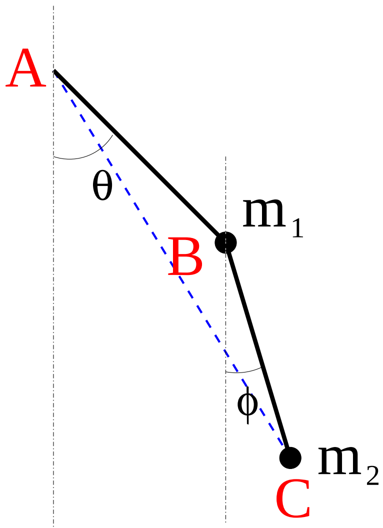
NATURAL SCIENCES TRIPOS Part II

Wednesday 16 January 2013 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains six sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 Consider a double pendulum composed of two masses m_1 and m_2 attached to two rigid massless rods of equal length ℓ, as illustrated in the figure. The two rods are connected by a frictionless hinge at point B and the other end of the first rod is pinned by a frictionless hinge to rotate about point A. A massless spring of elastic constant κ connects the end points A and C.



(a) Consider the case $m_1 = m_2 = m$. Derive the Lagrangian of the system as a function of the angles θ and ϕ. Expand it to second order assuming that both angles as well as their time derivatives are small. Show that the result can be written as

$$L = m\ell^2\left(\dot{\theta}^2 + \dot{\theta}\dot{\phi} + \frac{1}{2}\dot{\phi}^2\right) - mg\ell\left(\theta^2 + \frac{1}{2}\phi^2\right) + \frac{1}{2}\kappa\ell^2\left(\theta - \phi\right)^2,$$

up to irrelevant constants. [9]

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(b) From the Euler-Lagrange equations, derive the equations of motion. For what value(s) of the parameters is there a solution where both θ and ϕ oscillate with the same frequency, and satisfy the initial conditions $\theta(0) = -\phi(0) = \xi$, $\dot\theta(0) = \dot\phi(0) = 0$? [Note: the generic solution is much more involved!] Describe in words the resulting motion of the pendulum. [7]

(c) Obtain the Lagrangian for the case $m_1 = 0, m_2 = m$. Show that the Euler-Lagrange equations of motion in this case can be written in terms of the variables $\eta = \theta + \phi$ and $\nu = \theta - \phi$,

$$\begin{cases} \ddot\eta + \omega_0^2 \eta = 0 \\ (\omega_0^2 - \omega_1^2)\,\nu = 0 \end{cases},$$

where $\omega_0^2 = g/2\ell$ and $\omega_1^2 = \kappa/m$. Comment briefly on the nature of the resulting motion and what happens if $\omega_0 = \omega_1$. [5]

(d) We now add a friction term to the equations of motion in case (c) above, $\gamma\dot\theta + \gamma\dot\phi = \gamma\dot\eta$, $\gamma > 0$, and a time-dependent external force $\exp(-\alpha t)$, $\alpha > 0$, that couples only to the sum of the two angles for $t > 0$:

$$\ddot\eta + \gamma\dot\eta + \omega_0^2\eta = \begin{cases} 0 & t < 0 \\ Ae^{-\alpha t} & t \geq 0 \end{cases},$$

where A is a constant of dimensions $(\text{time})^{-2}$.

Find the Green's function for $\eta(t)$ by solving the equation

$$\ddot\eta + \gamma\dot\eta + \omega_0^2\eta = \delta(t - t')$$

via the Fourier transform

$$\hat\eta(\omega) = \int dt e^{-i\omega t}\eta(t), \qquad \eta(t) = \int \frac{d\omega}{2\pi} e^{i\omega t}\hat\eta(\omega),$$

assuming that $\omega_0 > \gamma/2$. Use it to obtain a solution to the equation

$$\ddot\eta + \gamma\dot\eta + \omega_0^2\eta = Ae^{-\alpha t}\Theta(t),$$

where $\Theta(t)$ is the Heaviside theta function, and show that the result corresponds to the choice of initial conditions $\eta(0) = 0$ and $\dot\eta(0) = 0$ in the expected general solution

$$\eta(t) = C_1\cos(\varpi t)e^{-\gamma t/2} + C_2\sin(\varpi t)e^{-\gamma t/2} + \frac{Ae^{-\alpha t}}{\alpha^2 - \gamma\alpha + \omega_0^2},$$

where $\varpi = \sqrt{\omega_0^2 - \gamma^2/4}$. [12]

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2 Consider two charged particles of mass m_1 and m_2, charge e_1 and e_2 with $e_1 = -e_2 = e$, and position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ that are constrained to move in the $x - y$ plane in the presence of a magnetic field perpendicular to the plane, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The two particles interact via the Coulomb potential $V(r) = -e^2/r$, $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.

(a) Introduce the centre of mass and relative position coordinates

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{M}, \qquad \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \qquad M = m_1 + m_2,$$

and write the Lagrangian of the system in the gauge $\boldsymbol{A}(\boldsymbol{r}) = (\boldsymbol{B} \times \boldsymbol{r})/2$. [Hint: it may be convenient to keep the electromagnetic potential $\boldsymbol{A}$ in its implicit vectorial form $\boldsymbol{A} = (\boldsymbol{B} \times \boldsymbol{r})/2$ rather than explicitly writing out each component.]

Show that, up to a total time derivative that can be neglected, the Lagrangian can be written as

$$L = \frac{M}{2}\dot{\boldsymbol{R}}^2 + \frac{\mu}{2}\dot{\boldsymbol{r}}^2 + \frac{e^2}{r} - \frac{e}{2}\frac{m_1 - m_2}{M}\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) - e\dot{\boldsymbol{R}} \cdot (\boldsymbol{B} \times \boldsymbol{r}),$$

where $\mu = m_1 m_2/M$ is the reduced mass. [10]

(b) Obtain the Hamiltonian of the system and show that it can be written as

$$H = \frac{[\boldsymbol{P} + e\,(\boldsymbol{B} \times \boldsymbol{r})]^2}{2M} + \frac{[\boldsymbol{p} + e^*\,(\boldsymbol{B} \times \boldsymbol{r})]^2}{2\mu} - \frac{e^2}{r},$$

where $e^* = e(m_1 - m_2)/2M$. Use the form of the Hamiltonian to show that the energy of the system and the momentum of the centre of mass are constants of the motion. [6]

(c) Working in the reference frame where $\boldsymbol{P} = 0$, derive Hamilton's equations of motion. [Note that since $\boldsymbol{r}$ lies in the $x - y$ plane and $\boldsymbol{B}$ is perpendicular to it, then $|\boldsymbol{B} \times \boldsymbol{r}| = Br$.] [6]

(d) Use the first order differential equations of motion to derive second order equations for x and y alone, $\boldsymbol{r} = (x, y)$:

$$\begin{cases} \mu\ddot{x} + 2e^*B\dot{y} = -\frac{e^2B^2}{M}x - \frac{e^2}{r^3}x \\ \mu\ddot{y} - 2e^*B\dot{x} = -\frac{e^2B^2}{M}y - \frac{e^2}{r^3}y. \end{cases}$$

Show that these equations admit a solution of the form

$$\begin{cases} x = R\cos(\omega t) \\ y = R\sin(\omega t), \end{cases}$$

with R, ω constants. Comment on the corresponding motion of the two particles: is it consistent with what you would expect for two particles moving in a magnetic field and interacting via a centrosymmetric potential? Compute the dependence of ω on B and R. [11]

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3 A dynamical system with Hamiltonian $\mathcal{H}$ is described by independent coordinates q_i $(i = 1, ..., n)$ and corresponding generalised (canonical) momenta p_i.

(a) Explain what is meant by the *Poisson Bracket* $\{f, g\}$ of two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ that depend on the generalised coordinates q_i and p_i and on time t. [3]

Show that if one of the functions coincides with a coordinate q_j or a momentum p_j, then the Poisson Bracket reduces to a partial derivative, and therefore that $\{q_i, p_j\} = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol. [3]

Starting from Hamilton's equations of motion, show that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\}.$$

[3]

Use the Jacobi Identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

to show that if f and g satisfy the relationships

$$\frac{\partial f}{\partial t} + \{f, \mathcal{H}\} = 0, \qquad \frac{\partial g}{\partial t} + \{g, \mathcal{H}\} = 0,$$

then so does h defined as $h = \{f, g\}$. [5]

(b) A new set of coordinates and momenta (Q_i, P_i) is defined by

$$Q_i = Q_i(q_j, p_j), \qquad P_i = P_i(q_j, p_j), \qquad i = 1, ..., n.$$

What condition must the new coordinates satisfy in order that this transformation is *canonical*, i.e. preserves the form of Hamilton's equations of motion? [3]

For a system with two degrees of freedom, two new coordinates are defined by

$$Q_1 = q_1^2, \qquad Q_2 = q_1 + q_2.$$

Find the most general expressions for the new generalised momenta $P_1(q_1, q_2, p_1, p_2)$ and $P_2(q_1, q_2, p_1, p_2)$ such that the transformation is canonical. [12]

Find a particular choice for the P_i that reduces the Hamiltonian

$$\mathcal{H} = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

to

$$\mathcal{H} = P_1^2 + P_2.$$

[4]

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4 The Lagrangian density for a triplet of real scalar fields in $3+1$ space-time dimensions, $\varphi_a(t, x_1, x_2, x_3)$ with $a = 1, 2, 3$, is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_a)(\partial^\mu \varphi_a) - \frac{1}{2}\lambda \varphi_a \varphi_a,$$

where $\partial^\mu = (\partial/\partial t, -\partial/\partial x_1, -\partial/\partial x_2, -\partial/\partial x_3)$. Use the Euler-Lagrange equations to derive the equations of motion for the fields φ_a. [5]

Show that $\mathcal{L}$ is invariant under the infinitesimal SO(3) rotation by an angle θ

$$\varphi_a \to \varphi_a + \theta \epsilon_{abc} n_b \varphi_c,$$

where n_a is an arbitrary unit vector and ϵ_{abc} is the three-dimensional Levi-Civita symbol, i.e. ϵ_{abc} is 1 if (a, b, c) is an even permutation of $(1, 2, 3)$, -1 if it is an odd permutation, and 0 if any index is repeated. [8]

Derive from first principles the Noether current J^μ corresponding to this symmetry of the Lagrangian density. [12]

Deduce that the three quantities

$$Q_a = \int d^3x \; \epsilon_{abc} \frac{\partial \varphi_b}{\partial t} \varphi_c$$

are all conserved and verify this directly using the field equations satisfied by the φ_a. You should state explicitly any assumptions needed for this result to hold. [8]

5 The non-linear version of the Klein-Gordon Lagrangian density for a scalar field $\phi(x, t)$ is given by

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + F(\phi) \; ,$$

where $F(\phi)$ is a differentiable function of its argument.

(a) Show that the Euler-Lagrange equation for the system leads to the equation of motion

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + f(\phi) \; ,$$

where $f(\phi) = F'(\phi)$. [4]

If $\phi = \phi(x, t)$ is a solution of this equation, show that the function

$$\phi_1 = \phi(x \cosh \beta + t \sinh \beta, t \cosh \beta + x \sinh \beta),$$

where β is an arbitrary constant, is also a solution. [8]

(b) Consider the particular case $f(\phi) = -a\phi + b\phi^n$, for positive constants a, b and integer $n > 1$. Determine the values of constants A and B for which the function

$$w(x) = \left[A \cosh^2(Bx)\right]^{\frac{1}{1-n}}$$

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is a (static) solution of the equation of motion. [12]

Sketch this solution for $-\infty < x < +\infty$ and several different values of n. [4]

Hence show that

$$\phi(x,t) = w(x \cosh\beta - t \sinh\beta),$$

where β is a positive constant, is a travelling-wave solution and describe its dependence on x and t. [5]

6 The Landau free energy expansion for a uniaxial ferromagnet in a magnetic field can be written as

$$F = F_0 - hm + \frac{a}{2}m^2 + \frac{b}{4}m^4,$$

where m is the magnetisation of the system and h represents an externally applied magnetic field.

(a) Briefly discuss the origin of this expansion and what you know *a priori* about (some of) the terms and their coefficients. [4]

(b) Define and compute the exponent δ along the critical isotherm. [6]

(c) Compute the susceptibility $\chi = (\partial m/\partial h)|_{h=0}$ as a function of $t = (T - T_c)/T_c$ both above ($t > 0$) and below ($t < 0$) the transition. Show that [8]

$$\lim_{t\to 0^+} \frac{\chi(t)}{\chi(-t)} = 2.$$

(d) Add the term $d\,m^3/3$ to the free energy F for a generic real parameter d and set $h = 0$. Discuss how the nature of the ordering transition is affected (you may restrict the discussion to values of $d < 9ab/2$ as the solution for larger values of d becomes more involved). [15]

END OF PAPER