}P_1 \\
1 &= \{Q_1, P_1\} = 2q_1\partial_{p1}P_1 \\
1 &= \{Q_2, P_2\} = \partial_{p1}P_2 + \partial_{p2}P_2 = \partial_{p2}P_2.
\end{aligned}$$

The first of these implies $P_2 = F(q_1, q_2, p_2)$ and the fourth then implies $P_2 = p_2 + f(q_1, q_2)$. The third implies $P_1 = p_1/(2q_1) + G(q_1, q_2, p_2)$ and then the second implies $P_1 = (p_1 - p_2)/(2q_1) + g(q_1, q_2)$.

We further need to impose

$$\begin{aligned}
0 &= \{Q_1, Q_2\} \qquad \text{(trivially satisfied)} \\
0 &= \{P_1, P_2\} = \partial_{q2}g - \frac{1}{2q1}\partial_{q1}f + \frac{1}{2q1}\partial_{q2}f.
\end{aligned}$$

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A particular choice that reduces the Hamiltonian

$$\mathcal{H} = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

to

$$\mathcal{H} = P_1^2 + P_2$$

is evidently

$$P_1 = (p_1 - p_2)/(2q_1), \qquad P_2 = p_2 + (q_1 + q_2)^2,$$

which corresponds to $g = 0$ and $f = (q_1 + q_2)^2$ (one can immediately verify that it satisfied all the conditions above).

4 (a) **[book work]** The Lagrangian density for a triplet of real scalar fields in $3+1$ space-time dimensions, $\varphi_a(t, x_1, x_2, x_3)$ with $a = 1, 2, 3$, is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_a)(\partial^\mu \varphi_a) - \frac{1}{2}\lambda \varphi_a \varphi_a,$$

where $\partial^\mu = (\partial/\partial t, -\partial/\partial x_1, -\partial/\partial x_2, -\partial/\partial x_3)$. We use the Euler-Lagrange equations in the form (see lectures)

$$\frac{\partial \mathcal{L}}{\partial \varphi_a} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial \varphi_a/\partial t)}\right) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial(\partial \varphi_a/\partial x_i)}\right), \qquad (a = 1, 2, 3),$$

to obtain the field equations of motion

$$\partial_\mu \partial^\mu \varphi_a \equiv \frac{\partial^2 \varphi_a}{\partial t^2} - \frac{\partial^2 \varphi_a}{\partial x_1^2} - \frac{\partial^2 \varphi_a}{\partial x_2^2} - \frac{\partial^2 \varphi_a}{\partial x_3^2} + \lambda \varphi_a = 0, \qquad (a = 1, 2, 3).$$

(b) To show that $\mathcal{L}$ is invariant under the infinitesimal SO(3) rotation by an angle θ

$$\varphi_a \to \varphi_a + \theta \epsilon_{abc} n_b \varphi_c$$

where n_a is an arbitrary unit vector and ϵ_{abc} is the three-dimensional Levi-Civita symbol, we consider

$$\delta \mathcal{L} = \frac{1}{2}(\partial_\mu \delta \varphi_a)(\partial^\mu \varphi_a) + \frac{1}{2}(\partial_\mu \varphi_a)(\partial^\mu \delta \varphi_a) - \lambda \varphi_a \delta \varphi_a,$$

where we have kept only first-order terms in $\delta \varphi_a$. Now using $\delta \varphi_a = \theta \epsilon_{abc} n_b \varphi_c$, we note that

$$\varphi_a \delta \varphi_a = \theta \epsilon_{abc} n_b \varphi_a \varphi_c = 0$$

because the product of the φ fields is symmetric in indices a, c, while the Levi-Civita tensor is antisymmetric. Similarly

$$\partial_\mu \varphi_a \partial^\mu \delta \varphi_a = \theta \epsilon_{abc} n_b \partial_\mu \varphi_a \partial^\mu \varphi_c = 0$$

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for the same reason. Hence $\delta\mathcal{L} = 0$ under this particular transformation.

(c) [**book work**] From lectures, the Noether current J^μ corresponding to a symmetry of the Lagrangian density is derived as follows. Suppose that the Lagrangian is invariant under the transformation symmetry under a transformation of the form

$$\varphi_a \to \varphi_a + \delta\varphi_a$$

where $\delta\varphi_a$ is infinitesimal. Symmetry means that $\mathcal{L}$ does not change:

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_a}\delta\varphi_a + \frac{\partial\mathcal{L}}{\partial(\partial\varphi_a/\partial x_i)}\delta(\partial\varphi_a/\partial x_i) + \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_a}\delta\dot{\varphi}_a,$$

where summation over i and a is understood. The Euler-Lagrange equations of motion (see above) then imply that

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\varphi_a/\partial x_i)}\right)\delta\varphi_a + \frac{\partial\mathcal{L}}{\partial(\partial\varphi_a/\partial x_i)}\frac{\partial}{\partial x_i}(\delta\varphi_a) \\
&+ \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_a}\right)\delta\varphi_a + \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_a}\frac{\partial}{\partial t}(\delta\varphi_a) = 0 \\
\Rightarrow & \quad \frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial(\partial\varphi_a/\partial x_i)}\delta\varphi_a\right) + \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_a}\delta\varphi_a\right) = 0 \\
\Rightarrow & \quad \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\varphi_a/\partial x_\mu)}\delta\varphi_a\right) = 0.
\end{aligned}$$

This then implies the existence of a conserved *Noether current*, $\partial_\mu J^\mu = 0$, where

$$J^\mu \propto \sum_j \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\delta\varphi_a.$$

By substituting for the $\delta\varphi_a$ given above, keeping only linear terms in θ, and dropping the overall factor of θ, we obtain the Noether current in this case,

$$J^\mu = \epsilon_{abc} n_b (\partial^\mu\varphi_a)\varphi_c,$$

which satisfies $\partial_\mu J^\mu = 0$. The corresponding continuity equation is

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0,$$

with

$$\rho = \epsilon_{abc} n_b \dot{\varphi}_a \varphi_c, \qquad \boldsymbol{J} = \epsilon_{abc} n_b (\boldsymbol{\nabla}\varphi_a)\,\varphi_c.$$

Now take $\int d^3x$ (over all space) of the continuity equation. The term $\int d^3x\, \boldsymbol{\nabla}\cdot\boldsymbol{J} = \int_{S_\infty} \boldsymbol{dS}\cdot\boldsymbol{J}$ vanishes by the assumed vanishing of the fields as

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$|\boldsymbol{x}| \to \infty$. Hence

$$\begin{aligned} 0 = \int d^3x \, \frac{\partial \rho}{\partial t} &= \frac{\mathrm{d}}{\mathrm{dt}} \int d^3x \, \epsilon_{abc} n_b \dot{\varphi}_a \varphi_c \\ &= n_b \frac{\mathrm{d}}{\mathrm{dt}} \int d^3x \, \epsilon_{abc} \dot{\varphi}_a \varphi_c \\ &= -n_b \frac{\mathrm{dQ_b}}{\mathrm{dt}} \quad \text{where} \quad Q_b = \int d^3x \, \epsilon_{bac} \dot{\varphi}_a \varphi_c \end{aligned}$$

Now n_a is an arbitrary unit vector, and hence $n_a X_a = 0 \Rightarrow X_a = 0$. Hence the three quantities

$$Q_a = \int d^3x \; \epsilon_{abc} \frac{\partial \varphi_b}{\partial t} \varphi_c$$

are all conserved.

To verify this directly using the field equations, we take

$$\begin{aligned} \frac{\mathrm{dQ_a}}{\mathrm{dt}} &= \int d^3x \; \epsilon_{abc} \frac{\partial}{\partial t} \left(\dot{\varphi}_b \varphi_c \right) \\ &= \int d^3x \; \epsilon_{abc} \left(\ddot{\varphi}_b \varphi_c + \dot{\varphi}_b \dot{\varphi}_c \right) \\ &= \int d^3x \; \epsilon_{abc} \ddot{\varphi}_b \varphi_c. \end{aligned}$$

where we have used the antisymmetry/symmetry of the tensors to drop the second term in the penultimate line. Substituting for $\ddot{\varphi}_b$ from the field equations, and again noting that the term proportional to $\varphi_b \varphi_c$ vanishes by antisymmetry/symmetry, we have

$$\begin{aligned} \frac{\mathrm{dQ_a}}{\mathrm{dt}} &= \int d^3x \; \epsilon_{abc} (\nabla^2 \varphi_b) \varphi_c \\ &= \int d^3x \; \epsilon_{abc} \left[\boldsymbol{\nabla} \cdot ((\boldsymbol{\nabla} \varphi_b) \varphi_c) - (\boldsymbol{\nabla} \varphi_b) \cdot (\boldsymbol{\nabla} \varphi_c) \right] \end{aligned}$$

Now the first term inside the integral vanishes because, by the Divergence Theorem, it corresponds to a surface integral at infinity, *under the assumption that the fields vanish there*, and the second term vanishes by antisymmetry/symmetry on the indices b, c. Thus we have verified directly that the quantities Q_a are indeed conserved.

5 The non-linear version of the Klein-Gordon Lagrangian density for a scalar field $\phi(x, t)$ is given by

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + F(\phi) \; ,$$

where $F(\phi)$ is a differentiable function of its argument.

(a) The Euler-Lagrange equations for the fields ϕ are

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0,$$

where $'$ and $\dot{}$ denote partial derivatives with respect to x and t respectively. Substituting for $\mathcal{L}$ then gives

$$f(\phi) + \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0,$$

where $f(\phi) = F'(\phi)$. This is the result to be proved.

Define

$$\phi_1 = \phi(x \cosh\beta + t \sinh\beta, t\cosh\beta + x\sinh\beta).$$

Then

$$\frac{\partial \phi_1}{\partial x} = \cosh\beta\phi' + \sinh\beta\dot{\phi}, \qquad \frac{\partial \phi_1}{\partial t} = \sinh\beta\phi' + \cosh\beta\dot{\phi}.$$

and

$$\frac{\partial^2 \phi_1}{\partial x^2} = \cosh^2\beta\phi'' + \sinh^2\beta\ddot{\phi} + 2\cosh\beta\sinh\beta\dot{\phi}'$$
$$\frac{\partial^2 \phi_1}{\partial t^2} = \sinh^2\beta\phi'' + \cosh^2\beta\ddot{\phi} + 2\cosh\beta\sinh\beta\dot{\phi}'.$$

Subtracting these equations and using $\cosh^2\beta - \sinh^2\beta = 1$ then gives

$$\frac{\partial^2 \phi_1}{\partial x^2} - \frac{\partial^2 \phi_1}{\partial t^2} = \phi'' - \ddot{\phi} = -f(\phi) = -f(\phi_1),$$

and so ϕ_1 satisfies the same partial differential equation as ϕ.

(b) Consider the particular case $f(\phi) = -a\phi + b\phi^n$, for positive constants a, b and integer $n > 1$, and the function

$$w(x) = \left[A\cosh^2(Bx)\right]^{\frac{1}{1-n}}.$$

To show that this is indeed a static solution of the equation we need to prove $w'' - aw + bw^n = 0$, where $w' = \mathrm{d}w/\mathrm{d}x$. Write $p = 2/(1-n) < 0$, then

$$w' = A^{p/2}pB\cosh^{(p-1)}(Bx)\sinh(Bx)$$

and

$$\begin{aligned} w'' &= A^{p/2}pB^2\left[(p-1)\cosh^{(p-2)}(Bx)\sinh^2(Bx) + \cosh^p(Bx)\right] \\ &= A^{p/2}pB^2\left[p\cosh^p(Bx) - (p-1)\cosh^{(p-2)}(Bx)\right]. \end{aligned}$$

Therefore

$$\begin{aligned} w'' - aw + bw^n &= A^{p/2}\left[(p^2B^2 - a)\cosh^p(Bx) - p(p-1)B^2\cosh^{(p-2)}(Bx)\right. \\ &\quad \left. + A^{p(n-1)/2}b\cosh^{pn}(Bx)\right]. \end{aligned}$$

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Now $pn = 2n/(1-n) = p-2$. Therefore to obtain a solution we require the coefficients of $\cosh^p(Bx)$ and $\cosh^{(p-2)}(Bx)$ to vanish, i.e.

$$\begin{aligned} B &= \pm\frac{\sqrt{a}}{|p|} = \pm\frac{1}{2}\sqrt{a}(n-1) \\ bA^{p(n-1)/2} = bA^{-1} &= -p(p-1)B^2 = a(p-1)/p = a(1+n)/2 \end{aligned}$$

i.e.

$$A = \frac{2b}{a(n+1)}, \qquad B = \pm\sqrt{a}(n-1)/2.$$

The function $w(x)$ has a maximum value of $w(0) = A^{1/(1-n)}$ at $x = 0$ and $w \to 0$ as $x \to \pm\infty$. As $n \to \infty$, $w(0) \to 1$.

By the first part of the question, the function

$$\phi(x,t) = w(x\cosh\beta - t\sinh\beta),$$

is a solution of the full equation for any constant β. This has a maximum at $x\cosh\beta - t\sinh\beta = 0$, i.e. $x = vt$ where $v = \tanh\beta > 0$, since β is a positive constant. This therefore corresponds to a travelling-wave solution moving with the same shape in x and travelling in the positive x direction with uniform velocity v.

6 The Landau free energy expansion for a uniaxial ferromagnet in a magnetic field can be written as

$$F = F_0 - hm + \frac{a}{2}m^2 + \frac{b}{4}m^4.$$

(a) [**book work**] Phase transitions occur when a new state (ordered) state develops from the disordered (high temperature) phase. The appearance of the new state is typically described using an order parameter, say the magnetisation in an Ising moded. Ginzburg-Landau theory provides a phenomenological description of critical phenomena based on an appropriately coarse grained order parameter m. It is constructed on the basis of symmetries rather than precise knowledge of the microscopic properties of the system.

Near the transition temperature, where the order parameter vanishes, one can expand the Ginzburg-Landau free energy in powers of m and its derivatives. The latter often penalise spatial variations of the parameter m and thus one can further simplify the free energy expansion by considering the case of uniform m, as in this example.

In an expansion of the form

$$F = F_0 - hm + \frac{a}{2}m^2 + \frac{b}{4}m^4$$

we require that $b > 0$ for the free energy to be bounded from below ($b = 0$ is acceptable if $a > 0$); we also recognise the second term on the right hand side as an

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externally applied magnetic field; finally, we also know that a transition in this free energy occurs when a changes sign (and therefore $a(T_c) = 0$). [4]

(b) [**book work**] The magnetisation of the system scales like a power law of the applied field along the critical isotherm (for small fields), and the exponent is $1/\delta$.

Along the critical isotherm $T = T_c$, $a(T_c) = 0$ and

$$F(T_c) = F_0 - hm + \frac{b}{4}m^4.$$

The equilibrium value of m can be obtained from the equation $\partial F/\partial m = 0$,

$$\frac{\partial F(T_c)}{\partial m} = -h + bm^3 = 0 \qquad \Rightarrow \qquad m = \sqrt[3]{\frac{h}{b}}$$

and therefore $\delta = 1/3$. [6]

(c) [**part book work, part new**] The dependence of the magnetisation on the applied field and temperature can be obtained as above,

$$\frac{\partial F(T)}{\partial m} = -h + a(T)m + b(T)m^3 = 0.$$

However, rather than solving for m, it is convenient to take a further derivative with respect to h:

$$\frac{\partial^2 F(T)}{\partial h \partial m} = -1 + a(T)\chi + 3b(T)m^2\chi = 0 \qquad \Rightarrow \qquad \chi = \frac{1}{a(T) + 3b(T)m^2}.$$

For $t > 0$ ($T > T_c$), we know that $m = 0$ and therefore $\chi = a(T)^{-1}$.

For $t < 0$, $m \neq 0$ and we can re-write the first equation as

$$b(T)m^2 = \frac{h - a(T)m}{m}.$$

Substituting into the expression for χ, one obtains

$$\chi = \left. \frac{1}{a(T) + 3(h/m - a(T))} \right|_{h=0} = \frac{1}{-2a(T)} = \frac{1}{2|a(T)|},$$

where we used the fact that $a(T)$ is negative for $T < T_c$.

Finally, it is straightforwad to combine these results and show that

$$\lim_{t \to 0^+} \frac{\chi(t)}{\chi(-t)} = \frac{2|a(-t)|}{a(t)} = 2,$$

where we used knoweldge of the fact that $a(T) \propto (T - T_c)/T_c$ for T close to T_c. [8]

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(d) [**new**] Once we add $d\,m^3/3$ to the free energy F and set $h = 0$,

$$F = F_0 + \frac{a}{2}m^2 + \frac{d}{3}m^3 + \frac{b}{4}m^4,$$

the equilibrium value of the magnetisation is given by

$$\frac{\partial F}{\partial m} = 0 = am + dm^2 + bm^3,$$

which admits solutions of the form

$$m = 0 \qquad \text{and} \qquad m = \frac{-d \pm \sqrt{d^2 - 4ab}}{2b}.$$

The latter of course are acceptable only if $d^2 > 4ab$.

By looking at the form of the second derivative with respect to m,

$$\frac{\partial^2 F}{\partial m^2} = a + 2dm + 3bm^2,$$

we see immediately that the sign at $m = 0$ is controlled solely by the parameter a: if $a > 0$ then $m = 0$ is a minimum; if $a < 0$, it is a maximum.

If $a < 0$, the other two solutions of $\partial F/\partial m = 0$ are acceptable ($d^2 > 4ab$) and therefore they both have to be minima (observe that in this case the two solutions lie on opposite sides of $m = 0$, and that $F(m) \to +\infty$ for $m \to \pm\infty$).

If $a > 0$, then the other two solutions (if they exist) are either both positive or both negative. They are thereofore a maximum and a minimum, respectively. So long as d is sufficiently small however, the global minimum remains at $m = 0$ (one can check that this is indeed the case for $d < 9ab/2$). The case of larger values of d can well be studied using this very same approach but it is more involved and beyond the scope of the question.

Similarly to the case $d = 0$ considered in the lectures, it is the sign of a that drives a transition from a state with $m = 0$ to a state with $m \neq 0$. However, the cubic term introduces two major changes: (i) the symmetry in the ordered phase is broken explicitely, since $m = (-d - \sqrt{d^2 - 4ab})/(2b)$ has a lower free energy than $m = (-d + \sqrt{d^2 - 4ab})/(2b)$; (ii) the change in the order parameter across the critical point $a = 0$ is discontinuous, hence the transition has become *first order*. [15]

END OF PAPER