Theoretical Physics 1
Answers to Examination 2003

Warning — these answers have been completely retyped... Please report any typos/errors.

steve@mrao.cam.ac.uk

Q1. Bookwork: Hamilton's principle is $\delta \int \mathrm{d}t\ L(q_i, \dot{q}_i, t) = 0$ and leads (via the calculus of variations) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{1}$$

i.e. N 2nd-order equations for the coordinates q_i. The position of the mass is

$$x = a\sin\omega t + l\sin\theta; \quad y = -a\cos\omega t - l\cos\theta \tag{2}$$

where $q = \theta(t)$ is the single variable of the problem. The Lagrangian is

$$L = T - V = \frac{m}{2}\left(a^2\omega^2 + l^2\dot{\theta}^2 + 2al\omega\dot{\theta}\cos(\omega t - \theta)\right) + mg(a\cos\omega t + l\cos\theta) \tag{3}$$

and the canonical momentum is

$$p_\theta = ml^2\dot{\theta} + mla\omega\cos(\omega t - \theta) \tag{4}$$

After considerable simplifications, the equation of motion is

$$ml^2\ddot{\theta} + mgl\sin\theta = ma\omega^2\sin(\omega t - \theta) \tag{5}$$

For small oscillations ($\theta \ll 1$) and in the limit $a\omega^2/lg \ll 1$ we can set $\sin\theta \approx \theta$ and $\sin(\omega t - \theta) \approx \sin\omega t$ so that the linearised equation is

$$l^2\ddot{\theta} + gl\theta \approx a\omega^2\sin\omega t \tag{6}$$

This has general solution

$$\theta = A\sin(\omega_0 t + \delta) + \frac{a\omega^2}{gl - l^2\omega^2}\sin\omega t \tag{7}$$

where $\omega_0^2 = g/l$ and A, δ are arbitrary constants. This shows resonance at $\omega = \omega_0$ as required.

Q2. Bookwork: the canonical momenta are $p_i \equiv \partial L/\partial \dot{q}_i$. The Hamiltonian is

$$H \equiv \sum_i p_i\dot{q}_i - L\ , \tag{8}$$

which is a function of (q_i, p_i) but not $\dot{q}_i$. Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}\ , \tag{9}$$

i.e. a set of $2N$ first-order equations for the coordinates and momenta. For a charged particle we add the scalar $-q(\phi - \boldsymbol{A}\cdot\dot{\boldsymbol{x}})$ to the Lagrangian. The canonical momentum is then $\boldsymbol{p} = m\dot{\boldsymbol{x}} + q\boldsymbol{A}$, but the Hamiltonian is still $H = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + q\phi$. Expressed as a function of $\boldsymbol{p}$ we have

$$H = \frac{(\boldsymbol{p} - q\boldsymbol{A})^2}{2m} + q\phi \tag{10}$$

The vector potential $(0, Bx, 0)$ has $\nabla\times\boldsymbol{A} = (0, 0, B)$ as required and $\boldsymbol{E} = -\nabla\phi$ is clearly OK. The Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{(p_y - qBx)^2}{2m} + \frac{p_z^2}{2m} - qEx \tag{11}$$

The Hamiltonian doesn't depend on y, z or t, so p_y, p_z and H are constants of the motion. The equations for p_x, x and y are

$$\dot{p}_x = \frac{qB}{m}(p_y - qBx) + qE \ ; \quad \dot{x} = \frac{p_x}{m} \ ; \quad \dot{y} = \frac{p_y - qBx}{m} \tag{12}$$

Differentiating the $\dot{x}$ equation and substituting we get the required result

$$\ddot{x} + \omega_0^2 x = \frac{qE}{m} + \frac{\omega_0 p_y}{m} \tag{13}$$

where $\omega_0 = qB/m$, the Larmor frequency. This has general solution

$$x = A\sin(\omega_0 t + \delta) + \frac{p_y}{\omega_0 m} + \frac{qE}{m\omega_0^2} \tag{14}$$

where A, δ are arbitrary constants. It shows that, in this gauge, the p_y parameter represents an offset in x. We complete the solution by substituting $x(t)$ into the $\dot{y}$ equation. The p_y term cancels and we have

$$\dot{y} = -\omega_0 A\sin(\omega_0 t + \delta) - \frac{E}{B} \tag{15}$$

which has general solution

$$y = A\cos(\omega_0 t + \delta) - \frac{Et}{B} \tag{16}$$

The path is a helix (free motion in z) that drifts at a rate $-E/B$ in the y direction.

Q3. The quantity $-\int m_i c^2 \mathrm{d}t/\gamma_i = -\int m_i c^2 \mathrm{d}\tau_i$, where τ_i is the proper time of the particle, so Lorentz invariance is assured. The canonical momentum $\boldsymbol{p_i} = m_i\gamma_i\dot{\boldsymbol{x}}_i$ as expected (also a way to derive the Lagrangian) and the Hamiltonian is $\sum m_i c^2\gamma_i$.

For the ring $L = -m_0 c^2(1 - \omega^2 a^2/c^2)^{1/2}$. The generalised coordinate is the rotation angle, so the angular momentum is the canonical momentum $J = \partial L/\partial\omega$. The Hamiltonian is $H = \omega\partial L/\partial\omega - L$. These evaluate to

$$J = \frac{ma^2\omega}{\sqrt{(1 - \omega^2 a^2/c^2)}}; H = \frac{mc^2}{\sqrt{(1 - \omega^2 a^2/c^2)}} \tag{17}$$

We have already seen that the action S is Lorentz invariant. The transformation of the time interval is $\mathrm{d}t' = \gamma_v \mathrm{d}t$, where γ_v is the Lorentz factor of the frame F' relative to F. The Lagrangian therefore is $L' = L/\gamma_v$. In frame f' the time dilation means that the ring rotates more slowly, so $\omega' = \omega/\gamma_v$. The Hamiltonian is the transformed energy, so $H' = H\gamma_v$. The angular momentum is $j' = \partial L'/\partial\omega'$ so is invariant $J' = J$.

Q4. Cauchy theorem says

$$\oint_C \mathrm{d}z f(z) = 2\pi i \sum(\text{residues}) \tag{18}$$

with the counterclockwise closed contour C. This is proved by expanding $f(z)$ in a Laurent series about a singular point z_0

$$f(z) = \sum_{n=-\infty}^{\infty} f_n (z - z_0)^n \tag{19}$$

and showing that only the f_{-1} term contributes (proof will not be required). The example has poles at $z = \pm i$. We convert to a closed contour by completion in (say) the upper half-plane. The residue at i is $1/2i$, hence result. Closing the contour in the lower half-plane is also possible, the residue is $-1/2i$ and the sign in Cauchy's theorem must be reversed (clockwise contour).
(a) Integrand has poles at $e^{\pm\pi i/4}, e^{\pm 3\pi i/4}$ and we can close over the upper half-plane (either way is fine). The residue at $x = e^{\pi i/4}$ is (draw a diagram!)

$$\frac{1}{(e^{\pi i/4} - e^{-\pi i/4})(e^{\pi i/4} - e^{3\pi i/4})(e^{\pi i/4} - e^{-3\pi i/4})} = \frac{1}{\sqrt{2}\sqrt{2}i\sqrt{2}(1+i)} = \frac{-1-i}{4\sqrt{2}} \tag{20}$$

Similarly the residue at $x = e^{-\pi i/4}$ is $(1-i)/4\sqrt{2}$. Using Cauchy's theorem we have the result $2\pi i \sum(\text{residues}) = \pi/\sqrt{2}$.
(b)

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\cos ax}{x^2 + b^2} = \Re\left(\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^{iax}}{x^2 + b^2}\right) \tag{21}$$

For $a > 0$ close over the upper half-plane. Residue of the pole ar $x = ib$ is $e^{-ab}/(2ib)$, so integral evaluates to $\pi e^{-ab}/b$.

Q5. Preferred version of (t, ω) Fourier transform and its inverse is

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \; f(t) e^{i\omega t} \; ; \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \; \tilde{f}(\omega) e^{-i\omega t} \tag{22}$$

For the (x,k) pair I prefer the opposite sign – the reason being the it is the convention in QM that $e^{i(kx-\omega t)}$ represents a positive energy wave travelling in the $+x$-direction (remember $i\hbar\dot{\psi} = E\psi$). The Green function can be calculated as

$$G(t;0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \; \frac{e^{-i\omega t}}{-\omega^2 - i\omega\gamma + \omega_0^2} \tag{23}$$

There are poles at $\omega = i\gamma/2 \pm i\Omega$, where $\Omega \equiv \sqrt{\omega_0^2 - \gamma^2/4}$. Complete using lower half-plane for $t < 0$) and upper half-plane for $t > 0$, generating a Heaviside function step $\theta(t)$ as required for causality. The residues are $\pm e^{-\frac{1}{2}\gamma t \pm i\Omega}/2\Omega$ so, by Cauchy's theorem we have (generalising to $G(t;t')$

$$G(t;t') = \theta(t-t')\frac{e^{-\frac{1}{2}\gamma(t-t')}\sin\Omega(t-t')}{\Omega} \tag{24}$$

We use the Green function to solve for the response to source $f(t)$ by calculating

$$y(t) = \int \mathrm{d}t' \; G(t;t')f(t') \tag{25}$$

For the present case we have $f(t') = f_0$ for $0 < t' < \tau$. We have to be careful about the step functions; for $t < \tau$ we need $\int_0^t \mathrm{d}t'$, for $t > \tau$ we use $\int_0^\tau \mathrm{d}t'$. Changing variable to $s \equiv t - t'$ we get for $t < \tau$ and $t > \tau$ respectively

$$f(t) = \int_0^t \mathrm{d}s \; \frac{e^{-\frac{1}{2}\gamma s}\sin\Omega s}{\Omega}; \quad f(t) = \int_{t-\tau}^t \mathrm{d}s \; \frac{e^{-\frac{1}{2}\gamma s}\sin\Omega s}{\Omega} \tag{26}$$

The final expressions for $f(t)$ are for $t < \tau$

$$y(t) = \frac{1}{2\omega_0^2}\left(2\Omega - e^{-\frac{1}{2}\gamma t}\left(2\Omega\cos\Omega t + \gamma\sin\Omega t\right)\right) \tag{27}$$

and for $t > \tau$

$$y(t) = \left[\frac{1}{2\omega_0^2}\left(-e^{-\frac{1}{2}\gamma t'}\left(2\Omega\cos\Omega t' + \gamma\sin\Omega t'\right)\right)\right]_{t-\tau}^{t} \tag{28}$$

Q6.

- Need to describe, for a discrete ID process with length scale a and timescale τ the idea that the transitions rates into $P_{N+1}(m)$ are given by $w(m,m')P_N(m')$
- Principle of detailed balance is $w(m,m')P(m') = w(m',m)P(m)$ for each pair m, m'
- The idea of the derivation presented in the notes was to consider the case when transitions are made only from m to $m \pm 1$, so that

$$\begin{aligned} P_{N+1}(m) &= w(m,m+1)P_N(m+1) - w(m+1,m)P_N(m) \\ &\quad + w(m,m-1)P_N(m-1) - w(m-1,m)P_N(m) \end{aligned} \tag{29}$$

- If the diffusion is symmetric $w = 1/2$, and we get the diffusion equation with coefficient $D = a^2/\tau$
- If there is a vertical asymmetry due to gravity, then transitions to $k-1$ are preferred over those to $k+1$, giving the first-derivative term in

$$\frac{\partial P}{\partial t} = \frac{1}{2} D \left(\frac{\partial^2 P}{\partial z^2} + \frac{\tilde{m} g}{k_{\mathrm{B}} T} \frac{\partial P}{\partial z} \right) \tag{30}$$

- The argument leading to the coefficient on this term will probably be circular (appeal to Boltzmann factors…), but never mind.

The steady-state solution of this equation is

$$P(z) \propto \exp(-\tilde{m} g z / kT) \tag{31}$$

The critical size of particle is that for which $\tilde{m} g a / kT \sim 1$. Evaluating this for the given parameters we find $a \sim 10^{-6}$ m.

(15 January 2003)