# Theoretical Physics 1
# Answers to Examination 2006

Warning — these answers have been completely retyped...
Please report any typos/errors to `emt1000@cam.ac.uk`

Q1. The gravity potential is $-GMm/R$ so we need to determine the distances of two masses to the centre. From the sketch, the projections are: $R_{1,2} = r \pm a\cos\theta$, so the total potential energy is

$$V = -GMm\left(\frac{1}{r + a\cos\theta} + \frac{1}{r - a\cos\theta}\right) \approx -\frac{2GMm}{r}\left(1 + \frac{a^2}{r^2}\cos^2\theta\right).$$

There are two separate motions, of a centre of mass of the pair and the rotation about this centre. The kinetic energy is made of these two parts:

$$T = \frac{1}{2}(2m)[(\dot{r})^2 + r^2(\dot{\phi})^2] + \frac{1}{2}(2ma^2)(\dot{\theta})^2.$$

Writing Euler-Lagrange equations for the Lagrangian $L = T - V$, depending on the three variables, is a straightforward task:

$$(r:) \qquad \ddot{r} - r\dot{\phi}^2 + \frac{GM(r^2 + 3a^2\cos^2\theta)}{r^4} = 0;$$

$$(\phi:) \qquad J_\phi = mr^2\dot{\phi} = \text{const};$$

$$(\theta:) \qquad \ddot{\theta} + \frac{GM}{r^3}\sin 2\theta = 0.$$

Obviously the positions of $\theta = 0$ and $\pi/2$ are equilibrium. There are several ways to determine stability, e.g., from the small increments of force, or from the second derivative of the potential. Just looking at the $-\cos^2\theta$ factor in the potential energy $V$ suggests that $\theta = 0$ is stable and $\pi/2$ is unstable.

In the vicinity of $\theta = 0$ we have the oscillation in $\theta(t)$,

$$\ddot{\theta} + \frac{2GM}{r^3}\theta = 0$$

with the frequency $\Omega = \sqrt{2GM/r^3}$. The question is, how is it related to the orbital period (or frequency $\omega = \dot{\phi}$)? For circular orbit we have $r =$const, and so $\omega \approx GM/r^3 + ...$ from the equation for $\ddot{r}$. Hence $\Omega = 2\omega$, or the period of $\theta$-oscillations is twice as short as the orbital period.

Q2. Bookwork: The generalised Bernoulli equation reads

$$\rho\dot{\phi} + \frac{1}{2}\rho v^2 + P + \rho g h = \text{const},$$

where $\rho$ is the mass density of a fluid, $\phi$ is a velocity potential (such that $\boldsymbol{v} = \nabla\phi$ in a fluid with zero vorticity), $\frac{1}{2}\rho v^2$ the density of kinetic energy, $P$ the local pressure and $\rho g h$ the density of gravitational potential energy. The equation remains a constant along the flow line. To be valid, the Bernoulli equation requires the potential flow condition, i.e. that the vorticity $\Omega = curl\boldsymbol{v}$ to be zero, exactly or approximately. In that case there is no viscous dissipation either and the equation represents the local conservation of energy density.

There are two separate short questions here, both exploring Bernoulli equation in action:

(a) In the bath, if we assume that the top surface is almost stationary (i.e. moves much slower than the liquid through the plug hole) and the process is steady-state, we have to conclude that $\dot{\phi} = 0$ and the pressure is constant at every point (equilibrated open system). Then taking the line of motion between the top free surface of water ($\rho g h$) and the plug hole ($\frac{1}{2}\rho v^2$) we obtain:

$$\rho g h = \text{const} = \frac{1}{2}\rho v^2,$$

or $v = \sqrt{2gh}$, as always in the "free fall". The change in the volume of incompressible liquid, $\dot{V} = A_{\text{bath}}\dot{h}$ is equal to the discharge rate through the plug hole $Q = \int v dA_{\text{plug}}$. Since we are asked to "estimate", let us crudely assume the velocity is constant across the hole, giving the equation for the depth $h(t)$:

$$\dot{h} = \frac{A_{\text{plug}}}{A_{\text{bath}}}\sqrt{2gh},$$

or $h = (A_{\text{plug}}/A_{\text{bath}})^2(g/2)t^2$. Putting in the values we estimate the time to reduce $h$ to zero as $t = 120\,\text{s}$!

(If you assumed a parabolic profile of $v$ across the plug hole, you may obtain a 4 min answer instead. This would earn you respect, but no extra points, since the questions asks for an estimate and $\sim 1$ min is accurate enough.)

(b) The two colliding jets are in a steady-state, and no gravity is evident in the question, so our Bernoulli equation reduces to $\frac{1}{2}\rho v^2 + P = \text{const}$ along any flow line. But the pressure has to be the same, atmospheric $P_0$, away from the collision zone. This means that connecting a flow line between two points in the original jet and in the spread sheet, we have on both sides: $\frac{1}{2}\rho v^2 + P_0 = \text{const} = \frac{1}{2}\rho v^2 + P_0$. In other words, the velocity remains $v$ in the sheet! Then we only need to consider the mass conservation: the initial flow rate, $2v\pi a^2$, is equal to the flow rate through the perimeter of the sheet, $vd2\pi r$, giving the required $d = a^2/r$.

Q3. Bookwork: There are two ways of showing the need for the revised electromagnetic potential energy in the Lagrangian dynamics. The one used in the lectures was to work back from the Lorentz force $\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ through the condition

$$\boldsymbol{F} = -\frac{\partial V}{\partial \boldsymbol{q}} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{\boldsymbol{q}}}\right)$$

for the velocity-dependent potential. The requirement of Lorentz invariance is an alternative way.

(a) The definition $\boldsymbol{B} = curl\boldsymbol{A}$ is frame independent. If you don't like evaluating differential operators in polar coordinates, take the flux over the circular area in the $(x, y)$ plane, $\int BdS_z = \pi r^2 B$, which gives you $\oint \boldsymbol{A} \cdot d\boldsymbol{l}$ around the circle $2\pi r$, with only $A_\theta = rB/2$ component in action.

(b) The Lagrangian in polar coordinates is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + e(r\dot{\theta})A_\theta$$

and the three equation of motion are, after the standard evaluation and noticing that two of the coordinates conserve their momentum:

$$\begin{aligned}
&(r:) \qquad m\ddot{r} - mr\dot{\theta}^2 - erB\dot{\theta} = 0; \\
&(\theta:) \qquad mr^2\dot{\theta} + \frac{1}{2}eBr^2 = \text{const}; \\
&(z:) \qquad \dot{z} = \text{const}.
\end{aligned}$$

The second equation gives the required condition. Note that we have no apriori knowledge of the sign of the constant $J$ (you may find an unexpected zero somewhere down the line if you accidentally choose it wrongly, because the $r$-equation is insensitive to this sign).

(c) Substituting $\dot{\theta}$ from the angular momentum balance into the $r$-equation gives

$$\ddot{r} = \frac{J^2}{r^3} - \frac{e^2B^2r}{4m^2}.$$

Note how increasing the magnitude of $B$ could change the sign of this acceleration. The stable helical orbit (helical - due to the constant velocity $v_z$ along the $B$-axis) is when $\ddot{r} = 0$, giving the condition for the radius:

$$r = \sqrt{\frac{2mJ}{eB}}$$

Substituting this back into the $\dot{\theta}$ gives the required angular frequency.

(d) This is straightforward: the angle of inclination is determined by the ratio of $v_\theta = r\dot{\theta}$ to the (constant) $v_z$. Substituting in what we know about $r$ and $\dot{\theta}$ gives the required tangent.

Q4. **Integral (1):** If a complex function $f(z)$ contains no poles within some contour then:

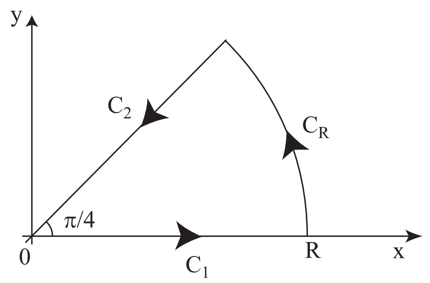
$$\oint f(z) = 0$$

In order to perform a definite integral using this relation you should: (1) choose a form for $f(z)$ that is simply related to the function in your integral. (2) choose a contour where one part of it will return your integral, or something simply related to it, and all other parts can be performed analytically.

For the suggested example consider the following contour integral

$$\oint e^{iz^2} dz = 0$$

on the the closed contour shown in the sketch below.




Along the real axis $C_1$ we have $z = x$, $dz = dx$ so that

$$\int_{C_1} e^{iz^2} dz = \int_0^\infty e^{ix^2} dx = \int_0^\infty \cos(x^2) dx + i \int_0^\infty \sin(x^2) dx$$

Along the contour $C_2$ at angle $\pi/4$ we have $z = re^{i\pi/4} = r(1+i)/\sqrt{2}$ with $dz = dr(1+i)/\sqrt{2}$ so that

$$\int_{C_2} e^{iz^2} dz = \int_\infty^0 e^{i(re^{i\pi/4})^2} dr(1+i)/\sqrt{2} = \int_\infty^0 e^{-r^2} dr(1+i)/\sqrt{2} = -\frac{\sqrt{\pi}}{2}.\frac{1+i}{\sqrt{2}}$$

Along the contour $C_R$ we have $z = Re^{i\theta}$ with $dz = iRe^{i\theta}d\theta$ so that

$$\int_{C_R} e^{iz^2} dz = \int_0^{\pi/4} e^{iR^2(\cos(2\theta)+i\sin(2\theta))} iRe^{i\theta} d\theta = \int_0^{\pi/4} e^{i\phi(R,\theta)}.e^{-\sin(2\theta)R^2} R d\theta$$

The function $\exp(i\phi)$ has modulus unity and $\sin(2\theta) \geq 0$ on the contour $C_R$. This implies that both the real and imaginary parts of this integral will be less than:

$$\int_0^{\pi/4} e^{-R^2} R d\theta = \pi e^{-R^2} R/4$$

In the limit $R \to \infty$ this integral tends to zero. Therefore

$$\lim_{R\to\infty} \int_{C_R} e^{iz^2} dz = 0$$

Collecting together the terms we have:

$$\oint e^{iz^2} dz = \int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz = 0$$

because the contour contains no poles. This implies

$$\int_0^\infty \cos(x^2) dx + i \int_0^\infty \sin(x^2) dx = \frac{\sqrt{\pi}}{2} . \frac{1+i}{\sqrt{2}}$$

Taking the real part of this relation we find:

$$\int_0^\infty \cos(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

**Integral (2):** If a contour passes through a pole half of its residue is included in the residue sum.

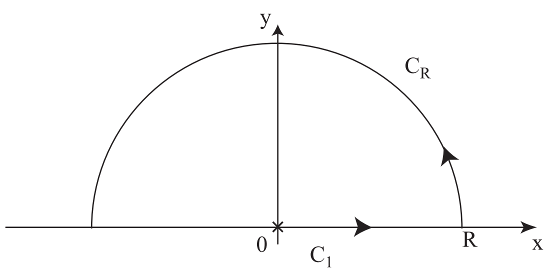
The integral in the suggested example does not have any poles but if it is rewritten as a complex integral there is a pole at the origin $z = 0$

$$\int_{-\infty}^\infty \frac{\sin(x)}{x} dx = \Im \int_{-\infty}^\infty \frac{e^{ix}}{x} dx$$

We perform the following contour integral

$$\oint \frac{e^{iz}}{z} dz$$

on the contour shown in the figure




Along $C_1$ we have $z = x$ so that the integral takes the form:

$$\int_{C_1} \frac{e^{iz}}{z} dz = \int_{-\infty}^\infty \frac{e^{ix}}{x} dx$$

Along $C_R$ we have $z = Re^{i\theta}$ so that the integral takes the form:

$$\int_{C_R} \frac{e^{iz}}{z} dz = \int_0^{\pi} e^{i.(\cos(\theta)+i\sin(\theta))R} iRe^{i\theta} d\theta$$

since $\sin(\theta) \geq 0$ along this contour in the limit $R \to \infty$ both the real and imaginary parts of this integral are zero.

The function $\exp(iz)/z$ has one pole at $z = 0$. The residue of this pole is:

$$\lim_{z \to 0} z.\frac{e^{iz}}{z} = e^{i.0} = 1$$

Collecting everything together we therefore have

$$\oint \frac{e^{iz}}{z} dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = 2\pi i.1/2 = i\pi$$

Taking the imaginary part of this integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \pi$$

**Integral (3):** No contour can pass across a branch cut but contours can pass infinitesimally above and below a branch cut. Since there is some freedom in choosing the location of a branch cut, this can sometimes be used to advantage in evaluating some definite integrals. The following is an example of this:

We consider the contour integral

$$\oint \frac{z^{\alpha}}{1 + \sqrt{2}z + z^2} dz$$

on contour shown in the figure below.

Along $C_R$ the integral is:

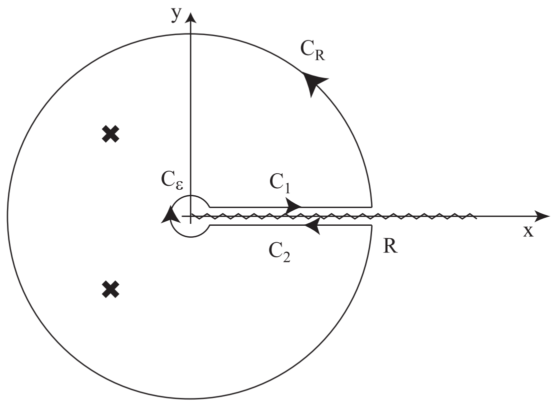
$$O(R^{\alpha}/R^2).2\pi R = O(R^{\alpha-1}) \to 0 \quad \text{as} \quad \text{R} \to \infty$$

Along $C_\epsilon$ the integral is

$$\int_{2\pi}^{0} \frac{\epsilon^{\alpha} e^{i\alpha\theta}}{1 + \sqrt{2}\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta} d\theta = O(\epsilon^{\alpha+1}) \to 0 \quad \text{as} \quad \text{epsilon} \to 0$$

Along $C_1$

$$\int_0^{\infty} \frac{x^{\alpha}}{1 + \sqrt{2}x + x^2} dx = I$$




Along $C_2$

$$\int_{\infty}^{0} \frac{x^\alpha e^{2\alpha\pi i}}{1+\sqrt{2}x+x^2} dx = -e^{2\alpha\pi i} I$$

The function in the contour integral has simple poles at $z = \exp(3\pi i/4)$ and $z = \exp(5\pi i/4)$ so by Cauchy's theorem

$$\oint \frac{z^\alpha}{1+\sqrt{2}z+z^2} dz = 2\pi i \left( \frac{e^{3\pi i/4}}{\sqrt{2}i} - \frac{e^{5\pi i/4}}{\sqrt{2}i} \right)$$

Collecting together the non-zero terms we have:

$$\oint \frac{z^\alpha}{1+\sqrt{2}z+z^2} dz = I - e^{2\alpha\pi i} I = 2\pi i \left( \frac{e^{3\pi i/4}}{\sqrt{2}i} - \frac{e^{5\pi i/4}}{\sqrt{2}i} \right)$$

Rearranging this expression we get:

$$I = \int_0^\infty \frac{x^\alpha}{1+\sqrt{2}x+x^2} dx = \sqrt{2}\pi \frac{\sin(\alpha\pi/4)}{\sin(\alpha\pi)}$$

Q5. The derivation of the Kramers-Kronig relations is given in the lecture notes.

The relationship between $x$,$G$ and $f$ can be found in the following way. Define the operator

$$L_t = \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2$$

so that the equation of motion has the form

$$L_t x(t) = f(t)$$

insert a delta function into the rhs

$$L_t x(t) = \int dt' \delta(t-t') f(t')$$

and substitute for it using the definition of the Green's function

$$L_t G(t-t') = \delta(t-t')$$

we get

$$L_t x(t) = \int dt' L_t G(t-t') f(t') = L_t \int dt' G(t-t') f(t')$$

which implies

$$x(t) = \int dt' G(t-t') f(t')$$

Note that from this we can deduce that $G(t-t')$ is a generalized susceptibility and therefore, its Fourier transform $G(\omega)$ will obey the Kramers-Kronig relations.

By definition

$$\left(\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2\right) G(t-t') = \delta(t-t')$$

Taking the Fourier transform we get

$$\left(-\omega^2 - i\gamma\omega + \omega_0^2\right) G(\omega) = 1$$

Hence

$$G(\omega) = \frac{1}{-\omega^2 - i\gamma\omega + \omega_0^2}$$

To simplify the algebra we rewrite $G(\omega)$ in the form

$$G(\omega) = \frac{1}{(\omega+b+ia)(\omega-b+ia)}$$

where $a = \gamma/2 > 0$ and $b = \sqrt{\omega_0^2 - \gamma^2/4} > 0$. This corresponds to damped oscillation. Splitting $G(\omega)$ into its real and imaginary parts we have

$$G(\omega) = \frac{(\omega^2 - b^2 - a^2) - 2ia\omega}{(\omega+b+ia)(\omega-b+ia)(\omega+b-ia)(\omega-b-ia)}$$

In order to show that $G(\omega)$ satisfies the first Kramers-Kronig relation we need to prove:

$$P \int_{-\infty}^{\infty} \Im G(\omega) . \frac{1}{\omega - \zeta} \frac{d\omega}{\pi} = \Re G(\zeta)$$

In the complex plane

$$f(z) = \Im G(z) . \frac{1}{z-\zeta} = \frac{-2az}{(z+b+ia)(z-b+ia)(z+b-ia)(z-b-ia)(z-\zeta)}$$

The easiest way to integrate this function is to rewrite as the sum of two functions

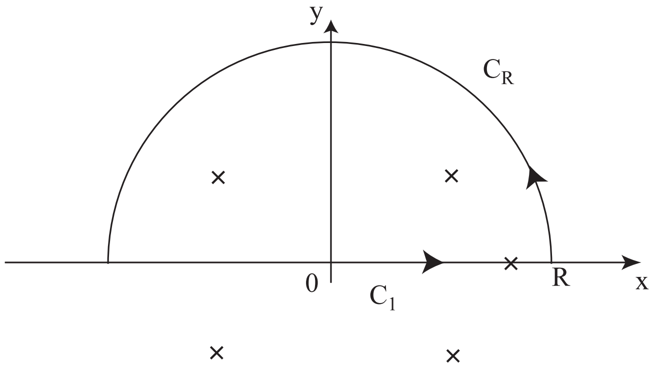
$$f_1(z) = \frac{-2a}{(z+b+ia)(z-b+ia)(z+b-ia)(z-b-ia)}$$

$$f_2(z) = -\frac{-2a\zeta}{(z+b+ia)(z-b+ia)(z+b-ia)(z-b-ia)(z-\zeta)}$$

So that

$$f(z) = f_1 + f_2$$

The poles of these functions are shown in the figure. The pole at $z = \zeta$ on the real axis is not present in $f_1(z)$.




For both functions the integral along the path $C_R$ vanishes as $R \to \infty$. Therefore for $f_1$:

$$\oint f_1(z) = P\int_{-\infty}^{\infty} f_1(z)dz = 2\pi i(Res_1 + Res_2)$$

$$= -2a.2\pi i.\left[\frac{1}{(2ia)(-2b+2ia)(-2b)} + \frac{1}{(2b+2ia)(2ia)(2b)}\right]$$

$$= -\frac{\pi}{b^2+a^2}$$

For $f_2$ we have:

$$\oint f_2(z) = P\int_{-\infty}^{\infty} f_1(z)dz = 2\pi i(Res_1 + Res_2 + \frac{Res_3}{2})$$

$$= -2a\zeta.2\pi i.\left[\frac{1}{(2ia)(-2b+2ia)(-2b)(ia-b-\zeta)} + \frac{1}{(2b+2ia)(2ia)(2b)(ia+b-\zeta)} + \right.$$

$$\left.\frac{1}{2}\frac{1}{(\zeta+b+ia)(\zeta-b+ia)(\zeta+b-ia)(\zeta-b-ia)}\right]$$

$$= -\frac{\zeta^2\pi(-b^2+3a^2+\zeta^2)}{(\zeta+b+ia)(\zeta-b+ia)(\zeta+b-ia)(\zeta-b-ia)(b^2+a^2)}$$

We therefore have

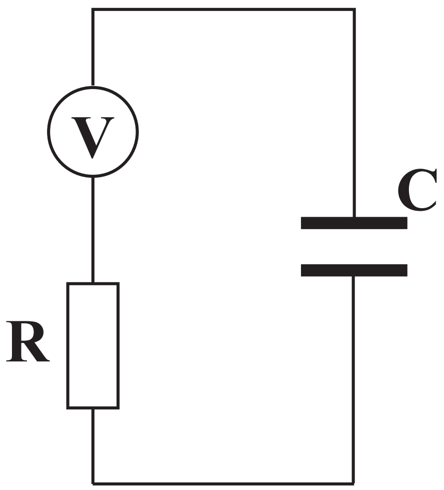
$$P\int_{-\infty}^{\infty} \Im G(\omega).\frac{1}{\omega-\zeta}d\omega = P\int_{-\infty}^{\infty} f_1 d\omega + P\int_{-\infty}^{\infty} f_2 d\omega$$

$$= -\frac{\pi}{b^2+a^2} - \frac{\zeta^2\pi(-b^2+3a^2+\zeta^2)}{(\zeta+b+ia)(\zeta-b+ia)(\zeta+b-ia)(\zeta-b-ia)(b^2+a^2)}$$

rearranging this we find

$$P\int_{-\infty}^{\infty} \Im G(\omega).\frac{1}{\omega-\zeta}d\omega = \frac{\zeta^2-a^a-b^2}{(\zeta+b+ia)(\zeta-b+ia)(\zeta+b-ia)(\zeta-b-ia)} = \Re G(\zeta)$$

Q6. The circuit is as shown in the figure below




The sum of the voltage across the resistor and capacitor is equal to the voltage across the voltage source.

$$V = IR + \frac{Q}{C}$$

since $I = dQ/dt$ we can rearrange this equation and write it in Langevin form

$$\frac{dQ}{dt} = -\gamma Q + \sqrt{\Gamma}A$$

where $\gamma = 1/RC$ and $\sqrt{\Gamma} = \sqrt{2Rk_BT}/R$. Here $A$ is uncorrelated white noise with mean zero and unit variance.

The next part of the question is book work and can be found in the lecture notes.

Making the correspondences between the Langevin and Fokker-Plank equations:

$$\begin{aligned} q &= Q \\ F(q) &= -\gamma Q \\ G(q) &= \sqrt{\Gamma} \\ K &= -\gamma Q \\ Q &= \Gamma \end{aligned}$$

This results in a Fokker-Plank equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial Q}(\gamma Q f) + \frac{\Gamma}{2}\frac{\partial^2 f}{\partial Q^2}$$

In equilibrium

$$\frac{\partial f}{\partial t} = 0$$

The equilibrium equation can then be rearranged to give

$$\gamma \frac{\partial}{\partial Q}(Q f) = -\frac{\Gamma}{2}\frac{\partial^2 f}{\partial Q^2}$$

Integrating w.r.t $Q$ we have

$$\gamma Q f = -\frac{\Gamma}{2}\frac{\partial f}{\partial Q}$$

assuming there is no net charge on the capacitor. Solving for $f$ we find

$$f = f_0 \exp\left(-\frac{\gamma}{\Gamma}Q^2\right)$$

Substituting for the definitions of $\gamma$ and $\Gamma$ from above, this becomes

$$f = f_0 \exp\left(-\frac{Q^2}{2Ck_BT}\right)$$

The energy dissipated in charging a capacitor is $W = Q^2/2C$ so the above equation had the form

$$f = f_0 \exp\left(-\frac{W}{k_BT}\right)$$

which is just the Boltzmann distribution.