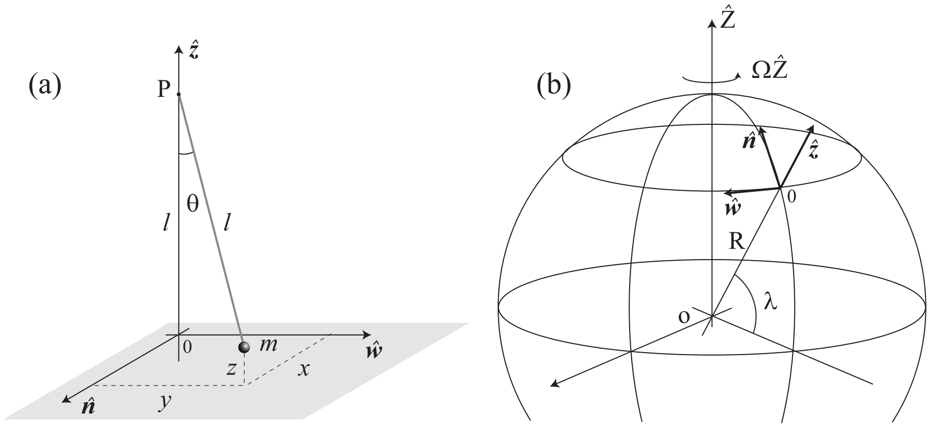
NATURAL SCIENCES TRIPOS Part II

Wednesday 19 January 2011 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains six sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 The pendulum shown in figure (a) below consists of a rigid massless rod of length l with a point mass m attached at the free end. The other end is attached to the fixed point P by means of a free hinge. The mass m moves above the two-dimensional plane $(\hat{\boldsymbol{n}}, \hat{\boldsymbol{w}})$ so that the rod makes an angle θ to the $\hat{\boldsymbol{z}}$ axis. The pendulum is situated at a latitude λ above the equator on the surface of the Earth. This is shown in figure (b), where $R = 6.38 \times 10^6$m is the radius of the Earth and $\Omega\hat{\boldsymbol{Z}}$ is the angular velocity of the Earth.



Consider the case where the pendulum exhibits small oscillations such that the velocity of the mass m in the Earth's rotating frame of reference may be approximated by $\boldsymbol{v}_r \approx \dot{x}\hat{\boldsymbol{n}} + \dot{y}\hat{\boldsymbol{w}}$.

(a) Calculate the velocity, $\boldsymbol{v}_s$, of the mass m in the Earth's stationary frame of reference and use it to show that the kinetic energy of pendulum may be approximated by

$$T \approx \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 - 2\Omega\dot{x}y\sin\lambda + 2\Omega\dot{y}(x\sin\lambda - R\cos\lambda) - 2\Omega^2 xR\sin\lambda\cos\lambda] + const.$$

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Explain the approximations you have made. [*Hint: You may find the identity* $\boldsymbol{v}_s = \boldsymbol{v}_r + \Omega \hat{\boldsymbol{Z}} \times \boldsymbol{r}$ *useful.*] [6]

(b) Show that the potential energy of the pendulum may be approximated by

$$V \approx \frac{m\tilde{g}}{2l}(x^2 + y^2),$$

where $\tilde{g} = g - \Omega^2 R \cos^2 \lambda$. The correction to g arises from consideration of the centrifugal force. [3]

(c) Using the approximations for T and V above, show that the equations of motion of the pendulum can be written in the form

$$\begin{pmatrix} \ddot{\tilde{x}} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} 0 & -2\alpha \\ 2\alpha & 0 \end{pmatrix} \begin{pmatrix} \dot{\tilde{x}} \\ \dot{y} \end{pmatrix} + \beta \begin{pmatrix} \tilde{x} \\ y \end{pmatrix} = 0. \qquad (1)$$

Give physical explanations for the parameters α, β and $\tilde{x}$. [9]

(d) Transform the equations of motion (1) to the rotating frame of reference defined by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \alpha t & -\sin \alpha t \\ \sin \alpha t & \cos \alpha t \end{pmatrix} \begin{pmatrix} \tilde{x} \\ y \end{pmatrix},$$

and show that they take the form

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \omega^2 \begin{pmatrix} X \\ Y \end{pmatrix} = 0. \qquad (2)$$

[10]

(e) Solve equation (2) and describe the characteristic motion of the pendulum in the two reference frames X, Y and x, y if the pendulum is: (i) at the equator, (ii) in Cambridge at 52° North or (iii) at the North Pole. [5]

2 Explain what is meant by a *canonical transformation.* [5]

(a) Show that the transformation

$$\begin{aligned} x &= \frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right) \\ y &= \frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right) \\ p_x &= \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) \\ p_y &= -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right) \end{aligned}$$

is canonical. [13]

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(b) The Hamiltonian for a particle of charge e moving in a two-dimensional plane (x, y) in a magnetic field $\mathbf{B} = B\hat{\boldsymbol{z}}$ can be written in the form

$$H = \frac{1}{2m}\left(p_x + eB\frac{y}{2}\right)^2 + \frac{1}{2m}\left(p_y - eB\frac{x}{2}\right)^2 , \tag{3}$$

where the symbols take their usual meanings. Transform this Hamiltonian to the coordinate system Q_1, P_1, Q_2, P_2 and choose a value for α to simplify the expression for the resulting Hamiltonian. [9]

(c) Derive and solve the equations of motion in the coordinates Q_1, P_1, Q_2, P_2. [3]

(d) Show that your solutions to part (c) satisfy Hamilton's equations of motion for the Hamiltonian in equation (3). [3]

3 The angular twisting $\phi(x, t)$ of a torsion bar along its length (in the x direction) can be described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\kappa\left(\frac{\partial\phi}{\partial x}\right)^2 - \zeta(1 - \cos\phi) ,$$

with constants $\rho, \kappa, \zeta > 0$. Show that the Euler-Lagrange equation for the system leads to the equation of motion

$$\frac{\partial^2\phi}{\partial t^2} - v^2\frac{\partial^2\phi}{\partial x^2} + \omega^2\sin\phi = 0 , \tag{4}$$

where $v^2 = \kappa/\rho$ and $\omega^2 = \zeta/\rho$. [5]

If the rod lies between $0 \le x \le L$ and is fixed at each end, $\phi(0, t) = \phi(L, t) = 0$, show that the general solution for small angular displacements, i.e. $\phi \ll 1$, can be written in terms of Fourier harmonics:

$$\phi(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin(\Omega_n t) + b_n \cos(\Omega_n t)\right]$$

and find the frequencies of oscillation Ω_n. [5]

Next consider the case of a rod of infinite extent, $-\infty < x < +\infty$, and switch to natural units in which $v = \omega = 1$. With $f(x, t) = \tan(\phi(x, t)/4)$, and relaxing the assumption that ϕ is small, show that the equation of motion (4) becomes [10]

$$\left(1 + f^2\right)\left(\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}\right) + f\left[1 - f^2 - 2\left(\frac{\partial f}{\partial t}\right)^2 + 2\left(\frac{\partial f}{\partial x}\right)^2\right] = 0 .$$

Regarding f as a function of the variable $y = (x + \alpha t)/\sqrt{1 - \alpha^2}$, with α a real parameter in the interval $-1 < \alpha < 1$, write down expressions for the partial

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derivatives $\partial f/\partial t$ and $\partial f/\partial x$ in terms of f and $f' \equiv \mathrm{d}f/\mathrm{d}y$, and show that the above partial differential equation for f becomes [6]

$$\left(1+f^2\right)f'' - f\left[1 - f^2 + 2\left(f'\right)^2\right] = 0\ .$$

Determine the values of λ for which $f = \exp(\lambda y)$ is a solution, and hence show that the original partial differential equation for $\phi(x,t)$ has two particular solutions [3]

$$\phi_\pm(x,t) = 4\,\arctan\left(\exp\left\{\pm\frac{x+\alpha t}{\sqrt{1-\alpha^2}}\right\}\right)\ ,$$

corresponding to boundary conditions $\phi_+(x=+\infty,t) = \phi_-(x=-\infty,t) = 2\pi$, $\phi_+(x=-\infty,t) = \phi_-(x=+\infty,t) = 0$. Taking the positive sign solution, interpret this result in terms of the evolution in time of a particular initial ($t=0$) angular displacement, which you should sketch. [4]

4 Consider the theory of a real vector field A^μ in *three* space-time dimensions, $\mu = 0,1,2$, i.e. $x^\mu = (t,\boldsymbol{x})$ with $\boldsymbol{x}$ the two-dimensional position vector. The dynamics of A^μ are determined by the 'Maxwell-Chern-Simons' Lagrangian

$$\mathcal{L}_{\mathrm{MCS}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda,$$

where $\partial_\mu = \partial/\partial x^\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\epsilon^{\mu\nu\lambda}$ is the completely antisymmetric tensor, $\epsilon^{012} = 1$, and g is a real constant.

Find the dimensions of the constant g. [3]

Show that the action $S = \int \mathrm{d}t\,\mathrm{d}^2\boldsymbol{x}\ \mathcal{L}_{\mathrm{MCS}}$ is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu f,$$

provided that the scalar function f and the field A_μ decrease sufficiently rapidly as $|t|, |\boldsymbol{x}| \to \infty$. [4]

Starting from the Euler-Lagrange equations, derive the field equations

$$\partial_\mu F^{\mu\alpha} + g\epsilon^{\alpha\rho\sigma}F_{\rho\sigma} = 0\ ,$$

and show that they are gauge invariant. [7]

The 'dual' vector field is defined by

$$\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}\ .$$

Show that $\partial_\mu\tilde{F}^\mu = 0$ and $F^{\mu\nu} = \epsilon^{\mu\nu\alpha}\tilde{F}_\alpha$. [6]

Show that the dual vector field satisfies the second-order partial differential equation [8]

$$\left[\partial_\mu\partial^\mu + (2g)^2\right]\tilde{F}^\nu = 0\ .$$

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Show that this equation has plane-wave solutions

$$\tilde{F}^\mu = \int \mathrm{d}^2\boldsymbol{k}\ \left[a^\mu(k)e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega(k)t} + (a^\mu(k))^* e^{-i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega(k)t}\right] .$$

and find an expression for $\omega(k)$ in terms of $k = |\boldsymbol{k}|$ and g. Interpret your result. [5]
[*Hint: You may find the identity* $\epsilon^{\alpha\mu\nu}\epsilon_{\alpha}^{\ \rho\sigma} = g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}$ *useful. Note that the metric tensor has its usual meaning:* $g^{00} = -g^{11} = -g^{22} = 1$ *with all other components zero.*]

5 The Lagrangian density for a self-interacting, complex, massless scalar field $\phi(\boldsymbol{r}, t)$ is given by

$$\mathcal{L} = (\partial^\mu\phi^*)(\partial_\mu\phi) - V(\phi),$$

where $\partial_\mu = \partial/\partial x^\mu$ and

$$V(\phi) = -m^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \qquad (\lambda > 0) .$$

Derive an expression for the Hamiltonian density $\mathcal{H}$ in terms of ϕ and its derivatives, and show that there is an infinite set of states $\phi = \phi_0 e^{i\theta}$, with $\phi_0 = m/\sqrt{\lambda}$ and $0 \leq \theta < 2\pi$, for which the energy is a minimum. [6]

Explain the concept of *spontaneous symmetry breaking*, using the above Lagrangian as an illustrative example. [6]

Consider the case when ϕ interacts with a real vector field A^μ through the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} + (D^\mu\phi)^*(D_\mu\phi) - V(\phi),$$

where $F_{A\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + ieA_\mu$, and e is a constant. By expanding ϕ about the ground state configuration, $\phi = \phi_0 + \chi_1 + i\chi_2$, where χ_1 and χ_2 are real fields, show that the excitation quanta of the A^μ field acquire a non-zero mass $em\sqrt{2/\lambda}$. [4]

A second real vector field B^μ is introduced into the system such that the Lagrangian density becomes

$$\mathcal{L} = -\frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} + (D^\mu\phi)^*(D_\mu\phi) - V(\phi) ,$$

where now $D_\mu = \partial_\mu + ieA_\mu + ie'B_\mu$. Show that under spontaneous symmetry breaking the term in the resulting Lagrangian density that is quadratic in the A^μ and B^μ fields is [4]

$$\mathcal{L}_{\text{quadratic}} = \frac{m^2}{\lambda}\left(e^2A_\mu A^\mu + e'^2B_\mu B^\mu + 2ee'A_\mu B^\mu\right) .$$

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The fields A^μ and B^μ are now 'rotated' into two new fields Z^μ and W^μ defined by $Z^\mu = \cos\alpha A^\mu + \sin\alpha B^\mu$ and $W^\mu = \sin\alpha A^\mu - \cos\alpha B^\mu$. Show that

$$-\frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} = -\frac{1}{4}F_{Z\mu\nu}F_Z^{\mu\nu} - \frac{1}{4}F_{W\mu\nu}F_W^{\mu\nu}$$

and that, for $\tan 2\alpha = 2ee'/(e^2 - e'^2)$, [9]

$$\mathcal{L}_{\text{quadratic}} = \frac{1}{2}m_Z^2 Z_\mu Z^\mu + \frac{1}{2}m_W^2 W_\mu W^\mu \, .$$

Interpret this result and determine m_Z and m_W. [4]

6 The Green's Function for a particle obeying the Klein-Gordon equation of motion in three dimensions is defined by

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m_0^2\right) G(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \boldsymbol{\delta}^3(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'),$$

where the symbols take their usual meanings.

(a) Use Fourier methods to derive the Green's function

$$G(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau e^{-i\omega\tau} G(\boldsymbol{r}, \boldsymbol{r}'; t, t')$$

for a free particle, where $\tau = t - t'$ and $\omega = E + i\epsilon$, in the three energy regimes (i) $E \geq m_0$ (ii) $|E| < m_0$ and (iii) $E \leq -m_0$. The parameter ϵ should be assumed to be real and small. [13]

(b) Use your results from (a) to calculate the quantity

$$\frac{\mathrm{d}n}{\mathrm{d}z} = \lim_{\boldsymbol{r}\to\boldsymbol{r}'} \lim_{\epsilon\to 0} \frac{G(\boldsymbol{r}, \boldsymbol{r}'; E + i|\epsilon|) - G(\boldsymbol{r}, \boldsymbol{r}'; E - i|\epsilon|)}{-2\pi i},$$

where $z = E^2$, and hence find the density of states $\mathrm{d}n/\mathrm{d}E$ in the same three energy regimes. [7]

(c) Use Fourier methods to derive the Green's function

$$G(\boldsymbol{k}; t, t') = \int_{-\infty}^{\infty} \mathrm{d}^3\boldsymbol{p} e^{-i\boldsymbol{k}.\boldsymbol{p}} G(\boldsymbol{r}, \boldsymbol{r}'; t, t'),$$

where $\boldsymbol{p} = \boldsymbol{r} - \boldsymbol{r}'$, for the two cases $t > t'$ and $t < t'$. [10]

(d) Comment on and give a physical explanation for your results in sections (b) and (c). [3]

END OF PAPER