Solitons in Ferromagnets and Quantum Hall Ferromagnets

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Overview

- Ferromagnets: Statics and Dynamics
- Topological Defects and Topological Invariants
- Topological Solitons in 2D and 3D
- Quantum Hall Ferromagnets
- Dynamical Solitary Waves in 2D
- Dynamical Solitary Waves in 3D
- Summary

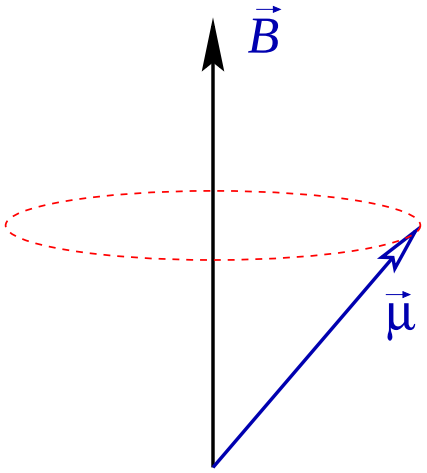
Continuum Ferromagnets

Local spin-orientation $\vec{n}(r)$ $\qquad (|\vec{n}|^2 = 1)$

$$E = \tfrac{1}{2}\rho_s \int d^d r \, (\nabla_\alpha n_i)^2 + \ldots \text{fields, anisotropies} \ldots$$

spin stiffness

Dynamics



$$\dot{\vec{\mu}}/\gamma = \vec{\mu} \times \vec{B}$$

The Landau-Lifshitz equation

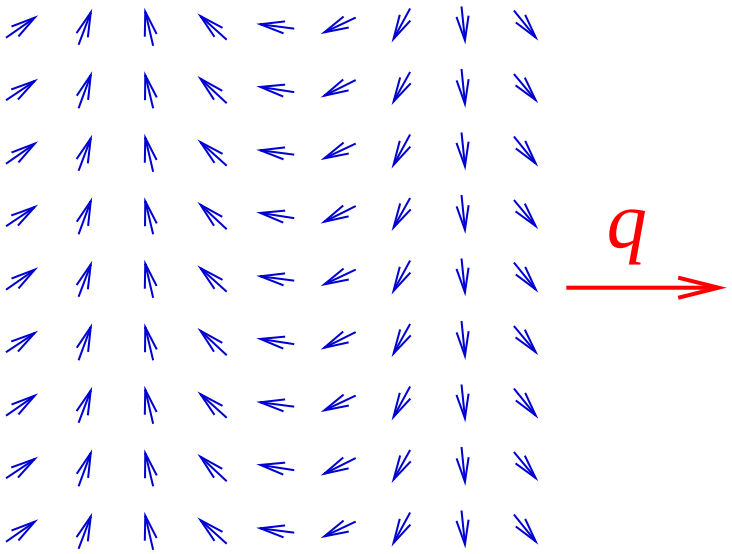
$$\rho J \dot{\vec{n}} = \vec{n} \times \frac{\delta E}{\delta \vec{n}} \qquad \text{[L.\&L., 1935]}$$

$$\rho J \dot{\vec{n}} = -\rho_s \vec{n} \times \nabla^2 \vec{n} \ \ldots$$

Some Known Properties

Linearized Theory ($n_z \simeq 1$)

Spin-waves:

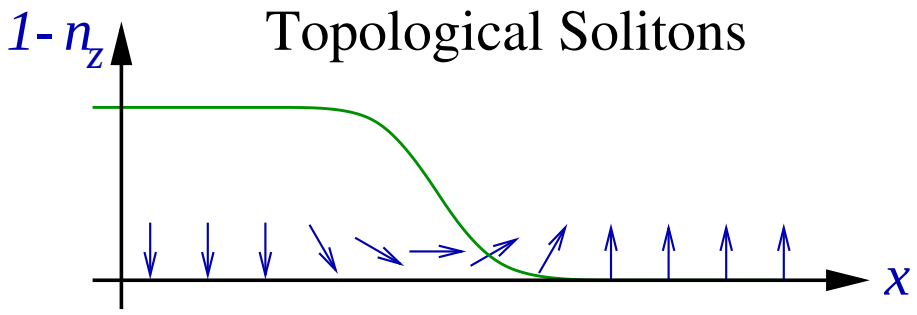


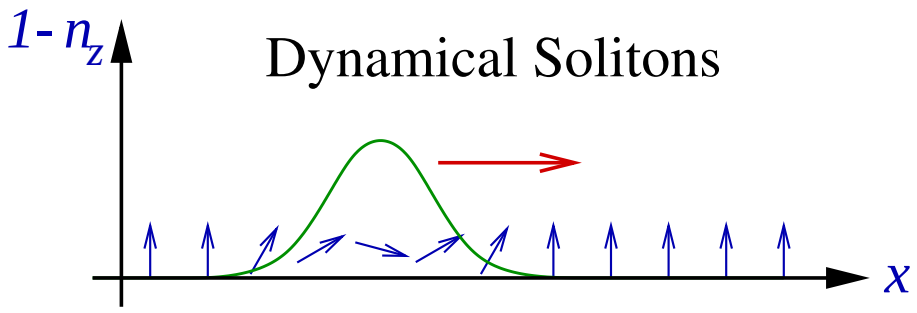
$\omega = (\rho_s/\rho J)|\vec{q}|^2$

Non-Linear Aspects

[Kosevich et al., Physics Reports **194**, 117 (1990)]

1D:



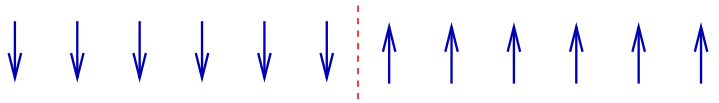


2D, 3D: Topological Solitons
& some Dynamical Solitons.

Aside: Topological Defects

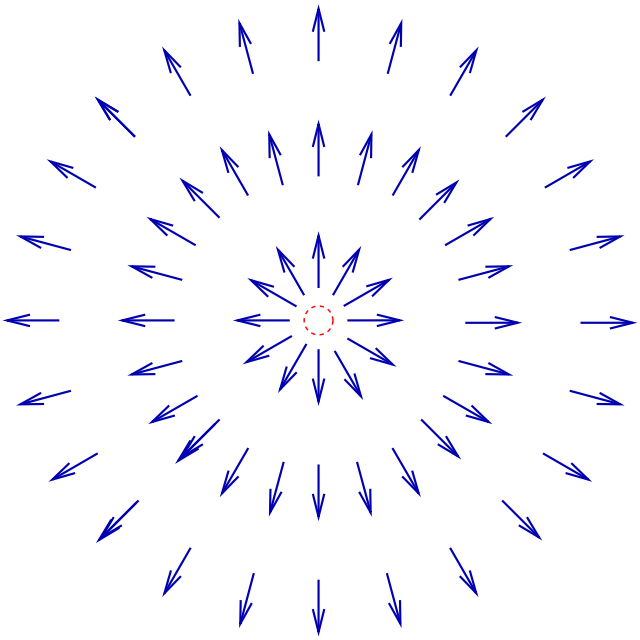
1D, Easy Axis

Domain Wall



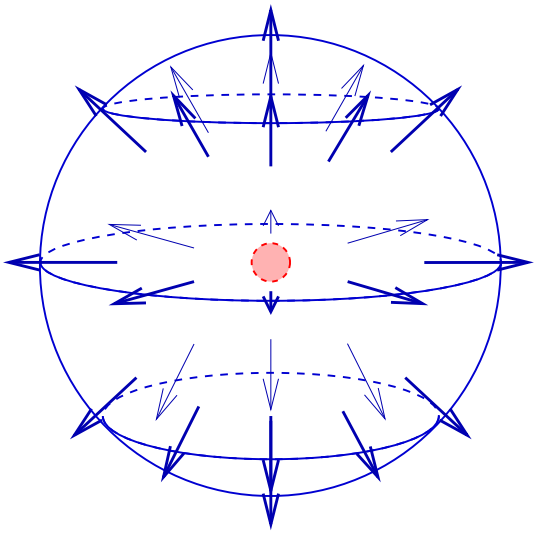
2D, Easy Plane

Vortex



3D, Isotropic

‘‘Hedgehog’’



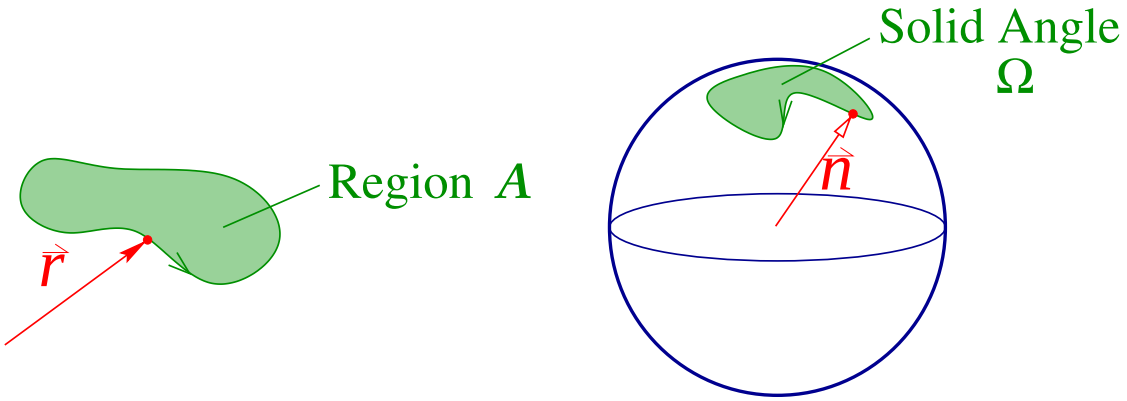
Topological Invariants

$\vec{n}(r)$ describes a mapping $S^d \mapsto S^2$

1D $\quad \pi_1(S^2) = 0.$

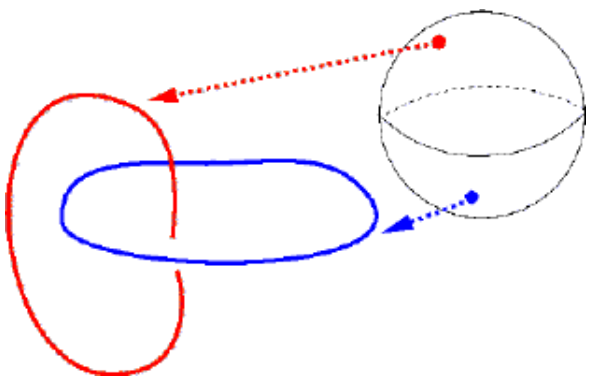
2D $\quad \pi_2(S^2) = \mathbb{Z} \Longrightarrow Q = \int d^2r\, q(r)$

"Topological density" $q(r) \equiv \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{\alpha\beta}n_i\nabla_\alpha n_j \nabla_\beta n_k$



$$\int_A q(r) d^2r = \Omega/4\pi$$

3D $\quad \pi_3(S^2) = \mathbb{Z} \Longrightarrow$ Hopf invariant, $\mathcal{H}$.



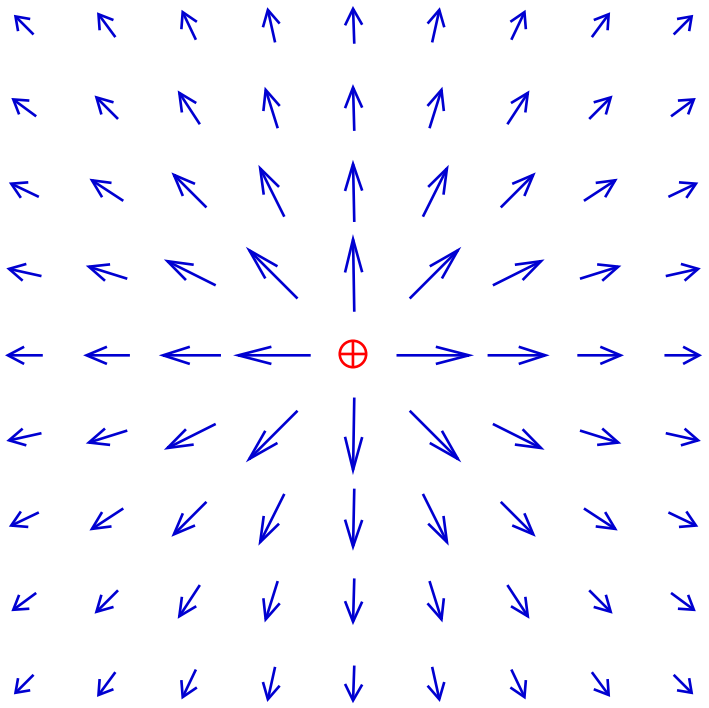
Topological Solitons in 2D

Minimize $E = \frac{1}{2}\rho_s \int d^2r \, (\nabla_\alpha n_i)^2$ at fixed Q.

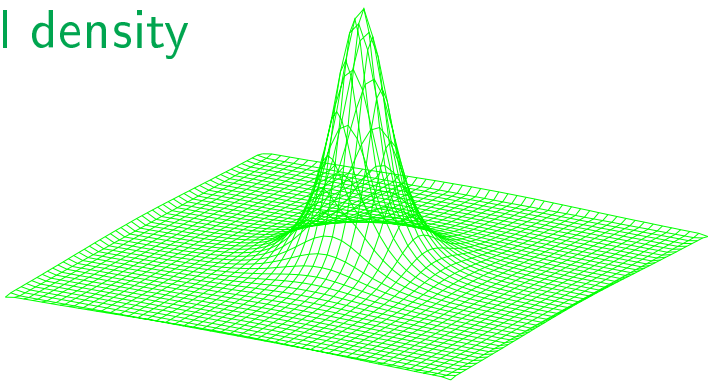
$$E_Q^* = 4\pi\rho_s|Q| \qquad \text{[Belavin \& Polyakov, 1975]}$$

$Q = 0 \rightarrow$ Uniform Ferromagnet

$Q = 1 \rightarrow$ "Skyrmion"



Topological density

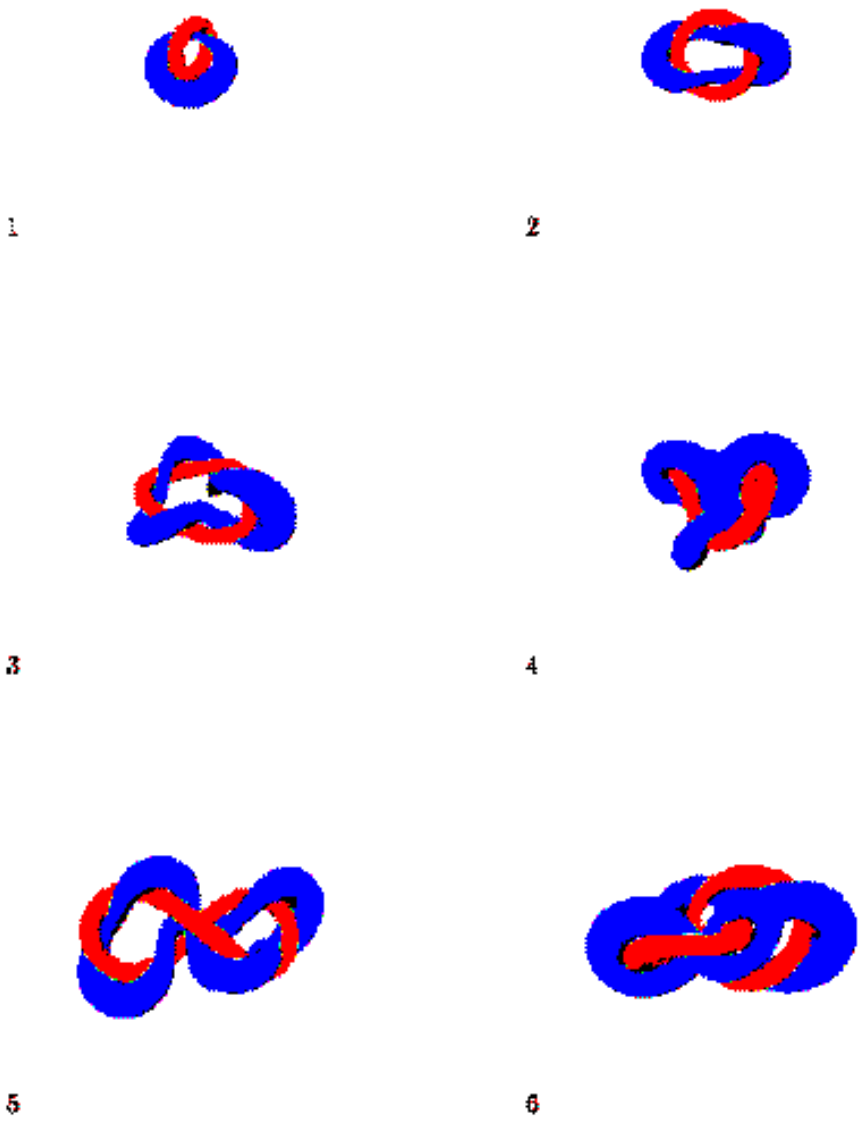


Topological Solitons in 3D

[Fadeev,1975/1997; Gladikowski & Hellmund,1997; Battye & Sutcliffe,1998]

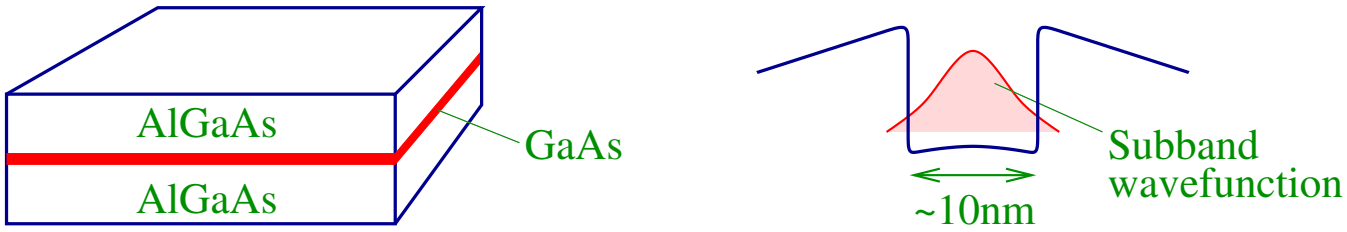
Minimize
$E = \frac{1}{2}\rho_s \int d^3r\ (\nabla_\alpha n_i)^2 - \frac{1}{2}(\epsilon_{ijk}\nabla_\alpha n_j \nabla_\beta n_k)^2$
at fixed $\mathcal{H}$.

$\rightarrow$ Linked & Knotted Solitons [Battye and Sutcliffe, 1998]



What is a Quantum Hall Ferromagnet?

Two-dimensional electron gas



$+$ strong perpendicular magnetic field B

Ferromagnetic states appear ($\nu = 1, 3, 5 \ldots; 1/3$)

- Described by a local spin-orientation $\vec{n}(r)$
- $E = \frac{1}{2}\rho_s \int d^2r \, (\nabla_\alpha n_i)^2 + g\mu_B BN$

spin-stiffness Zeeman energy

BUT!

Topological density $\Longleftrightarrow$ (physical) charge density

Skyrmions are *charged* excitations.

Linear Momentum $\Longleftrightarrow$ Electrical dipole moment

$$d_\alpha = 4\pi e \epsilon_{\alpha\beta} P_\beta$$

Dynamical Solitary Waves in 2D

$$\rho J \dot{\vec{n}} = \vec{n} \times \frac{\delta E}{\delta \vec{n}}$$

Conservation Laws [Papanicolaou & Tomaras, 1991]

$$Q \equiv \int d^2r \; q(r)$$

$$N \equiv \rho J/\hbar \int d^2r \; (1 - n_z)$$

$$P_\alpha \equiv 4\pi \rho J \epsilon_{\alpha\beta} \int d^2r \; r_\beta \; q(r)$$

Minimize Energy $\rightarrow$ $\boxed{E_Q^*(P_\alpha, N)}$

$$\mathcal{E} \equiv E - v_\alpha P_\alpha + \hbar\omega N$$

Lagrange multipliers

$$\dot{\vec{n}} = -v_\alpha \nabla_\alpha \vec{n} - \omega \vec{n} \times \vec{z}$$

$\Longrightarrow$ Travelling, precessing waves

$$v_\alpha = \left. \frac{\partial E_Q^*}{\partial P_\alpha} \right|_N \; ; \quad \omega = -\frac{1}{\hbar} \left. \frac{\partial E_Q^*}{\partial N} \right|_{P_\alpha} .$$

Some General Properties

$E = \frac{1}{2}\rho_s \int d^2r\ (\nabla_\alpha n_i)^2$
$N = \rho J/\hbar \int d^2r\ (1 - n_z)$
$P_\alpha = 4\pi\rho J \epsilon_{\alpha\beta} \int d^2r\ r_\beta\ q(r)$

- Dimensional analysis (+rotational invariance)

$$E^*_Q(P_\alpha, N) = \rho_s f_Q(|\vec{P}|/\sqrt{N\hbar\rho J})$$
$$= E^*_Q(p) \qquad (\rho_s = \rho J = \hbar = 1)$$

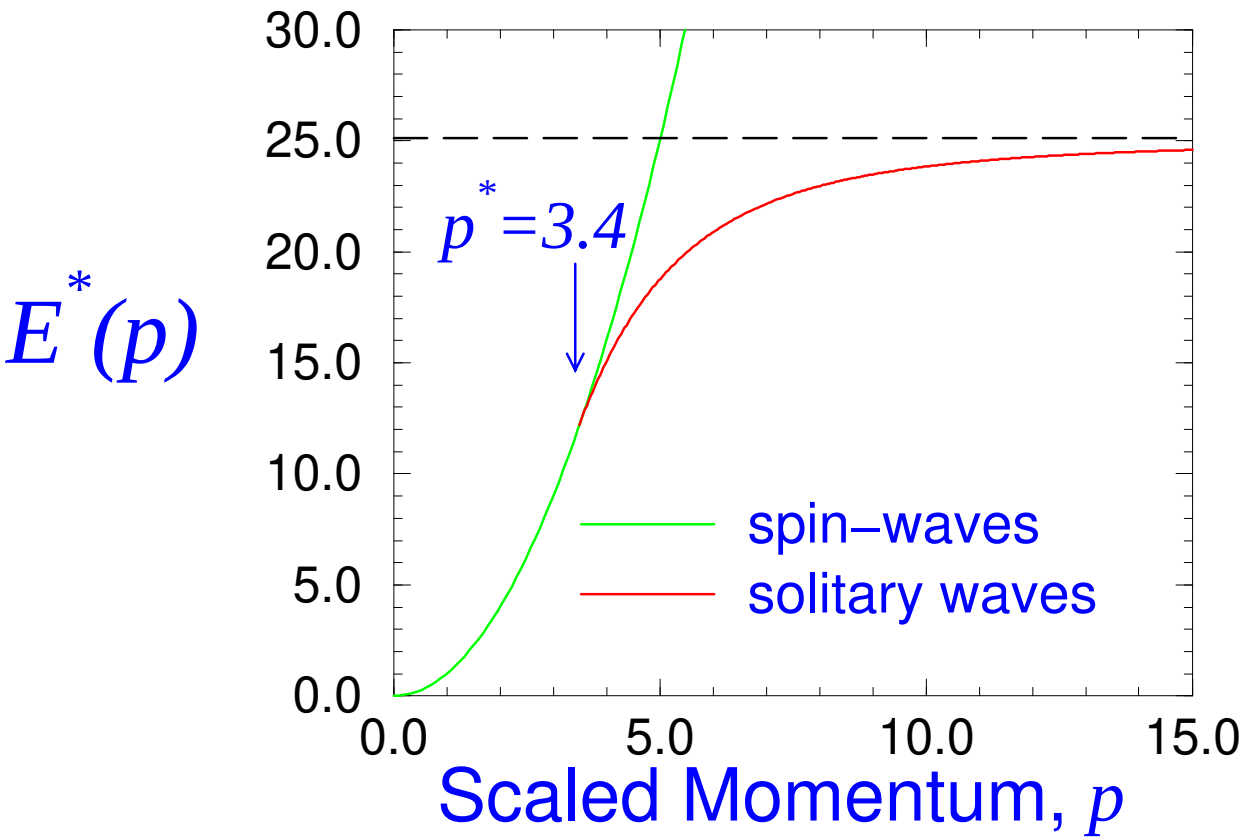
"Scaled momentum": $\boxed{p = |\vec{P}|/\sqrt{N}}$

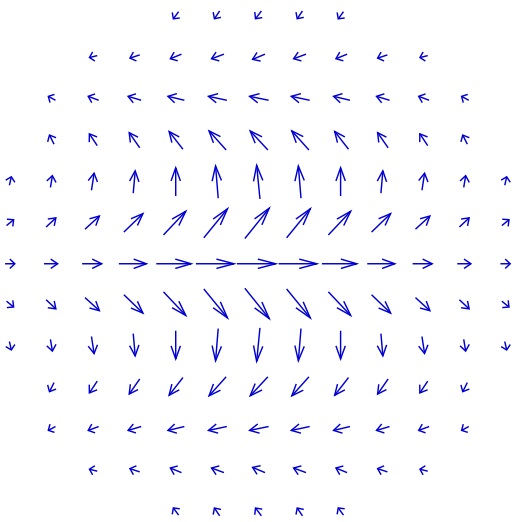
- Translational invariance

$\vec{n}(r_\alpha) \to \vec{n}(r_\alpha + R_\alpha) \Longrightarrow P_\alpha \to P_\alpha - 4\pi Q \epsilon_{\alpha\beta} R_\beta$

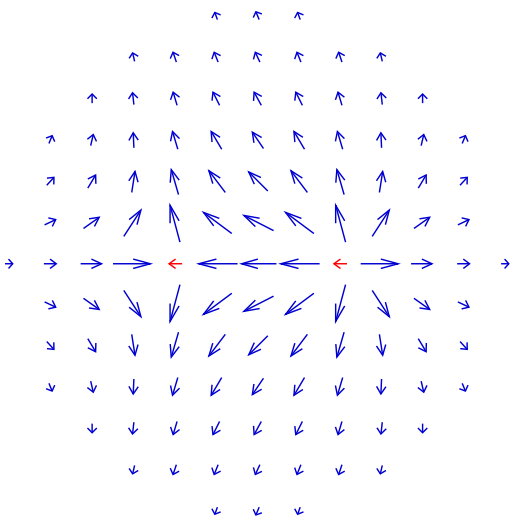
$\Longrightarrow \quad \boxed{E^*_{Q\neq 0}(p) = E^*_{Q\neq 0}(0)}$

Numerical Results





spin-wave droplet

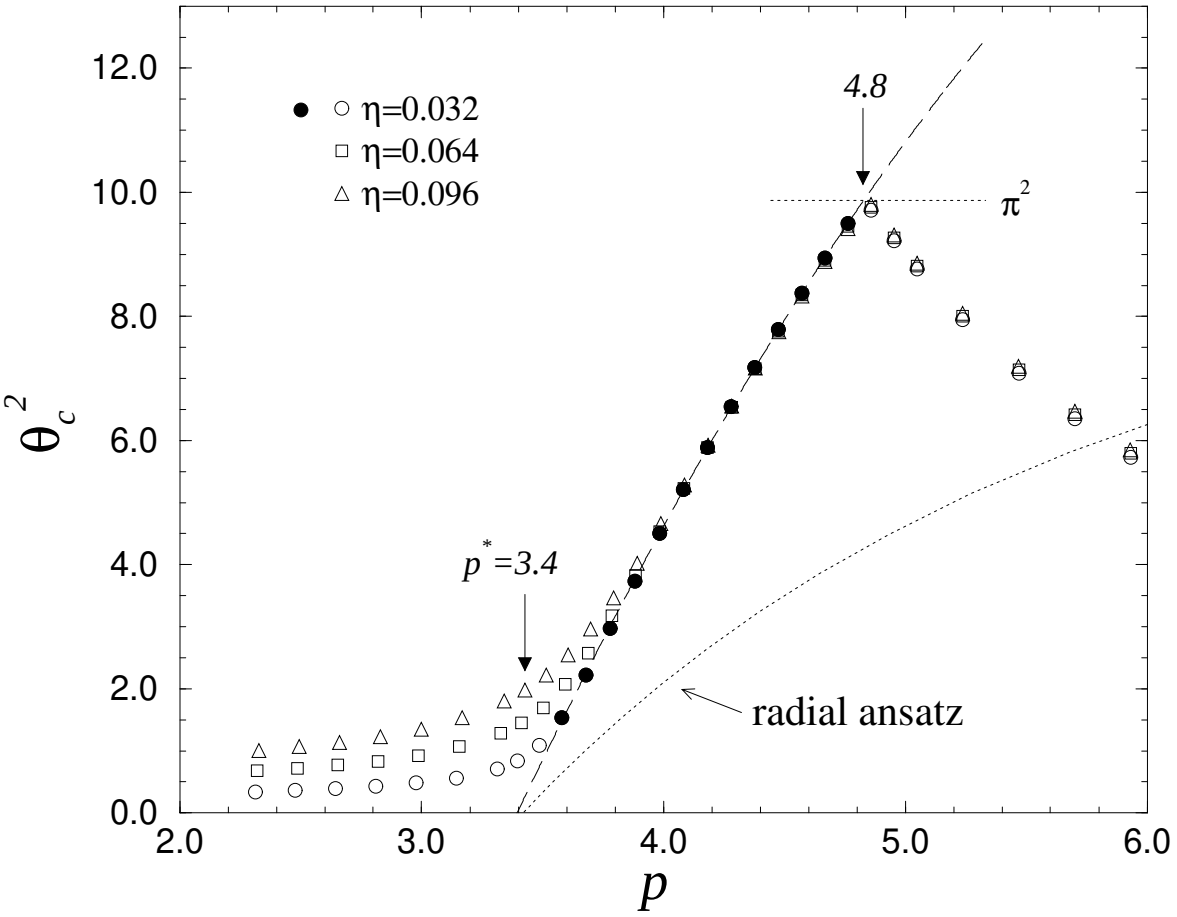


Skyrmion/antiSkyrmion pair

[cf. Papanicolaou & Zakrzewski, 1995; Piette & Zakrzewski, 1998]

[see also Papanicolaou & Spathis, 1999]

The Spin-Wave / Solitary Wave Transition



A transition occurs at $\boxed{p^* = 3.4}$.

"Smoke-Rings" in Ferromagnets

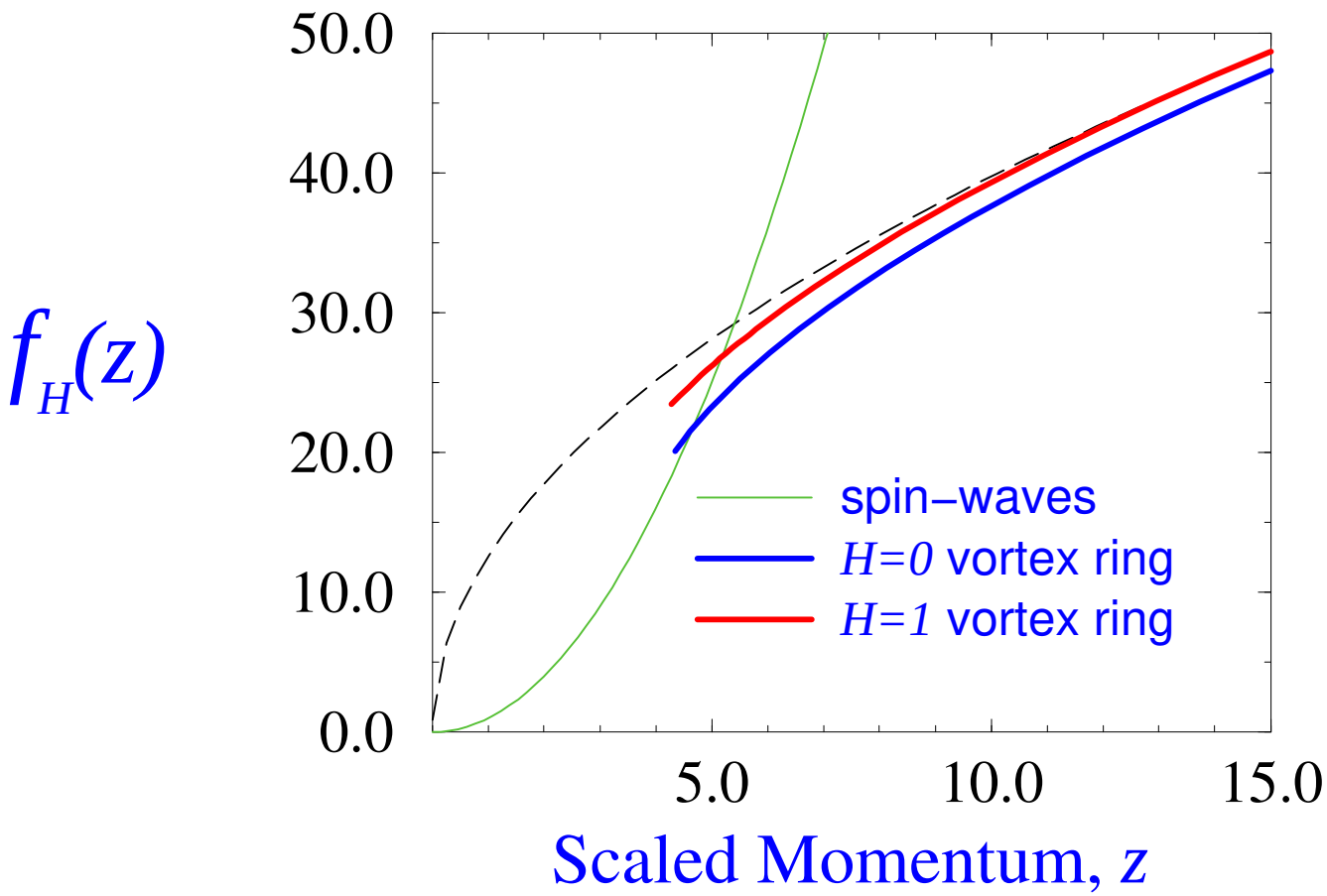
[N.R.C., unpublished]

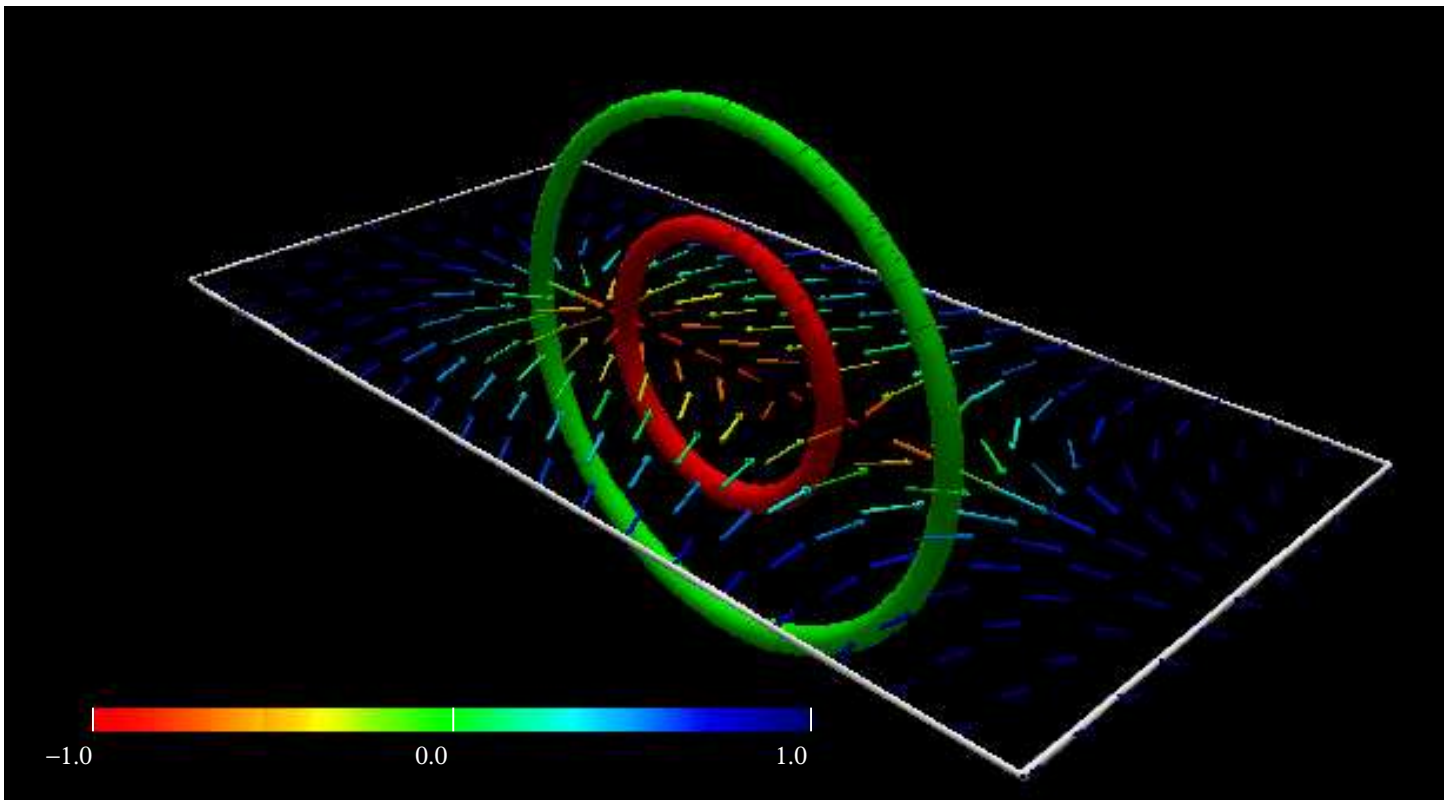
"Magnetic vorticity": $\Omega_\alpha \equiv \frac{1}{8\pi}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}n_i\nabla_\beta n_j\nabla_\gamma n_k$

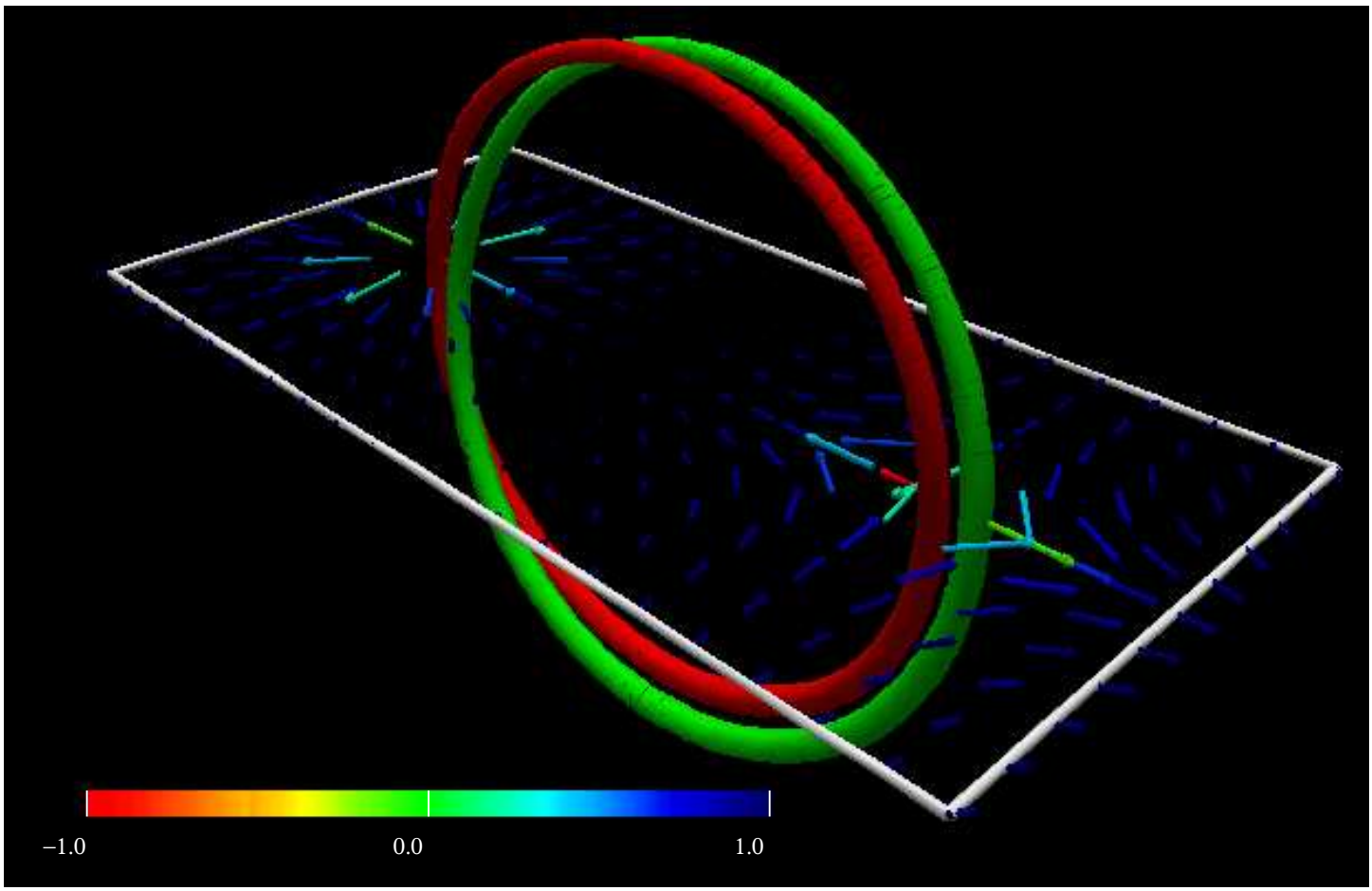
$N \equiv \rho J/\hbar \int d^3\vec{r}\,(1-n_z)$

$\vec{P} \equiv 2\pi\rho J \int d^3\vec{r}\,(\vec{r}\times\vec{\Omega})$

$E^*_{\mathcal{H}}(\vec{P},N) = N^{1/3} f_{\mathcal{H}}(z); \quad z = |\vec{P}|/N^{2/3}$




−1.0
0.0
1.0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Summary

• Propagating solitary waves exist for ferromagnets in $d > 1$.

• There are strong similarities between these excitations and vortex structures in fluid dynamics.

• Dynamical conservation laws stabilise topological solitons with non-zero Hopf charge (in 3D).

(• Solitary waves of this form can contribute to the breakdown of the spin-polarized QHE.)