NATURAL SCIENCES TRIPOS Part II

Wednesday 19 January 2011

THEORETICAL PHYSICS I

Answers

1 (a) Expressing the Earth's $\hat{\boldsymbol{Z}}$ axis in terms of the rotating frame of reference

$$\hat{\boldsymbol{Z}} = \cos\lambda\hat{\boldsymbol{n}} + \sin\lambda\hat{\boldsymbol{z}}.$$

The position of the mass m from the center of the Earth is

$$\boldsymbol{r} = x\hat{\boldsymbol{n}} + y\hat{\boldsymbol{w}} + (R+z)\hat{\boldsymbol{z}}.$$

Hence

$$\Omega\hat{\boldsymbol{Z}} \times \boldsymbol{r} = -y\Omega\sin\lambda\hat{\boldsymbol{n}} - ((R+z)\Omega\cos\lambda - x\Omega\sin\lambda)\hat{\boldsymbol{w}} + y\Omega\cos\lambda\hat{\boldsymbol{z}}$$

We are given that

$$\boldsymbol{v}_r = \dot{x}\hat{\boldsymbol{n}} + \dot{y}\hat{\boldsymbol{w}}$$

and

$$\boldsymbol{v}_g = \boldsymbol{v}_r + \Omega \times \boldsymbol{r}$$

Hence

$$\boldsymbol{v}_g = (\dot{x} - y\Omega\sin\lambda)\hat{\boldsymbol{n}} + (\dot{y} - ((R+z)\Omega\cos\lambda - x\Omega\sin\lambda))\hat{\boldsymbol{w}} + y\Omega\cos\lambda\hat{\boldsymbol{z}}$$

so that, keeping terms $\Omega x, \Omega y, \Omega^2 R \sim 10^{-5}$ and getting rid of terms $\Omega x^2, \Omega^2 y^2, \Omega xy \sim 10^{-10}$ as well as any constants we have

$$\boldsymbol{v}_g.\boldsymbol{v}_g \approx \dot{x}^2+\dot{y}^2-2\dot{x}y\Omega\sin\lambda+2\dot{y}\Omega(x\sin\lambda-(R+z)\cos\lambda-2\Omega^2Rx\sin\lambda\cos\lambda+const$$

We then have

$$T \approx \frac{1}{2}m(\dot{x}^2+\dot{y}^2-2\dot{x}y\Omega\sin\lambda+2\dot{y}\Omega(x\sin\lambda-(R+z)\cos\lambda-2\Omega^2Rx\sin\lambda\cos\lambda)+const$$

(b) Potential energy

$$V = mgz \approx mgl\theta^2/2 \approx mg\frac{x^2+y^2}{2l}$$

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The correction due to the centrifugal force is

$$\Omega\hat{\boldsymbol{Z}} \times \Omega\hat{\boldsymbol{Z}} \times \boldsymbol{r}.\hat{\boldsymbol{z}} \approx -\Omega^2 R\cos^2\lambda$$

Hence

$$V \approx m(g - \Omega^2 R\cos^2\lambda)\frac{x^2+y^2}{2l}$$

The Lagrangian has the form

$$L = T - V$$

Hence

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2-2\dot{x}y\Omega\sin\lambda+2\dot{y}\Omega(x\sin\lambda-(R+z)\cos\lambda-2\Omega^2 Rx\sin\lambda\cos\lambda))-m\tilde{g}\frac{x^2+y^2}{2l}$$

Euler Lagrange equations in x

$$\begin{aligned} \frac{\partial L}{\partial x} &= m\dot{y}\Omega\sin\lambda - mR\Omega^2\sin\lambda\cos\lambda - \frac{m\tilde{g}x}{l} && (1)\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} - \dot{y}\Omega\sin\lambda && (2)\end{aligned}$$

Hence we have

$$\ddot{x} - 2\dot{y}\Omega\sin\lambda + \frac{\tilde{g}}{l}x + R\Omega^2\sin\lambda\cos\lambda = 0$$

Euler Lagrange equations in y

$$\begin{aligned} \frac{\partial L}{\partial y} &= m\dot{x}\Omega\sin\lambda - \frac{m\tilde{g}x}{l} && (3)\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m\ddot{y} - 2m\dot{x}\Omega\sin\lambda && (4)\end{aligned}$$

Hence we have

$$\ddot{y} + 2\dot{x}\Omega\sin\lambda + \frac{\tilde{g}}{l}y = 0$$

Set $\tilde{x} = x - lR\Omega^2\cos\lambda\sin\lambda/\tilde{g}$ then equations of motion simplify to

$$\begin{pmatrix}\ddot{\tilde{x}}\\ \ddot{y}\end{pmatrix} + \begin{pmatrix}0 & -2\Omega\sin\lambda\\ 2\Omega\sin\lambda & 0\end{pmatrix}\begin{pmatrix}\dot{\tilde{x}}\\ \dot{y}\end{pmatrix} + \frac{\tilde{g}}{l}\begin{pmatrix}\tilde{x}\\ y\end{pmatrix} = 0 \qquad (5)$$

where, α is the projection of the Earth's rotational velocity onto the $\hat{\boldsymbol{z}}$ axis, $\beta = \tilde{g}/l$ is the square of the reduced frequency of the pendulum, and $\tilde{x}$ is the offset origin in the $\hat{\boldsymbol{n}}$ direction that results from the rotation of the Earth causing the equilibrium position of the pendulum to move south.

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Taking the inverse of the matrix in the question and defining vectors $\boldsymbol{r}$ and $\boldsymbol{R}$ (note the redefinition of $\boldsymbol{r}$ as the coordinate of the mass m in the rotating frame)

$$\begin{aligned} \boldsymbol{r} &= \begin{pmatrix} \tilde{x} \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = S\boldsymbol{R} && (6) \\ \dot{\boldsymbol{r}} &= \dot{S}\boldsymbol{R} + S\dot{\boldsymbol{R}} && (7) \\ \ddot{\boldsymbol{r}} &= -\alpha^2 S\boldsymbol{R} + 2\dot{S}\dot{\boldsymbol{R}} + S\ddot{\boldsymbol{R}} && (8) \end{aligned}$$

Substituting (6),(7) and (8) into (5) we get

$$\ddot{\boldsymbol{R}} + (\beta + \alpha^2)\boldsymbol{R} = 0$$

This is the equation for simple harmonic motion.

$$\omega^2 = \frac{g}{l} - \frac{\Omega^2 R \cos^2\lambda}{l} + (\Omega\sin\lambda)^2$$

Comparing orders of magnitude one has $\Omega^2 \sim 10^{-9}$ while $\Omega^2 R/l \sim 10^{-3}$ hence

$$\omega^2 \approx \frac{g}{l} - \frac{\Omega^2 R \cos^2\lambda}{l}$$

The pendulum therefore executes simple harmonic motion in the rotating frame X, Y with reduced frequency, this frame then slowly rotates at $\Omega\sin\lambda$.

(i) For $\lambda = 0$ we are at the equator and the two frames X, Y and x, y are the same. The pendulum therefore appears to maintain the orientation of its oscillation in the x,y frame but will have a lower frequency

$$\omega^2 \approx \frac{g}{l} - \frac{\Omega^2 R}{l}$$

(ii) For $\lambda = \pi/3.44$ the pendulum oscillates at frequency

$$\omega^2 = \frac{g}{l} - \frac{\Omega^2 R \cos^2\lambda}{l}$$

in the X,Y frame. The rotation in the x,y, frame is at a frequency of $\Omega\sin\lambda$.

(iii) For $\lambda = \pi/2$ we have

$$\omega^2 \approx \frac{g}{l}$$

The pendulum maintains its direction of oscillation in the Galilean reference frame - relative to the distant stars.

2 A canonical transformation is a (possibly) mixed position and coordinate transformation that preserves the Poisson Bracket relationships and therefore produces a valid Hamiltonian.

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(a) We need to prove that

$$\{Q_i, P_j\} = \delta_{i,j}$$

We solve equations (4) to get

$$
\begin{aligned}
P_1 &= \frac{1}{8}((\alpha x - 2p_y/\alpha)^2 + (\alpha y + 2p_x/\alpha)^2) && (9)\\
Q_1 &= \arctan(\frac{\alpha x - 2p_y/\alpha}{\alpha y + 2p_x/\alpha}) && (10)\\
P_2 &= (\alpha x + 2P_y/\alpha)/2 && (11)\\
Q_2 &= (\alpha y - 2P_x/\alpha)/2 && (12)
\end{aligned}
$$

We now take partial derivatives. Defining $A = \alpha x - 2p_y/\alpha$ and $B = \alpha y + 2p_x/\alpha$:

For P_1 we have

$$
\begin{aligned}
\frac{\partial P_1}{\partial x} &= \alpha A/4 && (13)\\
\frac{\partial P_1}{\partial y} &= \alpha B/4 && (14)\\
\frac{\partial P_1}{\partial p_x} &= B/2\alpha && (15)\\
\frac{\partial P_1}{\partial p_y} &= -A/2\alpha && (16)\\
& && (17)
\end{aligned}
$$

For Q_1 we have

$$
\begin{aligned}
\frac{\partial Q_1}{\partial x} &= \frac{1}{1 + A^2/B^2}.\alpha B && (18)\\
\frac{\partial Q_1}{\partial y} &= \frac{1}{1 + A^2/B^2}. - \alpha A/B^2 && (19)\\
\frac{\partial Q_1}{\partial p_x} &= \frac{1}{1 + A^2/B^2}. - 2A/\alpha B^2 && (20)\\
\frac{\partial Q_1}{\partial p_y} &= \frac{1}{1 + A^2/B^2}. - 2/\alpha B && (21)\\
& && (22)
\end{aligned}
$$

For P_2 we have

$$
\begin{aligned}
\frac{\partial P_2}{\partial x} &= \alpha/2 && (23)\\
\frac{\partial P_2}{\partial y} &= 0 && (24)
\end{aligned}
$$

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$$\frac{\partial P_2}{\partial p_x} = 0 \tag{25}$$
$$\frac{\partial P_2}{\partial p_y} = 1/\alpha \tag{26}$$
$$\tag{27}$$

For Q_2 we have

$$\frac{\partial Q_2}{\partial x} = 0 \tag{28}$$
$$\frac{\partial Q_2}{\partial y} = \alpha/2 \tag{29}$$
$$\frac{\partial Q_2}{\partial p_x} = -1/\alpha \tag{30}$$
$$\frac{\partial Q_2}{\partial p_y} = 0 \tag{31}$$
$$\tag{32}$$

From these we find

$$\{Q_i, P_j\} = \delta_{i,j}$$

(b) Substituting (4) into (3) we find

$$H = \omega P_1$$

if we make $\alpha^2 = eB$ and $\omega = eB/m$.

(c) Solving Hamilton's equations we get

$$Q_1 = \omega t + \phi \tag{33}$$
$$Q_2 = const \tag{34}$$
$$P_1 = const \tag{35}$$
$$P_2 = const \tag{36}$$
$$\tag{37}$$

which gives us

$$\begin{aligned} x &= \frac{1}{\alpha}\left(\sqrt{2P_1}\sin(\omega t + \phi) + P_2\right) \\ y &= \frac{1}{\alpha}\left(\sqrt{2P_1}\cos(\omega t + \phi) + Q_2\right) \\ p_x &= \frac{\alpha}{2}\left(\sqrt{2P_1}\cos(\omega t + \phi) - Q_2\right) \\ p_y &= -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin(\omega t + \phi) - P_2\right) \end{aligned} \tag{38}$$

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(d) For the Hamiltonian in equation (3) Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = P_x/m - \omega y/2 \tag{39}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = P_y/m + \omega x/2 \tag{40}$$

$$\dot{p_x} = -\frac{\partial H}{\partial x} = -\omega(p_y + m\omega x/2)/2 \tag{41}$$

$$\dot{p_y} = -\frac{\partial H}{\partial y} = \omega(p_x - m\omega y/2)/2 \tag{42}$$

$$\tag{43}$$

Substituting our solutions for x, y, p_x, p_y into these equations we find that they are readily satisfied.

3 The Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

$$-\zeta \sin\phi + \kappa \frac{\partial^2 \phi}{\partial x^2} - \rho \frac{\partial^2 \phi}{\partial t^2} = 0$$

and dividing through by ρ gives

$$\frac{\partial^2 \phi}{\partial t^2} - v^2 \frac{\partial^2 \phi}{\partial x^2} + \omega^2 \sin\phi = 0\ , \qquad (*)$$

where $v^2 = \kappa/\rho$ and $\omega^2 = \zeta/\rho$.

For small ϕ we may approximate $\sin\phi \approx \phi$, and the equation becomes

$$\frac{\partial^2 \phi}{\partial t^2} - v^2 \frac{\partial^2 \phi}{\partial x^2} + \omega^2 \phi = 0\ .$$

For a solution in $0 \le x \le L$, with $\phi(0,t) = \phi(L,t) = 0$, we use a Fourier sin series representation,

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) f_n(t)\ ,$$

and substituting gives

$$\ddot{f_n} + \left[\left(\frac{vn\pi}{L}\right)^2 + \omega^2\right] f_n = 0$$

with solutions proportional to $\sin(\Omega_n t)$ and $\cos(\Omega_n t)$, where the frequency of oscillation is

$$\Omega_n = \left(\frac{vn\pi}{L}\right)^2 + \omega^2\ , \qquad n \ge 1\ .$$

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Using natural units, the original equation of motion is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \sin\phi = 0 \ .$$

Then with $f(x,t) = \tan(\phi(x,t)/4)$,

$$\frac{\partial f}{\partial t} = \frac{1}{4}\frac{\partial\phi}{\partial t}\sec^2(\phi(x,t)/4) = \frac{1}{4}\frac{\partial\phi}{\partial t}(1+f^2)$$

and

$$\frac{\partial f}{\partial x} = \frac{1}{4}\frac{\partial\phi}{\partial x}(1+f^2) \ .$$

Taking second partial derivatives,

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{4}\left[\frac{\partial^2\phi}{\partial t^2}(1+f^2) + \frac{\partial\phi}{\partial t}2f\frac{\partial f}{\partial t}\right] \tag{44}$$

$$= \frac{1}{4}\left[\frac{\partial^2\phi}{\partial t^2}(1+f^2) + \frac{4f}{1+f^2}\left(\frac{\partial f}{\partial t}\right)^2\right] \ . \tag{45}$$

Likewise,

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{4}\left[\frac{\partial^2\phi}{\partial x^2}(1+f^2) + \frac{8f}{1+f^2}\left(\frac{\partial f}{\partial x}\right)^2\right] \ .$$

Next, take the difference of these two equations:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = \frac{1}{4}\left[(1+f^2)\left(\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2}\right) + \frac{8f}{1+f^2}\left\{\left(\frac{\partial f}{\partial t}\right)^2 - \left(\frac{\partial f}{\partial x}\right)^2\right\}\right] \ .$$

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = \frac{1}{4}\left[(1+f^2)\cdot -\sin\phi + \frac{8f}{1+f^2}\left\{\left(\frac{\partial f}{\partial t}\right)^2 - \left(\frac{\partial f}{\partial x}\right)^2\right\}\right] \ .$$

$$(1+f^2)\left(\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}\right) - 2f\left\{\left(\frac{\partial f}{\partial t}\right)^2 - \left(\frac{\partial f}{\partial x}\right)^2\right\} = \frac{1}{4}(1+f^2)^2\cdot -\sin\phi = -f(1-f^2) \ ,$$

where in the last line we have used the trignometric identity:

$$\sin\phi = \frac{4f(1-f^2)}{(1+f^2)^2} \ .$$

Now regard f as a function of the variable $y = (x+\alpha t)/\sqrt{1-\alpha^2}$, with α a real parameter in the interval $-1 < \alpha < 1$:

$$\frac{\partial f}{\partial t} = f'(y)\frac{\alpha}{\sqrt{1-\alpha^2}} \ , \quad \frac{\partial f}{\partial x} = f'(y)\frac{1}{\sqrt{1-\alpha^2}} \ .$$

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$$\frac{\partial^2 f}{\partial t^2} = f'(y)\frac{\alpha^2}{1-\alpha^2} , \quad \frac{\partial^2 f}{\partial x^2} = f'(y)\frac{1}{1-\alpha^2} .$$

Substituting into the above partial differential equation for f gives

$$-(1+f^2)f'' + f\left[1 - f^2 + 2(f')^2\right] = 0 ,$$

as required. Then substitute for $f = \exp(\lambda y)$ to get

$$(1+f^2)\lambda^2 f - f\left[1 - f^2 + 2\lambda^2 f^2\right] = f(\lambda^2 - 1)(1 - f^2) = 0 ,$$

which shows that $\lambda = \pm 1$ gives two solutions, i.e.

$$\phi_\pm(x,t) = 4 \arctan\left(\exp\left\{\pm\frac{x+\alpha t}{\sqrt{1-\alpha^2}}\right\}\right) .$$

Taking the positive sign, at $t = 0$ we have

$$\phi_\pm(x,0) = 4 \arctan\left(\exp\left\{\pm\frac{x}{\sqrt{1-\alpha^2}}\right\}\right) .$$

Thus $\phi_+(x,0) \to 0$ as $x \to -\infty$, $\phi_+(x,0) \to 2\pi$ as $x \to +\infty$, and $\phi_+(0,0) = \pi$. This 'soliton' solution – a complete 2π twist in the bar – moves to the right (i.e. increasing x) or left (i.e. decreasing x) according to whether α is less than or greater than 0 respectively, with uniform wave velocity $|\alpha|$.

4 The 'Maxwell-Chern-Simons' Lagrangian is

$$\mathcal{L}_{\rm MCS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda .$$

Using the first term on the right-hand side as calibration, there are two powers of the field in each term, two partial derivatives in the first and one in the second. Therefore g has dimensions $[L]^{-1} \equiv [M]$ in 'natural units'.

Consider the gauge transformation $A_\mu \to A_\mu + \partial_\mu f$ where f is a function of space-time co-ordinates. Then $F_{\mu\nu} \to F_{\mu\nu}$, since the gauge term cancels between the two terms in $F_{\mu\nu}$. The change in the second term in $\mathcal{L}_{\rm MCS}$ is

$$g\epsilon^{\mu\nu\lambda}\left[(A_\mu + \partial_\mu f)(\partial_\nu A_\lambda + \partial_\nu\partial_\lambda f) - A_\mu\partial_\nu A_\lambda\right] = g\epsilon^{\mu\nu\lambda}\partial_\mu(f\partial_\nu A_\lambda) .$$

However this is a pure divergence, which in the action integral corresponds to a surface integral at infinity. This vanishes provided that $f(x)$ and the field A_μ decrease sufficiently rapidly as $x \to \infty$. Hence the action is invariant under the gauge transformation.

The Euler-Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 .$$

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Now

$$\frac{\partial \mathcal{L}_{\mathrm{MCS}}}{\partial A_\nu} = g\epsilon^{\nu\alpha\beta}\partial_\alpha A_\beta = \frac{1}{2}g\epsilon^{\nu\alpha\beta}F_{\alpha\beta} \ ,$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}_{\mathrm{MCS}}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu F^{\mu\nu} + g\epsilon^{\alpha\mu\nu}\partial_\mu A_\alpha = -\partial_\mu F^{\mu\nu} - \frac{1}{2}g\epsilon^{\nu\alpha\beta}F_{\alpha\beta} \ ,$$

and hence

$$\partial_\mu F^{\mu\alpha} + g\epsilon^{\alpha\rho\sigma}F_{\rho\sigma} = 0 \ .$$

The gauge invariance of these equations is immediate, since the field A_μ appears only in the combination $F_{\mu\nu}$.

Next consider the 'dual' vector field

$$\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \ .$$

We have

$$\partial_\mu \tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta}\partial_\mu F_{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\alpha\beta}\left(\partial_\mu\partial_\alpha A_\beta - \partial_\mu\partial_\beta A_\alpha\right) = 0$$

by the symmetry of the partial derivatives on the field A_μ, and the antisymmetry of the ϵ tensor. Also

$$\epsilon^{\mu\nu\alpha}\tilde{F}_\alpha = \frac{1}{2}\epsilon^{\mu\nu\alpha}\epsilon_\alpha^{\ \rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)F_{\rho\sigma} = F^{\mu\nu} \ ,$$

as required. Now consider

$$\epsilon^{\mu\nu\lambda}\partial_\nu\tilde{F}_\lambda = \frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\nu\epsilon_\lambda^{\ \rho\sigma}F_{\rho\sigma} = \partial_\nu F^{\mu\nu} = g\epsilon^{\mu\rho\sigma}F_{\rho\sigma} = 2g\tilde{F}^\mu \ .$$

Then

$$\epsilon_\mu^{\ \beta\alpha}\partial_\beta\epsilon^{\mu\nu\lambda}\partial_\nu\tilde{F}_\lambda = 2g\epsilon_\mu^{\ \beta\alpha}\partial_\beta\tilde{F}^\mu = 2g\epsilon^{\mu\beta\alpha}\partial_\beta\tilde{F}_\mu = -(2g)^2\tilde{F}^\alpha \ .$$

Now the left-hand side can be simplified:

$$\epsilon_\mu^{\ \beta\alpha}\partial_\beta\epsilon^{\mu\nu\lambda}\partial_\nu\tilde{F}_\lambda = \left(g^{\beta\nu}g^{\alpha\lambda} - g^{\beta\lambda}g^{\alpha\nu}\right)\partial_\beta\partial_\nu\tilde{F}_\lambda = \partial^\nu\partial_\nu\tilde{F}^\alpha \ ,$$

using the previous result that $\tilde{F}$ is divergence free. Hence

$$\left[\partial_\nu\partial^\nu + (2g)^2\right]\tilde{F}^\alpha = 0$$

as required.

Substituting the (real) plane-wave representation

$$\tilde{F}^\mu = \int \mathrm{d}^2\boldsymbol{k}\ \left[a^\mu(k)e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega(k)t} + (a^\mu(k))^*e^{-i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega(k)t}\right]$$

we readily see that this is indeed a solution provided

$$-\omega(k)^2 + k^2 + (2g)^2 = 0 \ \Rightarrow\ \omega(k) = \sqrt{k^2+(2g)^2}$$

where $k = |\boldsymbol{k}|$. This dispersion relation shows that the excitation quanta of the $\tilde{F}$ field have acquired a mass $= 2g$.

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5 The Lagrangian density is

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - V(\phi) \ ,$$

where

$$V(\phi) = -m^2 \phi^* \phi + \frac{\lambda}{2}(\phi^* \phi)^2 \qquad (\lambda > 0) \ .$$

The Hamiltonian density is $\mathcal{H} = \pi \frac{\partial \phi}{\partial t} - \mathcal{L}$ where $\pi(x,t) = \frac{\partial \mathcal{L}}{\partial(\partial \phi / \partial t)} = \partial \phi^* / \partial t$, and therefore

$$\mathcal{H} = \left| \frac{\partial \phi}{\partial t} \right|^2 + |\boldsymbol{\nabla} \phi|^2 - m^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4 \ .$$

The minimium energy states correspond to the minimum of the potential $V(\phi)$. Regarding V as a function of $|\phi|$, we see that V is bounded from below and has a continuously infinite number of ground states $\phi = \phi_0 \exp(i\theta)$, where $\phi_0 = m/\sqrt{\lambda}$ is determined by solving

$$\frac{\mathrm{d}V}{\mathrm{d}|\phi|} = -2m^2|\phi| + 2\lambda|\phi|^3 = 0 \ ,$$

and $0 \le \theta < 2\pi$.

Spontaneous symmetry breaking: the system does not have a unique state of minimum energy but an infinite number of equivalent ones corresponding to different values of θ: it is said to be *degenerate*. However, if we take any particular configuration of the system and reduce its energy somehow to the minimum value, it will be in a state with a particular value of θ. The situation is like that of a thin rod initially balanced vertically on its tip on a horizontal plane: when it falls under gravity, it will lie at a particular angle on the plane, although all angles have equal energy. The dynamics and the initial state are symmetrical with respect to rotations about the vertical axis, but the final minimum-energy state is not: the rotational symmetry has been *spontaneously broken*. Similarly in the presence of the quartic interaction the scalar field will undergo spontaneous symmetry breaking by choosing some particular minimum-energy state, with a particular global value of θ. And since the dynamics has phase symmetry we may as well choose to label that state as $\theta = 0$ (like measuring angles with respect to the fallen rod).

Next consider the case when ϕ interacts with a real vector field A^μ through the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{A\mu\nu} F_A^{\mu\nu} + (D^\mu \phi)^* (D_\mu \phi) - V(\phi) \ ,$$

where $F_{A\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + ieA_\mu$, and e is a constant. Expand ϕ about the ground state configuration, $\phi = \phi_0 + \chi_1 + i\chi_2$, where χ_1 and χ_2 are real fields, and substitute into $\mathcal{L}$ (and neglecting high powers of the fields χ_i):

$$\begin{aligned} (D^\mu \phi)^* (D_\mu \phi) - V(\phi) \ &\rightarrow \ e^2 A_\mu A^\mu \phi_0^2 + \frac{1}{2}(\partial_\mu \chi_1)(\partial^\mu \chi_1) - V(\phi_0) - m^2 \chi_1^2 \\ &= \ \frac{e^2 m^2}{\lambda} A_\mu A^\mu + \frac{1}{2}(\partial_\mu \chi_1)(\partial^\mu \chi_1) - m^2 \chi_1^2 + ... \end{aligned}$$

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where an overall constant has been neglected and the gauge transformation on the field A^μ,

$$A_\mu \to A_\mu - \frac{1}{\sqrt{2}e\phi_0}\partial_\mu \chi_2 \ ,$$

has removed the remaining terms involving χ_2. The spontaneous symmetry breaking has led to the appearance of a new term quadratic in A^μ. In other words, the excitation quanta of the A^μ field have acquired a non-zero mass $M = em\sqrt{2/\lambda}$. The value of M can be deduced by deriving the Klein-Gordon (wave) equation for the components of A^μ and showing that the dispersion relation is $\omega(k) = \sqrt{k^2 + M^2}$.

A second vector field B^μ is now introduced into the system, such that the Lagrangian density becomes

$$\mathcal{L} = -\frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} + (D^\mu\phi)^*(D_\mu\phi) - V(\phi) \ ,$$

where now $D_\mu = \partial_\mu + ieA_\mu + ie'B_\mu$. The piece in the Lagrangian that is quadratic in the fields A^μ and B^μ is

$$(D^\mu\phi)^*(D_\mu\phi) \to -(ieA_\mu+ie'B_\mu)(ieA_\mu+ie'B_\mu)\phi^*\phi \to -(ieA_\mu+ie'B_\mu)(ieA_\mu+ie'B_\mu)\frac{m^2}{\lambda} \ ,$$

where ϕ has been replaced by its ground-state value after spontaneous symmetry breaking. Thus

$$\mathcal{L}_{\text{quadratic}} = \frac{m^2}{\lambda}\left(e^2 A_\mu A^\mu + e'^2 B_\mu B^\mu + 2ee' A_\mu B^\mu\right) \ .$$

as required.

Now with $Z^\mu = \cos\alpha A^\mu + \sin\alpha B^\mu$ and $W^\mu = \sin\alpha A^\mu - \cos\alpha B^\mu$ we have

$$A^\mu = \cos\alpha Z^\mu + \sin\alpha W^\mu \ , \quad B^\mu = \sin\alpha Z^\mu - \cos\alpha W^\mu \ ,$$

and

$$F_A^{\mu\nu} = \cos\alpha F_Z^{\mu\nu} + \sin\alpha F_W^{\mu\nu} \ , \quad F_B^{\mu\nu} = \sin\alpha F_Z^{\mu\nu} - \cos\alpha F_W^{\mu\nu} \ .$$

$$\begin{aligned}
F_{A\mu\nu}F_A^{\mu\nu} &= \cos^2\alpha F_{Z\mu\nu}F_Z^{\mu\nu} + \sin^2\alpha F_{W\mu\nu}F_W^{\mu\nu} + 2\cos\alpha\sin\alpha F_{Z\mu\nu}F_W^{\mu\nu} \\
F_{B\mu\nu}F_B^{\mu\nu} &= \sin^2\alpha F_{Z\mu\nu}F_Z^{\mu\nu} + \cos^2\alpha F_{W\mu\nu}F_W^{\mu\nu} - 2\cos\alpha\sin\alpha F_{Z\mu\nu}F_W^{\mu\nu}
\end{aligned}$$

and adding these two equations and multiplying by -1/4 gives the required result:

$$-\frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} = -\frac{1}{4}F_{Z\mu\nu}F_Z^{\mu\nu} - \frac{1}{4}F_{W\mu\nu}F_W^{\mu\nu} \ .$$

Now

$$\begin{aligned}
\frac{\lambda}{m^2}\mathcal{L}_{\text{quadratic}} &= (eA_\mu + e'B_\mu)^2 \\
&= \left([e\cos\alpha + e'\sin\alpha]\, Z_\mu + [e\sin\alpha - e'\cos\alpha]\, W_\mu\right)^2 \ .
\end{aligned}$$

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The coefficient of $Z_\mu W^\mu$ on the right-hand side is

$$2\left[e\cos\alpha + e'\sin\alpha\right]\left[e\sin\alpha - e'\cos\alpha\right] = 2\left[ee'(\sin^2\alpha - \cos^2\alpha) + (e^2 - e'^2)\sin\alpha\cos\alpha\right]$$
$$= -2ee'\cos(2\alpha) + (e^2 - e'^2)\sin(2\alpha)\ .$$

This evidently vanishes for $\tan(2\alpha) = 2ee'/(e^2 - e'^2)$, leaving the Lagrangian with only terms proportional to $Z_\mu Z^\mu$ and $W_\mu W^\mu$. The coefficients are

$$\frac{m^2}{\lambda}\left[e\cos\alpha + e'\sin\alpha\right]^2 \equiv \frac{1}{2}m_Z^2\ , \qquad \frac{m^2}{\lambda}\left[e\sin\alpha - e'\cos\alpha\right]^2 \equiv \frac{1}{2}m_W^2\ ,$$

Adding these gives

$$m_Z^2 + m_W^2 = \frac{2m^2}{\lambda}\left[e^2 + e'^2\right]\ ,$$

while subtracting gives

$$m_Z^2 - m_W^2 = \frac{2m^2}{\lambda}\left[(e^2 - e'^2)(\cos^2\alpha - \sin^2\alpha) + 4ee'\sin\alpha\cos\alpha\right]$$

$$= \frac{2m^2}{\lambda}\left[\frac{(e^2 - e'^2)^2}{2ee'} + 2ee'\right]\sin(2\alpha) = \frac{2m^2}{\lambda}\left[\frac{(e^2 + e'^2)^2}{2ee'}\right]\sin(2\alpha) = \pm\frac{2m^2}{\lambda}\left[e^2 + e'^2\right]\ .$$

So, we have two solutions depending on the choice of sign. Either $m_Z = M$ where

$$M = \sqrt{(2m^2)(e^2 + e'^2)/\lambda}$$

and $m_W = 0$ *or* $m_W = M$ and $m_Z = 0$. In any case, the net result of this particular rotation of fields is that one of the new fields acquires a non-zero mass M, while the other remains massless. Note that the $Z - W$ 'mass matrix' is diagonal, i.e. there are no $Z_\mu W^\mu$ terms remaining in the Lagrangian.

6 (a) Evaluate the Fourier transform

$$\int_{-\infty}^{\infty} \mathrm{d}\tau e^{-i\omega\tau}\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m_0^2\right)G(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \int_{-\infty}^{\infty} \mathrm{d}\tau e^{-i\omega\tau}\boldsymbol{\delta}^3(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'),$$

where $\tau = t - t'$ to get

$$\left(\omega^2 + \nabla^2 - m_0^2\right)G(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \boldsymbol{\delta}^3(\boldsymbol{r} - \boldsymbol{r}'),$$

Now Evaluate the Fourier transform

$$\int_{-\infty}^{\infty} \mathrm{d}^3\boldsymbol{p} e^{-i\boldsymbol{k}.\boldsymbol{p}}\left(\omega^2 + \nabla^2 - m_0^2\right)G(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \int_{-\infty}^{\infty} \mathrm{d}^3\boldsymbol{p} e^{-i\boldsymbol{k}.\boldsymbol{p}}\boldsymbol{\delta}^3(\boldsymbol{r} - \boldsymbol{r}'),$$

where $\boldsymbol{p} = \boldsymbol{r} - \boldsymbol{r}'$, to get

$$G(\boldsymbol{k}, \omega) = \frac{1}{\omega^2 - \boldsymbol{k}^2 - m^2} \qquad (46)$$

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We now take the inverse Fourier transform to find

$$G(\boldsymbol{p},\omega) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^2} \frac{e^{i\boldsymbol{k}.\boldsymbol{p}}}{\omega^2 - k^2 - m^2} \tag{47}$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{k^2 \mathrm{d}k}{\omega^2 - k^2 - m^2} \int_0^{\pi} \mathrm{d}\theta \sin(\theta) e^{ikp\cos(\theta)} \tag{48}$$

$$= \frac{1}{(2\pi)^2} \frac{1}{ip} \int_{-\infty}^{\infty} \frac{k}{(\omega^2 - m^2) - k^2} e^{ikp} \mathrm{d}k \tag{49}$$

We can solve this integral using Contour integration.

For $\omega = E + i|\epsilon|$ we close the contour in the UHP and find the result

$$G(\boldsymbol{p}, E + i|\epsilon|) = -\frac{2\pi i}{(2\pi)^2 ip} \frac{e^{i(E^2-m^2)^{1/2}p}}{2}$$

For $\omega = E - i|\epsilon|$ we close the contour in the LHP and find the result

$$G(\boldsymbol{p}, E - i|\epsilon|) = -\frac{2\pi i}{(2\pi)^2 ip} \frac{e^{-i(E^2-m^2)^{1/2}p}}{2}$$

valid in all three regimes.

(b) For $|E| > m$ we have

$$\Delta G(\boldsymbol{p}, E) = G(\boldsymbol{p}, E + i|\epsilon|) - G(\boldsymbol{p}, E - i|\epsilon|) \tag{50}$$

$$= -2\pi i \frac{\sin\left[(E^2 - m^2)^{1/2}|\boldsymbol{r} - \boldsymbol{r}'|\right]}{4\pi^2 |\boldsymbol{r} - \boldsymbol{r}'|} \tag{51}$$

Hence using the formula in the question

$$\frac{dn}{d(E^2)} = \frac{(E^2 - m^2)^{1/2}}{4\pi^2}$$

and so

$$\frac{\mathrm{d}n}{\mathrm{d}E} = 2E \frac{dn}{d(E^2)} = 2E \frac{(E^2 - m^2)^{1/2}}{4\pi^2}$$

in regimes (i) and (iii). In regime (ii) $\mathrm{d}n/\mathrm{d}E = 0$.

(c) We have

$$G(\boldsymbol{k}, \tau) = \int_{-\infty}^{\infty} e^{i\omega t} \frac{1}{\omega^2 - \boldsymbol{k}^2 - m^2} \frac{\mathrm{d}\omega}{2\pi}$$

We can solve this integral using contour integration.

Setting

$$\omega^2 = (\boldsymbol{k}^2 + m^2) e^{i|\delta|}$$

where δ is a small parameter we have two poles - one in the UHP and one in the LHP.

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For $t > t'$ we close the contour in the UHP to find

$$G(\boldsymbol{k}, |\tau|) = i\frac{e^{i\sqrt{k^2+m^2}|\tau|}}{2\sqrt{k^2+m^2}}$$

For $t < t'$ we close the contour in the LHP to find

$$G(\boldsymbol{k}, -|\tau|) = -i\frac{e^{-i\sqrt{k^2+m^2}|\tau|}}{2\sqrt{k^2+m^2}}$$

(d) We have Green's Functions for propagating waves for both $\tau > 0$ and $\tau < 0$. Nominally this violates causality but we can think of $\tau < 0$ as representing antiparticles for the negative energy branch of $\mathrm{d}n/\mathrm{d}E$.

END OF PAPER