NATURAL SCIENCES TRIPOS Part II

Wednesday 13 January 2010 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answers

1 (a) The kinetic energy of the rolling cylinder is

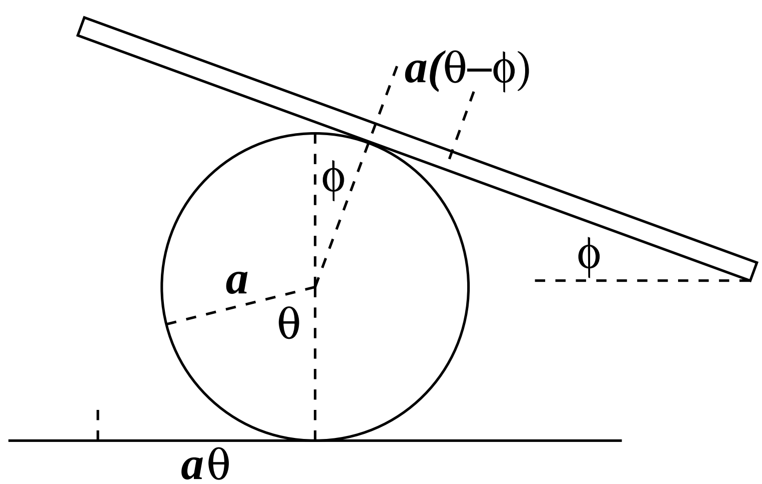
$$T_c = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}I_c\dot{\theta}^2$$

where $I_c = ma^2/2$ is its moment of inertia about its axis. The kinetic energy of the plank is

$$T_p = \frac{1}{2}m^2(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_p\dot{\phi}^2$$

where $I_p = ml^2/12$ is its moment of inertia about its centre. The potential energy of the Plank is $V_p = mgy$. Therefore

$$L = T_c + T_p - V_p = \frac{3}{4}ma^2\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\phi}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$



(b) The top of the cylinder is at $(a\theta, 0)$. Therefore the point of contact of plank and cylinder is at $a(\theta + \sin\phi, \cos\phi - 1)$. If there is no slipping, the vector from there to the centre of the plank is $a(\theta - \phi)(\cos\phi, -\sin\phi)$. Hence

$$\begin{aligned} x/a &= \theta + \sin\phi + (\theta - \phi)\cos\phi \,, \\ y/a &= -1 + \cos\phi - (\theta - \phi)\sin\phi \,. \end{aligned}$$

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(c) We have

$$\begin{aligned} \dot{x}/a &= \dot\theta + \dot\phi\cos\phi + (\dot\theta - \dot\phi)\cos\phi + (\phi-\theta)\dot\phi\sin\phi \ , \\ \dot{y}/a &= -\dot\phi\sin\phi - (\dot\theta - \dot\phi)\sin\phi + (\phi-\theta)\dot\phi\cos\phi \ . \end{aligned}$$

Hence

$$\begin{aligned} (\dot{x}^2+\dot{y}^2)/a^2 &= [\dot\theta(1+\cos\phi) + (\phi-\theta)\dot\phi\sin\phi]^2 + [\dot\theta\sin\phi - (\phi-\theta)\dot\phi\cos\phi]^2 \\ &= 2\dot\theta^2(1+\cos\phi) + (\phi-\theta)^2\dot\phi^2 + 2(\phi-\theta)\dot\theta\dot\phi\sin\phi \end{aligned}$$

The canonical momentum p_θ is

$$\begin{aligned} p_\theta = \frac{\partial L}{\partial\dot\theta} &= \frac{3}{2}ma^2\dot\theta + \frac{1}{2}ma^2[4\dot\theta(1+\cos\phi) + 2(\phi-\theta)\dot\phi\sin\phi] \\ &= \frac{1}{2}ma^2[\dot\theta(7+4\cos\phi) + 2(\phi-\theta)\dot\phi\sin\phi] \end{aligned}$$

The canonical momentum p_ϕ is

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = \frac{1}{12}ml^2\dot\phi + ma^2(\phi-\theta)[(\phi-\theta)\dot\phi + \dot\theta\sin\phi]$$

(d) To second order in θ and ϕ,

$$y/a \simeq \frac{1}{2}\phi^2 - \theta\phi \ , \quad (\dot{x}^2+\dot{y}^2)/a^2 \simeq 4\dot\theta^2$$

so that

$$L \simeq \frac{11}{4}ma^2\dot\theta^2 + \frac{1}{24}ml^2\dot\phi^2 - \frac{1}{2}mga\phi(\phi-2\theta)$$

To find the Hamiltonian to second order, we only need the canonical momenta to first order:

$$p_\theta \simeq \frac{11}{2}ma^2\dot\theta \ , \quad p_\phi \simeq \frac{1}{12}ml^2\dot\phi$$

Hence

$$H = p_\theta\dot\theta + p_\phi\dot\phi - L \simeq \frac{1}{11}\frac{p_\theta^2}{ma^2} + 6\frac{p_\theta^2}{ml^2} + \frac{1}{2}mga\phi(\phi-2\theta) \ .$$

Hamilton's equations are

$$\begin{aligned} \dot\theta &= \frac{\partial H}{\partial p_\theta} = \frac{2}{11}\frac{p_\theta}{ma^2} \ , \quad \dot{p}_\theta = -\frac{\partial H}{\partial\theta} = mga\phi \\ \dot\phi &= \frac{\partial H}{\partial p_\phi} = 12\frac{p_\phi}{ml^2} \ , \quad \dot{p}_\phi = -\frac{\partial H}{\partial\phi} = -mga(\phi-\theta) \end{aligned}$$

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(e) Hence

$$\ddot{\theta} = \frac{2}{11}\frac{g}{a}\phi\ , \quad \ddot{\phi} = 12\frac{ga}{l^2}(\theta - \phi)\ .$$

Writing $\theta = Ae^{i\omega t}$, $\phi = Be^{i\omega t}$,

$$-\omega^2 A = \frac{2}{11}\frac{g}{a}B\ , \quad -\omega^2 B = 12\frac{ga}{l^2}(A - B)$$

so

$$-\omega^2 B = -12\frac{ga}{l^2}B - \frac{24}{11}\frac{g^2}{l^2\omega^2}B\ ,$$

$$\omega^4 - 12\frac{ga}{l^2}\omega^2 - \frac{24}{11}\frac{g^2}{l^2} = 0\ ,$$

and therefore

$$\omega^2 = 6\frac{ga}{l^2}\left(1 \pm \sqrt{1 + \frac{2}{33}\frac{l^2}{a^2}}\right)$$

The negative root corresponds to a runaway solution – the plank falls off. The positive root ω_+ is an oscillation with θ and ϕ in antiphase and

$$\frac{B}{A} = -\frac{11}{2}\frac{a}{g}\omega_+^2 = -33\frac{a^2}{l^2}\left(1 + \sqrt{1 + \frac{2}{33}\frac{l^2}{a^2}}\right)\ .$$

2 (a) We have

$$\begin{aligned} A^\mu &= \left(\frac{Q}{rc}, -\frac{1}{2}Br\sin\theta, \frac{1}{2}Br\cos\theta, 0\right) \\ &= \left(\phi(r)/c, -\frac{1}{2}By, \frac{1}{2}Bx, 0\right) \end{aligned}$$

where $\phi(r)$ is the electrostatic potential due to the charge at the origin. Now $A^\mu = (\phi/c, \boldsymbol{A})$ where $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial\boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. Thus in this case $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ and $\boldsymbol{B} = (0, 0, B)$ as required.

(b) The Lagrangian is

$$L = T - V = \frac{1}{2}mv^2 - e(\phi - \boldsymbol{v}\cdot\boldsymbol{A})$$

In plane polar coordinates $\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ and here $\boldsymbol{A} = Br\hat{\boldsymbol{\theta}}/2$, so

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{eQ}{r} + \frac{1}{2}eBr^2\dot{\theta}$$

(c) Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}\ , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

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Hence

$$m\ddot{r} = mr\dot{\theta}^2 + \frac{eQ}{r^2} + eBr\dot{\theta} ,$$
$$\frac{d}{dt}\left(mr^2\dot{\theta} + \frac{1}{2}eBr^2\right) = 0 .$$

(d) We see from the equation of motion for θ that the angular momentum

$$p_\theta = mr^2\dot{\theta} + \frac{1}{2}eBr^2 = J$$

is a constant of the motion. Furthermore the Lagrangian does not depend explicitly on time, so the Hamiltonian is also constant:

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{eQ}{r} = E$$

(e) Writing

$$\theta = \phi - \omega_L t = \phi - \frac{eB}{2m}t$$

the equations of motion become

$$\begin{aligned} m\ddot{r} &= mr\left(\dot{\phi} - \frac{eB}{2m}\right)^2 + \frac{eQ}{r^2} + eBr\left(\dot{\phi} - \frac{eB}{2m}\right) \\ &= mr\dot{\phi}^2 + \frac{eQ}{r^2} - \frac{e^2B^2}{4m}r , \\ \frac{d}{dt}\left(mr^2\dot{\phi}\right) &= 0 . \end{aligned}$$

Therefore, to first order in B, the effect of the field is cancelled in a frame rotating with the Larmor frequency.

3 (a) We are given

$$V = \frac{S_0}{\gamma+1}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma+1} - 1\right]$$

where $\rho = \rho_0(1 - \boldsymbol{\nabla}\cdot\boldsymbol{\xi})$, i.e. $\rho/\rho_0 = 1 - \delta$, where $\delta = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ is small. Expanding

$$\begin{aligned} V &= \frac{S_0}{\gamma+1}\left[(1-\delta)^{\gamma+1} - 1\right] \\ &= \frac{S_0}{\gamma+1}\left[1 - (\gamma+1)\delta + \frac{1}{2}\gamma(\gamma+1)\delta^2 + \ldots - 1\right] \\ &= S_0\left[-\delta + \frac{\gamma}{2}\delta^2 + \mathcal{O}(\delta^3)\right] \\ &= S_0\left[-\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \frac{\gamma}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2\right] \end{aligned}$$

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(b)

$$\mathcal{L} = T - V = \frac{1}{2}\rho_0 \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} + S_0 \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \frac{\gamma}{2} (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 \right)$$

Lagrange's equation of motion for ξ_i is

$$\frac{\partial \mathcal{L}}{\partial \xi_i} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial(\partial \xi_i / \partial x_j)} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\xi}_i} = 0$$

Hence in this case

$$0 - \frac{\partial}{\partial x_i} S_0 (1 - \gamma \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - \rho_0 \ddot{\xi}_i = 0$$

i.e.

$$\rho_0 \ddot{\boldsymbol{\xi}} - \gamma S_0 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) = 0.$$

(c) The canonical momentum density $\pi_i = \partial \mathcal{L} / \partial \dot{\xi}_i = \rho_0 \dot{\xi}_i$. Hence

$$\begin{aligned} \mathcal{H} &= \sum_i \pi_i \dot{\xi}_i - \mathcal{L} \\ &= \rho_0 \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} - \frac{1}{2} \rho_0 \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} - S_0 \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \frac{\gamma}{2} (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 \right) \\ &= \frac{\boldsymbol{\pi}^2}{2\rho_0} - S_0 \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \frac{\gamma}{2} (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 \right) \end{aligned}$$

The term involving the total derivative $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}$ gives a contribution to the total Hamiltonian equal to an integral of the field over a surface at infinity,

$$-S_0 \int \mathrm{d}^3\boldsymbol{r} \; \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = -S_0 \int \mathrm{d}^2\boldsymbol{S} \cdot \boldsymbol{\xi} = 0$$

since the field vanishes at infinity. Hence

$$H = \int \mathrm{d}^3\boldsymbol{r} \; \mathcal{H}(\boldsymbol{r}, t)$$

where

$$\mathcal{H}(\boldsymbol{r}, t) = \frac{1}{2} \rho_0 \boldsymbol{\pi}^2 + \frac{\gamma}{2} S_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 \,.$$

(d) We have

$$\rho_0 \ddot{\boldsymbol{\xi}}^{T,L} - \gamma S_0 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^{T,L}) = 0$$

where

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^T = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{\xi}^L = 0 \,.$$

Thus $\boldsymbol{\xi}^T$ obeys the free-particle equation of motion $\ddot{\boldsymbol{\xi}}^T = 0$, while for $\boldsymbol{\xi}^L$ we have

$$\rho_0 \ddot{\boldsymbol{\xi}}^L - \gamma S_0 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^L) = 0 \,.$$

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Now we need the identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$$

which since $\boldsymbol{\nabla} \times \boldsymbol{\xi}^L = 0$ gives

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^L) = \boldsymbol{\nabla}^2 \boldsymbol{\xi}^L$$

so that $\boldsymbol{\xi}^L$ obeys

$$\rho_0 \ddot{\boldsymbol{\xi}}^L - \gamma S_0 \boldsymbol{\nabla}^2 \boldsymbol{\xi}^L = 0$$

which is the wave equation with wave velocity $\sqrt{\gamma S_0/\rho_0}$.

4 (a) Hamilton's principle of least action states that $\delta S = 0$ for variations of the motion around the classical path, where the action S is

$$S = \int L\, dt = \int \mathcal{L}\, dx\, dt$$

for a field in one dimension. In this case we have

$$\mathcal{L} = \mathcal{L}\left(\varphi_t, \varphi_{xx}\right)$$

where $\varphi_t \equiv \partial\varphi/\partial t$ and $\varphi_{xx} \equiv \partial^2\varphi/\partial x^2$. Therefore

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \varphi_t}\delta\varphi_t + \frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\delta\varphi_{xx}\right] dx\, dt$$

Now

$$\delta\varphi_t = \frac{\partial}{\partial t}\delta\varphi\ , \quad \delta\varphi_{xx} = \frac{\partial^2}{\partial x^2}\delta\varphi\ .$$

Therefore, integrating the first term by parts once w.r.t t and the second by parts twice w.r.t x, and dropping boundary terms since the field must vanish at $\pm\infty$:

$$\delta S = \int \left[-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \varphi_t} + \frac{\partial^2}{\partial x^2}\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right] \delta\varphi\, dx\, dt = 0$$

The variation in φ is arbitrary and therefore

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \varphi_t} - \frac{\partial^2}{\partial x^2}\frac{\partial \mathcal{L}}{\partial \varphi_{xx}} = 0$$

or in this case

$$\rho A \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2\varphi}{\partial x^2}\right) = 0\ .$$

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(b) The canonical momentum per unit length is

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \varphi_t} = \rho A \frac{\partial \varphi}{\partial t}$$

and the Hamiltonian per unit length is

$$\mathcal{H} = \pi \frac{\partial \varphi}{\partial t} - \mathcal{L} = \frac{1}{2}\rho A \left(\frac{\partial \varphi}{\partial t}\right)^2 + \frac{1}{2} EI \left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 \ .$$

(c) We have

$$\frac{\partial \mathcal{H}}{\partial t} = \rho A \frac{\partial \varphi}{\partial t}\frac{\partial^2 \varphi}{\partial t^2} + EI \frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^3 \varphi}{\partial t \partial x^2} \ .$$

Applying the equation of motion,

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \varphi}{\partial t}\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \varphi}{\partial x^2}\right) + EI \frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^3 \varphi}{\partial t \partial x^2} \ .$$

Writing $EI\partial^2\varphi/\partial x^2 \equiv \psi$, the r.h.s. has the form

$$-\frac{\partial \varphi}{\partial t}\frac{\partial^2 \psi}{\partial x^2} + \psi \frac{\partial^3 \varphi}{\partial t \partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial x} - \psi \frac{\partial^2 \varphi}{\partial t \partial x}\right)$$

and so

$$\mathcal{J} = \frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial x} - \psi \frac{\partial^2 \varphi}{\partial t \partial x} \ .$$

(d) For $EI =$ constant, substituting a wave solution $\varphi = C\cos\phi$ with $\phi = kx - \omega t$ in the equation of motion we have

$$-\rho A C \omega^2 + EICk^4 = 0$$

and hence the dispersion relation is

$$\omega = \sqrt{\frac{EI}{\rho A}} k^2 \ .$$

The wave and group velocities are

$$c_w \equiv \frac{\omega}{k} = \sqrt{\frac{EI}{\rho A}} k \ , \quad c_g \equiv \frac{d\omega}{dk} = 2\sqrt{\frac{EI}{\rho A}} k = 2c_w \ .$$

The energy per unit length is

$$\mathcal{H} = \frac{1}{2}\rho A C^2 \omega^2 \sin^2\phi + \frac{1}{2} EIC^2 k^4 \cos^2\phi = \frac{1}{2} EIC^2 k^4$$

and $\psi = -EICk^2\cos\phi$, so the energy current is

$$\mathcal{J} = C\omega \sin\phi \, EICk^3 \sin\phi + EICk^2 \cos\phi \, C\omega k \cos\phi = EIC^2 \omega k^3 \ .$$

Therefore the velocity of energy transfer is $\mathcal{J}/\mathcal{H} = 2\omega/k = 2c_w = c_g$.

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5 (a) We have

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - V(\phi)$$

The Euler-Lagrange equation for ϕ is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial(\partial \phi / \partial x_j)} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$$

which gives (in units where $c = 1$)

$$-\frac{\partial V}{\partial \phi} + \boldsymbol{\nabla}^2 \phi^* - \ddot{\phi}^* = 0$$

i.e.

$$\partial^\mu \partial_\mu \phi^* + \frac{\partial V}{\partial \phi} = 0.$$

Similarly the Euler-Lagrange equation for ϕ^* gives

$$\partial^\mu \partial_\mu \phi + \frac{\partial V}{\partial \phi^*} = 0.$$

(b) Setting $c = 1$, the canonical momentum densities are $\pi = \partial \mathcal{L} / \partial \dot{\phi} = \partial \phi^* / \partial t$ and $\pi^* = \partial \mathcal{L} / \partial \dot{\phi}^* = \partial \phi / \partial t$. The corresponding Hamiltonian density is

$$\mathcal{H} = \pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L} = \pi^* \pi + \boldsymbol{\nabla} \phi^* \cdot \boldsymbol{\nabla} \phi + V(\phi)$$

(c) A global phase change $\phi \to \phi e^{i\epsilon}$, $\phi^* \to \phi^* e^{-i\epsilon}$ leaves $\phi^* \phi$ and $(\partial^\mu \phi^*)(\partial_\mu \phi)$ unchanged. Therefore if the potential V is a function of $\phi^* \phi$, the Lagrangian density is invariant under this change.

(d) We are given the *Coleman-Weinberg* potential,

$$V(\phi) = (\phi^* \phi)^2 \left[\ln \left(\frac{\phi^* \phi}{\Lambda^2} \right) - \kappa \right],$$

where Λ and κ are real, positive constants. Let $X = \phi^* \phi$ and consider V as a function of X. Then V is continuous for $X > 0$, $V = 0$ at $X = 0$ and $V \to +\infty$ as $X \to \infty$. We have

$$\frac{dV}{dX} = 2X \left[\ln(X/\Lambda^2) - \kappa \right] + X = 2X \left[\ln(X/\Lambda^2) - \kappa + \frac{1}{2} \right]$$

This is zero for $X = 0$ and $X = X_0$ where $\ln(X_0/\Lambda^2) = \kappa - 1/2$, i.e.

$$X_0 = \Lambda^2 e^{\kappa - 1/2}$$

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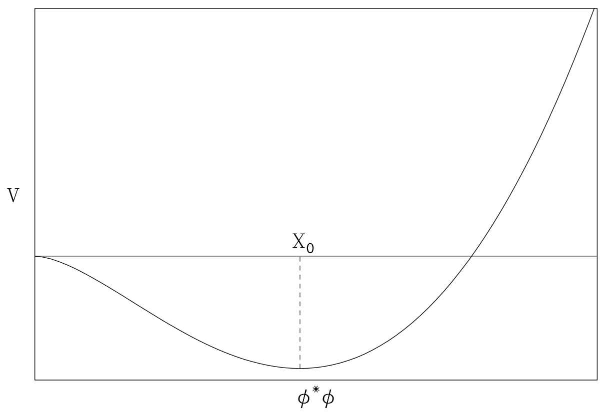
At $X = X_0$ we have

$$\frac{d^2V}{dX^2} = 2\left[\ln(X_0/\Lambda^2) - \kappa + \frac{1}{2}\right] + 2 = 2$$

Hence X_0 is a minimum of V and the Hamiltonian is bounded from below.



(e) The states of minimum energy correspond to the circle in the complex ϕ plane where $\phi^*\phi = X_0$, i.e. $\phi = \phi_0$ where

$$\phi_0 = r_0 \mathrm{e}^{i\theta}\ , \quad r_0 = \Lambda \mathrm{e}^{(2\kappa-1)/4}.$$

(f) Considering small field variations around $\phi = r_0$, i.e. $\phi = r_0 + (\chi_1 + i\chi_2)/\sqrt{2}$, we have

$$X = \phi^*\phi = r_0^2 + \sqrt{2}r_0\chi_1 + \frac{1}{2}(\chi_1^2 + \chi_2^2)$$

Then

$$V = V(X_0) + \frac{1}{2}(X - X_0)^2 \frac{d^2V}{dX^2} + \ldots = V(X_0) + 2r_0^2\chi_1^2 + +\mathcal{O}(\chi^3)$$

i.e.

$$V(\phi) = V(\phi_0) + \frac{1}{2}m^2\chi_1^2 + \mathcal{O}(\chi^3)\,.$$

where $m = 2r_0$. Thus the field χ_1 satisfies a Klein-Gordon equation with mass (in natural units) $2r_0$, while the field χ_2 satisfies a massless Klein-Gordon equation. This is an example of *Goldstone's theorem*: the global phase symmetry is spontaneously broken when the field chooses a particular ground state on the circle, and there is an associated massless Goldstone boson χ_2, while the other degree of freedom of the field χ_1 is massive.

6 The propagator $G(t)$ must vanish for $t < 0$ and so we can write

$$G(t) = \Theta(t)g(t)$$

where $g(t)$ can be chosen to be either an odd or even function of t. By the convolution theorem, it follows that

$$\begin{aligned}\widetilde{G}(\omega) &= \int \frac{d\omega'}{2\pi}\widetilde{\Theta}(\omega-\omega')\widetilde{g}(\omega') \\ &= \int \frac{d\omega'}{2\pi}\left[\pi\delta(\omega-\omega') + iP\frac{1}{\omega-\omega'}\right]\widetilde{g}(\omega')\end{aligned}$$

where

$$\widetilde{g}(\omega) = \int_{-\infty}^{+\infty} dt\, g(t)(\cos\omega t + i\sin\omega t)$$

Hence if we choose $g(t)$ to be an odd function, $\widetilde{g}(\omega)$ will be purely imaginary, say $\widetilde{g}(\omega) = i\widetilde{h}(\omega)$. Then equating real parts in the above equation

$$\mathrm{Re}\,\widetilde{G}(\omega) = -P\int \frac{d\omega'}{2\pi}\frac{\widetilde{h}(\omega')}{\omega-\omega'}$$

and equating imaginary parts

$$\mathrm{Im}\,\widetilde{G}(\omega) = \int \frac{d\omega'}{2\pi}\pi\delta(\omega-\omega')\widetilde{h}(\omega') = \frac{1}{2}\widetilde{h}(\omega)\ .$$

Substituting this in the above then gives the Kramers-Kronig relation.

(a) We have

$$\begin{aligned}\mathrm{Re}\,\widetilde{G}(\omega) &= \frac{\omega-\omega_0}{(\omega-\omega_0)^2+\gamma^2/4} \\ \mathrm{Im}\,\widetilde{G}(\omega) &= -\frac{1}{2}\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2/4}\ .\end{aligned}$$

The r.h.s. of the Kramers-Kronig relation is thus

$$-P\int \frac{d\omega'}{2\pi}\frac{\gamma}{(\omega'-\omega)}\frac{1}{(\omega'-\omega_0)^2+\gamma^2/4}\ .$$

The integrand has poles at $\omega' = \omega$ and at $\omega' = \omega_0 \pm i\gamma/2$. However, it vanishes rapidly at ∞, so we can complete the contour with a large semicircle in either the upper or the lower half-plane. Using

$$P\int = \lim_{\epsilon\to 0}\frac{1}{2}\left(\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} + \int_{-\infty-i\epsilon}^{+\infty-i\epsilon}\right)\ ,$$

and choosing the upper half plane, the first integral encloses only the pole at $\omega' = \omega_0 + i\gamma/2$, which gives

$$-i\pi\frac{1}{2\pi}\frac{\gamma}{(\omega_0+i\gamma/2-\omega)}\frac{1}{(i\gamma)} = -\frac{1}{2}\frac{1}{(\omega_0+i\gamma/2-\omega)}\ .$$

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Choosing the lower half plane for the second integral encloses only the pole at $\omega' = \omega_0 - i\gamma/2$ (in a negative sense), giving

$$i\pi \frac{1}{2\pi} \frac{\gamma}{(\omega_0 - i\gamma/2 - \omega)} \frac{1}{(-i\gamma)} = -\frac{1}{2} \frac{1}{(\omega_0 - i\gamma/2 - \omega)} .$$

The sum of these is

$$-\frac{1}{2} \frac{1}{(\omega_0 - i\gamma/2 - \omega)} - \frac{1}{2} \frac{1}{(\omega_0 + i\gamma/2 - \omega)} = \frac{\omega - \omega_0}{(\omega - \omega_0)^2 + \gamma^2/4} = \mathrm{Re}\, \widetilde{G}(\omega)$$

as required.

(b) The propagator is $G(\boldsymbol{r} - \boldsymbol{r}', t - t')$ where

$$G(\boldsymbol{r}, t) = \int \frac{d^3\boldsymbol{k}\, d\omega}{(2\pi)^4} \frac{\exp(i\boldsymbol{k} \cdot \boldsymbol{r} - i\omega t)}{\omega - \hbar \boldsymbol{k}^2/2m + i\gamma/2} .$$

For $t < 0$ we can complete the contour with a large semicircle in the upper half-plane, which encloses no singularities and so gives $G(\boldsymbol{r}, t) = 0$ for $t < 0$. For $t > 0$ we must instead choose the lower half-plane, which encloses the pole at $\omega = -\hbar \boldsymbol{k}^2/2m - i\gamma/2$ (in a negative sense), giving

$$G(\boldsymbol{r}, t) = -i \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \exp\left(i\boldsymbol{k} \cdot \boldsymbol{r} - \frac{i\hbar \boldsymbol{k}^2 t}{2m} - \frac{\gamma}{2} t \right)$$

for $t > 0$. Now we can write

$$i\boldsymbol{k} \cdot \boldsymbol{r} - \frac{i\hbar \boldsymbol{k}^2 t}{2m} = -\frac{i\hbar t}{2m} \left(\boldsymbol{k} - \frac{m}{\hbar t} \boldsymbol{r} \right)^2 + \frac{im}{2\hbar t} \boldsymbol{r}^2$$

Hence, changing the variable of integration to

$$\boldsymbol{k}' = \boldsymbol{k} - \frac{m}{\hbar t} \boldsymbol{r}$$

we have

$$G(\boldsymbol{r}, t) = -i \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} \exp\left(-\frac{i\hbar t}{2m} \boldsymbol{k}'^2 + \frac{im}{2\hbar t} \boldsymbol{r}^2 - \frac{\gamma}{2} t \right)$$

But

$$\int d^3\boldsymbol{k}' \exp\left(-a\boldsymbol{k}'^2 \right) = \left(\frac{\pi}{a} \right)^{3/2}$$

so

$$\begin{aligned} G(\boldsymbol{r}, t) &= -\frac{i}{(2\pi)^3} \left(\frac{2\pi m}{i\hbar t} \right)^{3/2} \exp\left(\frac{im}{2\hbar t} \boldsymbol{r}^2 - \frac{\gamma}{2} t \right) \\ &= \left(\frac{im}{2\pi\hbar t} \right)^{3/2} \exp\left(\frac{im}{2\hbar t} \boldsymbol{r}^2 - \frac{\gamma}{2} t \right) \end{aligned}$$

for $t > 0$ and $G(\boldsymbol{r}, t) = 0$ for $t < 0$. Correspondingly

$$G(\boldsymbol{r}, \boldsymbol{r}', t, t') = \left[\frac{im}{2\pi\hbar(t-t')}\right]^{3/2} \exp\left(\frac{im(\boldsymbol{r}-\boldsymbol{r}')^2}{2\hbar(t-t')} - \frac{\gamma}{2}(t-t')\right)$$

for $t > t'$ and $G(\boldsymbol{r}, \boldsymbol{r}', t, t') = 0$ for $t < t'$.

END OF PAPER