Theoretical Physics 1
Answers to Examination 2005

Warning — these answers have been completely retyped...
Please report any typos/errors to `emt1000@cam.ac.uk`

Q1. Bookwork: Hamilton's principle is $\delta \int \mathrm{d}t\ L(q_i, \dot{q}_i, t) = 0$ and leads (via the calculus of variations) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{1}$$

i.e. N 2nd-order equations for the coordinates q_i.

The Lagrangian is

$$L = T - V = \frac{m}{2}\left(l^2\dot{\theta}^2 + l^2\omega^2\sin^2\theta\right) + mgl\cos\theta \tag{2}$$

Evaluating the Euler−Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{3}$$

gives

$$ml^2\ddot{\theta} = -mgl\sin\theta + ml^2\omega^2\sin\theta\cos\theta \tag{4}$$

For small oscillations around $\theta = 0$ this may be rewritten as

$$ml^2\ddot{\theta} = -mgl\theta + ml^2\omega^2\theta \tag{5}$$

For stability this requires that

$$ml^2\omega^2 > mgl \tag{6}$$

The rotation rate for which $\theta = 0$ is no-longer stable is then

$$\omega_C = \sqrt{\frac{g}{l}} \tag{7}$$

For the stable point with $\theta > 0$ at frequencies $\omega > \omega_C$ we assume that the system performs small oscillations around the angle θ_0 so that $\theta = \theta_0 + \delta$. Substituting this into equation 4 we find

$$l\ddot{\delta} = \left(-g\sin\theta_0 + l\omega^2\sin\theta_0\cos\theta_0\right) + \left(-g\cos\theta_0 + l\omega^2\left(\cos^2\theta - \sin^2\theta\right)\right)\delta \tag{8}$$

Simple harmonic motion only occurs if

$$g\sin\theta_0 - ml\omega^2\sin\theta_0\cos\theta_0 = 0 \tag{9}$$

Since we know that $0 < \theta_0 < \pi$ then $\sin\theta_0 > 0$ and therefore from Eqn. 9

$$\cos\theta_0 = \frac{g}{l\omega^2} \tag{10}$$

This reduces Eqn. 8 to

$$l\ddot{\delta} = \left(-g\cos\theta_0 + l\omega^2\left(\cos^2\theta - \sin^2\theta\right)\right)\delta \tag{11}$$

Substituting from Eqn. 10 into Eqn. 11

$$l\ddot{\delta} = -l\omega^2\left(1 - \frac{g^2}{\omega^4 l^2}\right)\delta \tag{12}$$

Small oscillations around $\theta = \theta_0$ therefore have frequency

$$\Omega = \omega\sqrt{1 - \frac{g^2}{\omega^4 l^2}} \tag{13}$$

Q2. To write the given Lagrangian in components (with the convention of summation over pairs of repeated indices): $L = \frac{1}{2}a_{ij}\dot{q}_i\dot{q}_j - V(q)$. Strictly following the definition of the canonical momentum, we obtain [8]

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2}a_{ij}\delta_{ik}\dot{q}_j + \frac{1}{2}a_{ij}\dot{q}_i\delta_{jk} = a_{kj}\dot{q}_j.$$

The Hamiltonian is $H = p_i\dot{q}_i - L$, with all $\dot{q}_j$ substituted by $\dot{q}_j = a^{-1}_{jk}p_k$. This gives, after a little algebra, the required answer [6]

$$H = \frac{1}{2}a^{-1}_{ij}p_ip_j + V(q).$$

For the particular case of matrix $\boldsymbol{A}$ given in the question you'll have

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_3^2 + 2q_1^2\dot{q}_2^2 - 2q_1\dot{q}_2\dot{q}_3) + \frac{1}{2}\log q_1.$$

This you need, if you prefer not to invert the matrix a_{ij} to write down the Hamiltonian directly. Either way you should obtain

$$H = \frac{1}{2}\left(p_1^2 + 2p_3^2 + \frac{p_2^2}{q_1^2} + \frac{2p_2p_3}{q_1}\right) - \frac{1}{2}\log q_1.$$

Now write down the Hamilton equations for the components of momentum:

$$\dot{p}_1 = -\partial H/\partial q_1 = \frac{p_2^2}{q_1^3} + \frac{p_2p_3}{q_1^2} + \frac{1}{2q_1}$$

$$\dot{p}_2 = -\partial H/\partial q_2 = 0$$
$$\dot{p}_3 = -\partial H/\partial q_3 = 0.$$

The last two conditions prove that the corresponding components are the constants of motion. [10]

Now we are told that p_1 is fixed (and equal to zero), so the first of the equations gives the condition [5]

$$\frac{p_2^2}{q_1^3} + \frac{p_2 p_3}{q_1^2} + \frac{1}{2q_1} = 0.$$

Resolving this to find the required p_3^2, we obtain

$$p_3^2 = \left(-\frac{2p_2^2 + q_1^2}{2p_2 q_1}\right)^2 = 1 + \frac{p_2^2}{q_1^2} + \frac{q_1^2}{4p_2^2}.$$

This has a minimum with respect to either of its variables, q_1 or p_2; a sketch would be nice but not necessary. [5]

Q3. First of all, let's write down the Lagrangian in the simplifying case. Now $(dx^0, dx^1) = (cdt, dx)$ and

$$g_{\mu\nu} = \begin{pmatrix} g(x) & 0 \\ 0 & -g(x) \end{pmatrix}$$

which gives, after multiplication under the root, [4]

$$L = -m_0\sqrt{c^2 g(x) - \dot{x}^2 g(x)} = -m_0 c\sqrt{g}\sqrt{1 - v^2/c^2}$$

The l.h.s. of the Euler-Lagrange equation will then take the form [4]

$$\frac{d}{dt}\left(m_0\sqrt{g}\frac{\dot{x}}{\sqrt{c^2 - \dot{x}^2}}\right) = \frac{d}{dt}\left(m_0 v\frac{\sqrt{g}}{\sqrt{c^2 - v^2}}\right)$$

(the factor following the $m_0 v$ is therefore denoted as Γ in the question. The r.h.s. is

$$\frac{\partial L}{\partial x} = -m_0 c\sqrt{1 - v^2/c^2}\left(\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x}\right) = -\frac{m_0}{\Gamma}\frac{\partial}{\partial x}[\frac{1}{2}g(x)]$$

where ϕ is the expression in square brackets. [8]

For the general case of $L = -m_0\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ we just need to be careful with components and indices. For the three spatial components of the 4-vector variable, we'll have in the l.h.s. of the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d}{dt}\left(-m_0\frac{2g_{i\mu}\dot{x}^\mu}{2\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\right) \equiv \frac{d}{dt}\left(\gamma g_{i\mu}\dot{x}^\mu\right)$$

Here $i = (1, 2, 3)$ and $\mu, \nu = (0, 1, 2, 3)$. Now evaluating the derivatives in the r.h.s. we should group terms together into $\gamma = -m_0/\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ (or, equivalently, without m_0 as this cancels on both sides of the linear equation): [10]

$$\frac{\partial L}{\partial x_i} = -m_0 \frac{(\partial g_{\mu\nu}/\partial x_i)\dot{x}^\mu\dot{x}^\nu}{2\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \equiv \frac{1}{2}\gamma\left(\frac{\partial g_{\mu\nu}}{\partial x_i}\right)\dot{x}^\mu\dot{x}^\nu.$$

[8]

Q4. Cauchy theorem says

$$\oint_C \mathrm{d}z f(z) = 2\pi i \sum(\text{residues}) \tag{14}$$

with the counterclockwise closed contour C. This is proved by expanding $f(z)$ in a Laurent series about a singular point z_0

$$f(z) = \sum_{n=-\infty}^{\infty} f_n (z - z_0)^n \tag{15}$$

and showing that only the f_{-1} term contributes (proof will not be required).

The solution of each of the three integrals is based on noticing that the denominator is a quadratic of a quadratic. The first integral has double poles at $z = \pm i$. These are found easily because the denominator is a quadratic in x^2. We convert to a closed contour by completion in (say) the upper half-plane.

$$\text{Res}(x = i) = \lim_{x\to i} \frac{1}{(2-1)!}\frac{\mathrm{d}}{\mathrm{d}x}(x-i)^2 \frac{1}{(x-i)^2(x+i)^2} = -2(2i)^{-3} \tag{16}$$

This imples that the integral is

$$I = 2\pi i. - 2(2i)^{-3} = \frac{\pi}{2} \tag{17}$$

The third integral also has a quadratic form for the denominator. It may be be rewritten

$$(1+x)^2 + (1+1/x)^2 + 1 = (x+1/x)^2 + 2(x+1/x) + 1 = (x+1/x+1)^2 = 0 \tag{18}$$

The solutions are therefore

$$x = -\exp(\pm i\frac{\pi}{3}) \tag{19}$$

As before each root is doubly degenerate. We close the contour in the u.h.p

$$\text{Res}(x = i) = \lim_{x\to -\exp(-i\frac{\pi}{3})} \frac{1}{(2-1)!}\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{(x+\exp(i\frac{\pi}{3}))^2} = -\frac{2}{9}\sqrt{3}i \tag{20}$$

The integral then becomes

$$I = \frac{4}{9}\sqrt{3}\pi \tag{21}$$

The second integral should be rewritten with the substitution $z = \exp(i\theta)$ so that $\mathrm{d}z = iz\mathrm{d}\theta$. The cosine terms should be rewritten $\cos\theta = (z + 1/z)^2$. A contour around the unit circle is used

$$I = \oint \frac{1}{\frac{1}{4}(z+1/z)^2 + \frac{1}{2}(z+1/z) + 1}.\frac{\mathrm{d}z}{iz} \tag{22}$$

The denominator is a quadratic that has the solution

$$z + \frac{1}{z} = -1 - p\sqrt{3}i \qquad p = \pm 1 \tag{23}$$

This is a second quadratic which has the solution

$$z = \frac{1 + p\sqrt{3}i}{2} + \frac{q}{2}\sqrt{(1+p\sqrt{3}i)^2 - 4} \qquad q = \pm 1 \tag{24}$$

The term in the square root can be rewritten

$$(1 + p\sqrt{3}i)^2 - 4) = -4\sqrt{3}(\frac{\sqrt{3}}{2} - p\frac{i}{2}) = -4\sqrt{3}\exp(-ip\frac{\pi}{6}) \tag{25}$$

The poles therefore occur at

$$z = \frac{1 + p\sqrt{3}i}{2} + q3^{1/4}i\exp(-ip\frac{\pi}{12}) \tag{26}$$

These may be evaluated on a calculator to be
$z = 0.159 + 0.4052i, 0.1593 - 0.4052i, 0.840 + 2.173i, 0.840 - 2.173i$.
Evaluation is considerably aided by noticing that if z is a solution then so are $z^*, 1/z, 1/z^*$. Only the first two poles are in the unit circle. The integral is then

$$I = 2\pi i\left(\mathrm{Res}(z_1) + \mathrm{Res}(z_2)\right) = \pi\frac{3^{1/4}\sqrt{2}}{3}\left(1 + \sqrt{3}\right) \tag{27}$$

Q5. In the opening "essay part" must mention that a potential flow (that is, a potential $\phi(\boldsymbol{x}, t)$ exists such that $\boldsymbol{v} = \nabla\phi$) requires that $\mathrm{curl}\boldsymbol{v} = 0$. This means zero vorticity – as a consequence: (a) no viscous dissipation, (b) Bernoulli equation applicable. Incompressible fluid then satisfies $\nabla^2\phi = 0$. [8]

Looking at the circular sandbank from above, there are two separate regions, in which we must solve the Laplacian condition $\nabla^2\phi = 0$, for ϕ_1 inside (over the bank) and for ϕ_2 outside the bank. 2D polar coordinates are recommended in the question, which means ignoring the vorticity region around the edge of the bank.

The boundary conditions are (require 2 for each of $\phi_{1,2}$): At $r \to \infty$ the radial component $\partial_r \phi(2) v_r(2) = u_0 \cos\theta$ for the uniform flow. At $r \to 0$ we will require no singularity in the solution ϕ_1 (see below). At the interface, $r = a$, we want to match the potentials, $\phi_1 = \phi_2$ (which is a similar level of approximation as ignoring the z-nonuniformity near the edge). Finally, we need to consider the mass conservation, i.e. the water flowing into the bank must have the same volume as that flowing over it: matching the flow rate $Q_1 = v_r(1) \cdot [area] = v_r(1)[2\pi a d/2]$ inside, with the flow rate $Q_2 = v_r(2) \cdot [area] = v_r(2)[2\pi a d]$ outside (where the depth d is twice that over the bank). This gives $\partial_r \phi_1 = 2\partial_r \phi_2$ at $r = a$. [8]

The solution of the Laplacian in 2D polars should be well known to you, as the multipole expansion:

$$\phi_{1,2} = a_0 \ln r + \sum_{n=0}^{\infty} \left(a_n r^n + \frac{b_n}{r^n} \right) \cos n\theta.$$

(This format is equivalent, but easier than using formal Legendre polynomials.)

Now, outside the bank, at $r \to \infty$, we have

$$v_r(2) = \partial_r \phi_2|_{r\to\infty} = \sum_{n=0}^{\infty} n a_n r^{n-1} \cos n\theta.$$

Matching this with the required $u_0 \cos\theta$ only leaves the mode $n = 1$, with $a_1 = u_0$. Since we'll need to match $\phi_1 = \phi_2$, the requirement of only single harmonic ($n = 1$) applies to the inside as well. So,

$$\phi_1 = \left(A_1 r + \frac{B_1}{r} \right) \cos\theta; \qquad \phi_2 = \left(u_0 r + \frac{b_1}{r} \right) \cos\theta$$

Now, at $r = 0$ we don't want the singularity in ϕ_1 and, therefore, $B_1 = 0$. This means $\phi_1 = A_1 r \cos\theta$, or the velocity $v_r(1) = A_1 \cos\theta$. This is a uniform flow over the bank! (as in many "dielectric problems" you've seen). [8]

We only need to match the solutions at the bank edge, $r = a$. Here

$$[\phi :] \;\; A_1 a \cos\theta = (u_0 a + b_1/a)\cos\theta; \quad [v_r :] \;\; A_1 \cos\theta = 2(u_0 - b_1/a^2)\cos\theta$$

These are two linear equations for the unknowns A_1, b_1. Resolving them we obtain that A_1 (which is the value of uniform flow velocity over the bank) is equal to $\frac{4}{3} u_0$. [10]

Q6. The Langevin equation can be written for each of the Cartesian components of particle coordinate/velocity:

$$m\ddot{x} = -\gamma \dot{x} + qE + \xi_x(t)$$

$$m\ddot{y} = -\gamma\dot{y} + \xi_y(t)$$

$$m\ddot{z} = -\gamma\dot{z} - mg + \xi_z(t)$$

with the identical properties of the delta-correlated stochastic force in each direction: $\langle\xi_i\rangle = 0, \langle\xi_i^2\rangle = \Gamma$. [8]

In the overdamped limit we can neglect the inertial (ballistic) component of motion, that is set the acceleration to zero $\ddot{\boldsymbol{x}} = 0$. This also means losing the memory of the initial condition for the particle velocity and only consider the balance of forces in the l.h.s. of the Langevin equations. [6]

Formal stochastic solution in the y-direction, which is not affected by any force, is $y(t) = (1/\gamma)\int \xi(\tau)d\tau$. So $\langle y\rangle = 0$ and $\langle y^2\rangle = (\Gamma/\gamma^2)t$, the basic diffusion. [5]

Solution in the x-direction is $x(t) = (1/\gamma)\int \xi(\tau)d\tau + (qE/\gamma)t$. So $\langle x\rangle = (qE/\gamma)t$, the constant drift velocity. The mean square has the cross-term vanishing: $\langle x^2\rangle = (\Gamma/\gamma^2)t + \langle x\rangle^2$, which means the deviations from the drift velocity are diffusive, with $D = \Gamma/\gamma^2$. [5]

Strictly, the same applies to the z-motion, since it also has the constant force applied. However, since there is a restriction (the impenetrable bottom of the vessel at $z = 0$), the constant-velocity drift is not going to happen indefinitely. Instead the system will approach the steady state. The equilibrium (t-independent) probability requires writing the kinetic equation. There are several methods suggested in lectures, the most comprehensive (Fokker-Planck) is to convert

$$\dot{z} = -(mg/\gamma) + (1/\gamma)\xi \quad \text{into} \quad \partial_t f(z,t) = \partial_z[(mg/\gamma)f] + (\Gamma/2\gamma^2)\partial_z^2 f$$

The steady state is obtained by setting the r.h.s. to zero and integrating the l.h.s. over z:

$$\frac{mg}{\gamma}f = -\frac{\Gamma}{2\gamma^2}\partial_z f \quad \text{or} \quad \frac{df}{f} = -\frac{2\gamma mg}{\Gamma}dz$$

and obtaining the required Boltzmann distribution

$$f(z) = \text{const}\ e^{-mgz/kT}$$

(because $\Gamma = 2\gamma kT$ in the present notations). [10]