NATURAL SCIENCES TRIPOS Part II

Wednesday 13 January 2010 10.30am to 12.30pm

THEORETICAL PHYSICS I

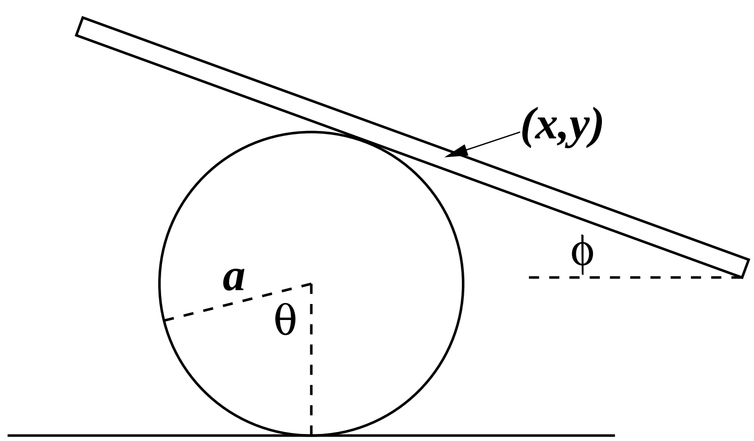
*Answer* **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A thin uniform plank of length $l$ and mass $m$ rests, initially in equilibrium, on a uniform cylinder of radius $a$, also of mass $m$, which can roll on a horizontal plane.

(a) Show that the Lagrangian of this system is

$$L = \frac{3}{4}ma^2\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\phi}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

where $\theta$ is the angular displacement of the cylinder, $\phi$ is the angle of inclination of the plank, and $x, y$ are the horizontal and vertical displacements of the centre of the plank, as shown in the figure. [8]




(b) Show further that if there is no slipping then [5]

$$\begin{aligned} x/a &= \theta + \sin\phi + (\theta - \phi)\cos\phi \,, \\ y/a &= -1 + \cos\phi - (\theta - \phi)\sin\phi \,. \end{aligned}$$

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(c) Deduce the canonical momenta $p_\theta$ and $p_\phi$ conjugate to the generalized coordinates $\theta$ and $\phi$. [5]

(d) Working to second order in small quantities, find the Hamiltonian of the system and hence show that Hamilton's equations for small displacements are [7]

$$\begin{aligned} \dot{\theta} &= \frac{2}{11}\frac{p_\theta}{ma^2}\ , \quad \dot{p}_\theta = mga\phi \\ \dot{\phi} &= 12\frac{p_\phi}{ml^2}\ , \quad \dot{p}_\phi = -mga(\phi - \theta)\ . \end{aligned}$$

(e) Show that there is a mode of small oscillation and find the corresponding angular frequency and relationship between $\theta$ and $\phi$. [8]

2 A particle of mass $m$ and charge $e$ moves non-relativistically in a plane under the influence of a charge $4\pi\epsilon_0 Q$ fixed at the origin and a constant magnetic field $B$ perpendicular to the plane.

(a) Show that a suitable form for the 4-vector potential is

$$A^\mu = \left(\frac{Q}{rc}, -\frac{1}{2}Br\sin\theta, \frac{1}{2}Br\cos\theta, 0\right)$$

where $r, \theta$ are plane polar coordinates.s [5]

(b) Hence show that the Lagrangian can be written as [5]

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{eQ}{r} + \frac{1}{2}eBr^2\dot{\theta}\ .$$

(c) Deduce the equations of motion. [8]

(d) Find two constants of the motion. [8]

(e) Show that, to first order in $B$, the effect of the magnetic field is to cause the orbit of the charge to precess with the Larmor frequency, $\omega_L = eB/2m$. [7]

3 The potential energy density for sound vibrations in an ideal classical gas at density $\rho$ is

$$V = \frac{S_0}{\gamma+1}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma+1} - 1\right]$$

where $S_0$, $\rho_0$ and $\gamma$ are constants.

(a) Writing $\rho = \rho_0(1 - \boldsymbol{\nabla}\cdot\boldsymbol{\xi})$, where $\boldsymbol{\xi}(\boldsymbol{r}, t)$ is the vector field describing the (small) amplitude of vibration, show that

$$V = S_0\left(-\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \frac{\gamma}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2\right)$$

where terms cubic and higher in $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ have been neglected. [5]

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(b) Assuming that the kinetic energy density of the gas can be approximated by

$$T = \frac{1}{2}\rho_0 \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}},$$

write down the expression for the Lagrangian density $\mathcal{L}$ and show that Lagrange's equations of motion lead to the field equation [12]

$$\rho_0 \ddot{\boldsymbol{\xi}} - \gamma S_0 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) = 0.$$

(c) Calculate the canonical momentum density $\boldsymbol{\pi}(\boldsymbol{r}, t)$ conjugate to $\boldsymbol{\xi}$, and show that the total Hamiltonian is

$$H = \int \mathrm{d}^3\boldsymbol{r}\ \mathcal{H}(\boldsymbol{r}, t)$$

where

$$\mathcal{H}(\boldsymbol{r}, t) = \frac{\boldsymbol{\pi}^2}{2\rho_0} + \frac{\gamma}{2} S_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2$$

and you may assume that the field $\boldsymbol{\xi}$ vanishes at $\infty$ in all spatial directions. [8]

(d) If $\boldsymbol{\xi}^T$ and $\boldsymbol{\xi}^L$ are solutions of the equation of motion that are solenoidal and irrotational respectively, i.e.

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi}^T = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{\xi}^L = 0,$$

show that $\boldsymbol{\xi}^T$ obeys a free-particle equation of motion while $\boldsymbol{\xi}^L$ obeys a wave equation. Find the wave velocity for the latter. [8]

4 The Lagrangian per unit length for bending of a stiff elastic rod is

$$\mathcal{L} = \frac{1}{2}\rho A \left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{1}{2} EI \left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2$$

where $\varphi(x, t)$ is the transverse displacement, $\rho$ the density, $A$ the cross-sectional area, $E$ Young's modulus and $I$ the moment of area of the rod.

(a) State Hamilton's principle of least action and use it to deduce the equation of motion [9]

$$\rho A \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \varphi}{\partial x^2}\right) = 0 .$$

(b) Derive the canonical momentum and the Hamiltonian per unit length, $\mathcal{H}$. [6]

(c) By considering the conservation equation

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{J}}{\partial x}$$

show that the current of energy is [9]

$$\mathcal{J} = \frac{\partial \varphi}{\partial t}\frac{\partial}{\partial x}\left(EI \frac{\partial^2 \varphi}{\partial x^2}\right) - EI \frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^2 \varphi}{\partial t \partial x} .$$

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(d) By considering a wave solution of the form $\varphi = C\cos(kx - \omega t)$ for a uniform rod, find the dispersion relation and the wave and group velocities, and show that energy is transferred at the group velocity. [9]

5 The Lagrangian density for a self-interacting, complex, massless scalar field $\phi$ is given by

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - V(\phi)$$

where $V$ is a real-valued function of the scalar field.

(a) Using the Euler-Lagrange equations, show that the equations of motion are [6]

$$\partial^\mu \partial_\mu \phi + \frac{\partial V}{\partial \phi^*} = \partial^\mu \partial_\mu \phi^* + \frac{\partial V}{\partial \phi} = 0.$$

(b) Show that the corresponding Hamiltonian density, in units where $c = 1$, is

$$\mathcal{H} = \pi^* \pi + \boldsymbol{\nabla} \phi^* \cdot \boldsymbol{\nabla} \phi + V(\phi)$$

where $\pi = \partial \phi^* / \partial t$ and $\pi^* = \partial \phi / \partial t$ are the canonical momentum densities conjugate to $\phi$ and $\phi^*$ respectively. [5]

(c) Show that if the potential $V$ is a function of $\phi^* \phi$, the Lagrangian density is invariant under a global phase change in $\phi$. [3]

(d) Consider the case of the *Coleman-Weinberg* potential,

$$V(\phi) = (\phi^* \phi)^2 \left[ \ln \left( \frac{\phi^* \phi}{\Lambda^2} \right) - \kappa \right],$$

where $\Lambda$ and $\kappa$ are real, positive constants. Sketch the potential $V$ as a function of $\phi^* \phi \geq 0$, and hence show that the Hamiltonian is bounded from below. [7]

(e) Show that the states of minimum energy correspond to a circle in the complex $\phi$ plane $\phi_0 = r_0 e^{i\theta}$ where [5]

$$r_0 = \Lambda e^{(2\kappa - 1)/4}.$$

(f) By considering small field variations around the state of minimum energy on the positive real $\phi$ axis, i.e. $\phi = r_0 + (\chi_1 + i\chi_2)/\sqrt{2}$, show that

$$V(\phi) = V(\phi_0) + \frac{1}{2} m^2 \chi_1^2 + \mathcal{O}(\chi^3).$$

Comment on the significance of this result and find the value of $m$. [7]

6 Outline the derivation of the first Kramers-Kronig relation

$$\mathrm{Re}\,\widetilde{G}(\omega) = P \int \frac{d\omega'}{\pi} \frac{\mathrm{Im}\,\widetilde{G}(\omega')}{\omega' - \omega}$$

where $\widetilde{G}(\omega)$ represents the Fourier transform of the time dependence of a causal propagator. [10]

[*You may assume that the Fourier transform of the Heaviside step function is* $\widetilde{\Theta}(\omega) = \pi\delta(\omega) + iP(1/\omega)$.]

(a) The propagator for the wavefunction of an unstable particle with energy $\hbar\omega_0$ and mean lifetime $1/\gamma$ has

$$\widetilde{G}(\omega) = \frac{1}{\omega - \omega_0 + i\gamma/2} \ .$$

Show explicitly that this satisfies the above Kramers-Kronig relation. [10]

[*Hint: Interpret the principal value as* $P\int = \lim_{\epsilon\to 0} \frac{1}{2}\left(\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} + \int_{-\infty-i\epsilon}^{+\infty-i\epsilon}\right)$.]

(b) Derive the propagator $G(\boldsymbol{r}, \boldsymbol{r}', t, t')$ of a free non-relativistic unstable particle, for which $\omega_0 = \hbar\boldsymbol{k}^2/2m$ where $\boldsymbol{k}$ is the wave vector. [13]

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