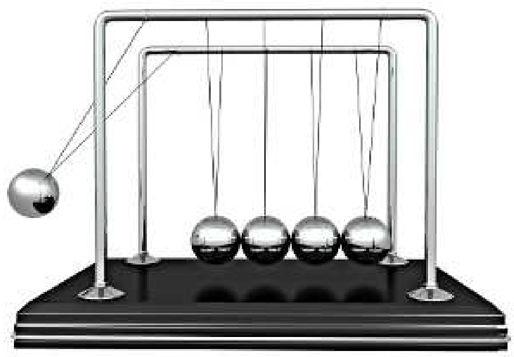
UNIVERSITY OF CAMBRIDGE SUMMER PROGRAMME 2016

AT THE CANADIAN INTERNATIONAL SCHOOL OF HONG KONG

ENERGY AND MOMENTUM



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1 About this course

This is a course on classical Newtonian dynamics. It differs slightly from most courses on this topic by being focussed on energy and momentum rather than forces and acceleration. The motivation for this focus is twofold. Firstly, conservation of energy and momentum allow us to tackle some otherwise very difficult looking problems that would require advanced calculus to solve using forces. Secondly, as one goes further in physics, one finds that quantum mechanics, relativity, statistical physics, and many other branches of physics, are better described in terms of energy and momentum than in terms of forces.

This course will be accompanied by material from Cambridge University's new school physics website isaacphysics.org.

2 Momentum

2.1 What is momentum

We start with a definition. The momentum of an object, $\mathbf{p}$, is the product of its mass and its velocity:

$$\mathbf{p} = m\mathbf{v}. \tag{1}$$

Pinning down exactly what momentum "is", is a bit tricky, but a good start is that the amount of momentum an object has describes how difficult it would be to stop it moving. Imagine I send two balls down a bowling alley, one a proper bowling ball and one a ping-pong ball . Even if I send them at identical speeds, the pins will easily stop the ping-pong ball, without falling over, whereas the bowling ball will scatter them, hopefully leading to strike. Why? Because the bowling ball weighs several kilograms while the ping-pong ball only weighs a few grams, so, although I launched them at the same speed, the bowling ball had much more momentum and thus was much harder for the pins to stop. It is also

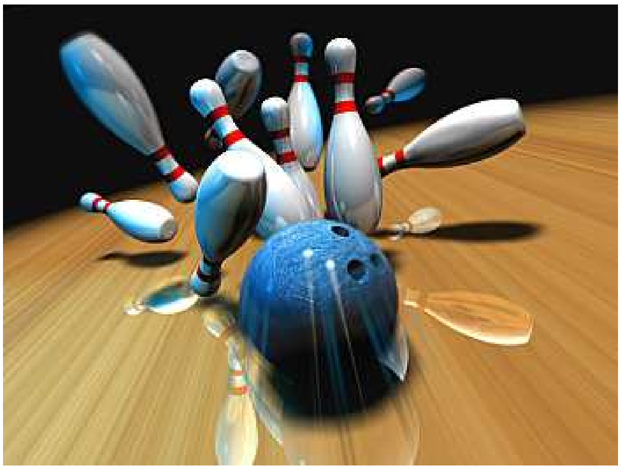

Figure 1: Ten pin bowling with a real bowling ball.

intuitive that the faster something is going, the harder it is to stop. Would you rather be hit by a car moving at 1mph or 70mph? In both cases the cars have the same mass, but the faster car has more momentum, so it will knock you over much more effectively.

Exercise 1. *What are the units of momentum?*

Exercise 2. *Estimate the momentum of a speeding bullet, a car on the highway, a bowling ball and a flying fly. Which would be hardest to stop?*

2.2 What is a force

A force is simply a push or a pull — an action that tends to cause an object to start or stop moving. There are many different types of forces, for example spring forces, magnetic forces, frictional forces and gravitational forces, but they all try to bring objects into motion. It is important to understand that force is a vector, with both a magnitude and a direction; if I push you north you will start to move northward, and if I push you south you will start to move southward. This provides a simple way of comparing the size of different forces: we simply apply both forces to an object in opposite directions and see which direction it actually starts moving in. If the two forces have exactly the same magnitude but opposite directions, they will cancel out completely, and the object will not start moving.

For the rest of the course, we will repeatedly use two important examples of forces, those caused by springs and those caused by gravity. A spring is characterized by a "spring constant" k, which describes how strong the spring is. To stretch a spring by a distance x you must apply a force

$$F = kx, \tag{2}$$

to its end. Secondly, gravity is characterized by a "gravitational field strength", g. If an object with mass m sits in a gravitational field g it will experience a gravitational force

$$F = mg. \tag{3}$$

On Earth $g \approx 9.81\text{ms}^{-2}$, and the resulting force points towards the ground.

Exercise 3. *A mass m hangs in force balance on the end of a spring with spring constant k. How much has the spring extended? What is pulling the mass up and what is pulling it down?*

Exercise 4. *What are the units of force?*

2.3 Newton's first and second laws

Newton's first law of motion is rather dull. It simply says that if an object is in motion, and nothing applies a force to it, it's velocity does not change. However, when Newton (or rather Galileo) came up with this, it was a complete revelation. Previously, everyone thought that things only move while you apply a force to them, so the more force you apply the faster they go, and if you stop applying the force they stop moving. This is a pretty good description of high friction environments. However, if you think about low friction environments, you can easily find examples of objects that continue moving long after all force has been removed. For example, if I get some speed up on by bicycle, it will then roll a considerable distance even after I stop pedaling. If I throw a ball up into the air, it will continue rising even though I stopped pushing it upwards as soon as I let go of it.

Exercise 5. *Think of some more examples of objects that keep moving even when no force is applied.*

Exercise 6. *I push a box in a muddy field (a very high friction environment) and it moves. I then stop pushing and it stops moving. Why does this look like a violation of Newton's first law? Why isn't it?*

On Earth, in the end friction always gets the better of you and stops moving objects, but in space there is almost no friction so you can observe Newton's first law in all its glory. Voyager 1 is a space-probe launched by NASA in 1977, that is currently beyond the solar-system traveling through interstellar space at 17 kilometers per second, despite the fact that absolutely nothing is pushing it along. Unless we decide to recapture it, it will probably continue to move for billions of years.

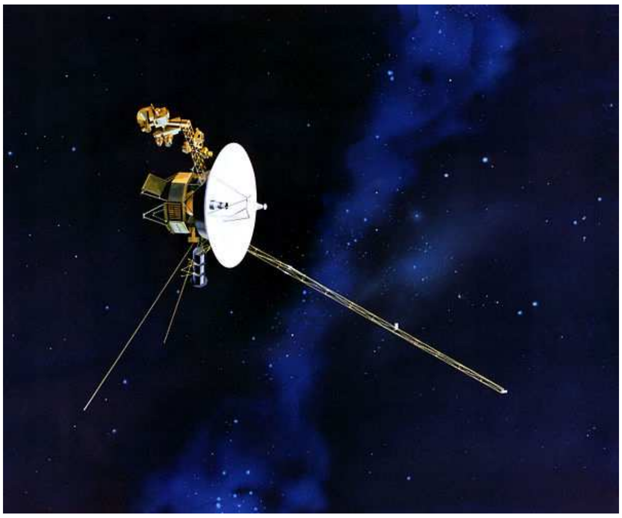

Figure 2: Voyager 1: An interstellar example of Newton's first law.

We can also state Newton's first law in terms of momentum: if no force acts on an object its momentum does not change. This leads us nicely to Newton's second law, which states that the total force on an object is equal to the rate at which its momentum changes.

2.3.1 Aside: Rates of change

An object's speed tells you how far something moves in a given time. If it is at position s_1 at time t_1 and position s_2 at time t_2, you work it's speed as

$$v = \frac{s_2 - s_1}{t_2 - t_1} \equiv \frac{\Delta s}{\Delta t}, \tag{4}$$

where $\Delta s = s_2 - s_1$ is how far the object has moved, and $\Delta t = t_2 - t_1$ is how long it took. We say that speed is the rate-of-change of position as it tells us how fast the position of an object is changing. In SI units, speed is measured in ms^{-1}, and tells you how many meters the object's position changes by every second.

One can apply exactly the same idea for an object's temperature. Suppose an object is heating up. We could define the rate of increase of its temperature as

$$r = \frac{\Delta T}{\Delta t}, \tag{5}$$

which would be measured in $^\circ\mathrm{Cs}^{-1}$, and tell us how many degrees the temperature rises by every second.

We can similarly define the rate of change for any object. We often use "dot" notation, where a letter with a dot over it means the rate of change of this object, so we would write r as $\dot{T}$, and v as $\dot{s}$.

This way of calculating rates of change only works if the rate itself is not changing. To deal with changing rates, one needs to use calculus. Essentially one can always define a rate of change by examining two points very close together in time (i.e. two points with a very small Δt) which, if you think about it, is exactly how the speedometer in your car works. However, for this course we will restrict ourselves to constant rate changes, where we do not need such sophisticated machinery. To learn calculus, take Prof Korner's class "what is calculus?"

2.3.2 Newton's Second Law

As stated above, Newton's second law says that the total force on an object is equal to the rate at which its momentum changes

$$\mathbf{F} = \dot{\mathbf{p}}. \tag{6}$$

We note that this law actually encompasses the first law. If F is zero, then the momentum is not changing, so the object simply moves along at constant momentum and hence constant velocity. However, the second law also tells us how the motion of an object will change if we apply a force.

Exercise 7. *Escape velocity from the Earth is 11 176 ms^{-1}. If a* 2kg *rocket is initially at rest. If its thrusters generate 500N of force, how long must they be run for, for the rocket to reach escape velocity?*

Professional physicists almost always write Newton's second law in terms of momentum, as I have above. However, if the mass of the moving object isn't changing (which usually it isn't) then we can rewrite the rate-of-change of momentum as

$$\dot{p} = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{mv_2 - mv_1}{t_2 - t_1} = m\frac{v_2 - v_1}{t_2 - t_1} = m\frac{\Delta v}{\Delta t} = m\dot{v} \equiv ma, \tag{7}$$

where $a \equiv \dot{v}$ is the rate of change of the objects velocity, otherwise known as its acceleration. Thus we can write

$$\mathbf{F} = m\mathbf{a}, \tag{8}$$

which is arguably the more famous, though less useful, version of Newton's second law. We note it encodes the same physics: if there is no force, the object does not accelerate, so its velocity does not change. If a force is applied, the object accelerates, meaning the velocity of the object changes. However, the momentum version of the law is better since it can also handle situations where the mass of an object is changing.

Exercise 8. *A ball is dropped from a great hight and falls under gravity towards the ground. As it falls, it experiences a drag force from the air of the form $F \propto v^2$. Sketch graphs of the position, velocity and acceleration of the ball as it falls. When it hits the ground, it bounces elastically, meaning its velocity is simply reversed. What is the ball's acceleration just after it bounces?*

2.4 Newton's third law

Newton's third law is normally paraphrased as "to every action there is an equal and opposite reaction". What this rather unhelpful phrase is trying to say, is that whenever object a exerts a force on object b, object b exerts a force of equal size but opposite direction on a. This applies to all types of forces, and whether or not anything is moving.

Exercise 9. *A book rests on the floor. Identify all the forces acting on the book, and their equal but opposite reactions.*

2.5 Conservation of Momentum

At first glance Newton's third law doesn't seem to have much to do with momentum, but actually it tells us something extremely useful. Suppose we have two objects, a and b, with masses m_a and m_b, velocities $\mathbf{v}_a$ and $\mathbf{v}_b$ and momenta $\mathbf{p}_a$ and $\mathbf{p}_b$. The total momentum of our system (that is of both particles) is simply

$$\mathbf{P} = \mathbf{p}_a + \mathbf{p}_b. \tag{9}$$

Now, suppose there are no external forces acting on the objects, but that they are exerting forces on each other. They could be galaxies pulling on each other gravitationally, or and iron filing and a magnet, or any other pair of interacting objects. The force on object b caused by object a is $\mathbf{F}_{ba}$, and the force on object a caused by object b is $\mathbf{F}_{ab}$. Newton's third law tells us these are equal and opposite

$$\mathbf{F}_{ab} = -\mathbf{F}_{ba}. \tag{10}$$

However, from Newton's second law, we have $\mathbf{F}_{ab} = \dot{\mathbf{p}}_a$ and $\mathbf{F}_{ba} = \dot{\mathbf{p}}_b$, so we now have

$$\dot{\mathbf{p}}_a = -\dot{\mathbf{p}}_b. \tag{11}$$

What this means is that $\mathbf{p}_a$ goes up at exactly the same rate as $\mathbf{p}_b$ goes down (or vice-versa), so their sum is not changing:

$$\dot{\mathbf{P}} = \dot{\mathbf{p}}_a + \dot{\mathbf{p}}_b = \dot{\mathbf{p}}_a - \dot{\mathbf{p}}_a = 0. \tag{12}$$

We say that the total momentum is "conserved". What this means is that when two objects interact, they can swop momentum from one to the other, but the total amount of momentum cannot change.

Exercise 10. *Generalize the above argument for two particles to a system of 3 particles, and show that conservation of momentum still holds.*

Exercise 11. *Tricky! Generalize further to a system of N particles*

One example of a system containing many objects but with no external forces acting on it is the Universe. Thus we have just learnt that the total momentum of the Universe is conserved.

A recurring theme of this course is going to be that we can analyze many physical situations rather easily using conservation of momentum, when a more direct analysis involving forces would be very challenging.

3 Completely inelastic collisions

A completely inelastic collision is a process in which two objects collide and stick together rather than bouncing apart. They provide good practice for conservation of momentum.

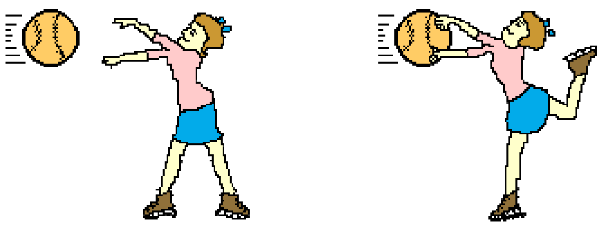

Figure 3: Diagram of an ice-skater catching a ball.

Exercise 12. *An ice-skater weighs* 50kg. *Someone throws a 1kg ball at her at* 10ms^{-1}, *and she catches it. Use conservation of momentum to work out how fast she is moving after she catches the ball.*

Exercise 13. *The same ice-skater eventually comes back to rest. She then herself throws the ball away from herself at* 10ms^{-1}. *How fast is she moving after she throws the ball? (Incidentally, this is how rockets work)*

Exercise 14. *Two ice-skaters are at rest. One has mass M and the other mass m. They push off each other, and move apart. If the one with mass M moves off with velocity* $\mathbf{V}$, *what is the velocity,* $\mathbf{v}$, *of the other? Is it possible to achieve* $\mathbf{V} = \mathbf{v}$*?*

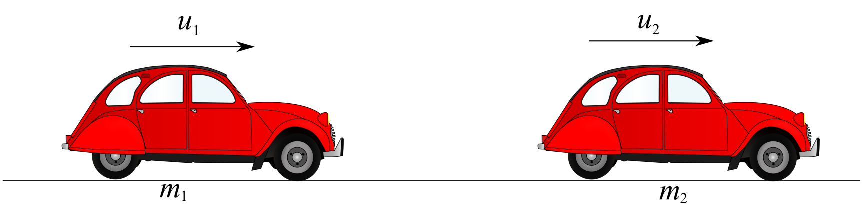

Figure 4: Two cars collide. What happens next?

Exercise 15. *Two cars drive along the road, one with mass m_1 and velocity u_1, the other with mass m_2 and velocity u_2. They then crash, and crumple into a single moving object. How fast is this object moving? Under what condition is the resulting tangled mess stationary? What happens if $m_1 \gg m_2$? Is it safer to drive a heavy car or a light car?*

Exercise 16. *An alien and an astronaut meet in a space-station, where there is no gravity. Both have mass M. The alien fires a bullet of mass m and velocity v at the man. After the incident, what is the motion of the alien and the man?*

Exercise 17. *Only if you have studied vectors. Re-analyze exercise 15 for two cars that collide at a junction that makes an angle of* $90°$, *as shown above. What direction do the cars move in after collision?*

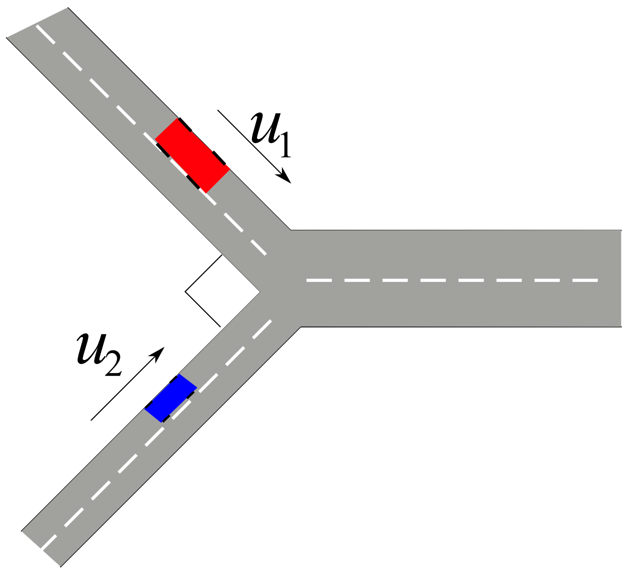

Figure 5: Two cars collide at an intersection. What happens next?

4 Newton's law of impact

If you fire a bullet (or any other high-velocity projectile) into a block of stuff, it pushes its way in for a certain depth but then is brought to rest. Newton applied conservation of momentum to estimate how far in the projectile would get. To be concrete, let us imagine we are firing a bullet into the ground. Newton's idea was simple: as the bullet penetrates the ground, it has to push ground out of the way, leaving a cylindrical shaped hole, as shown below. Newton guessed that all the material moved out of the way of the bullet gained a downwards velocity v equal to the initial velocity of the bullet.

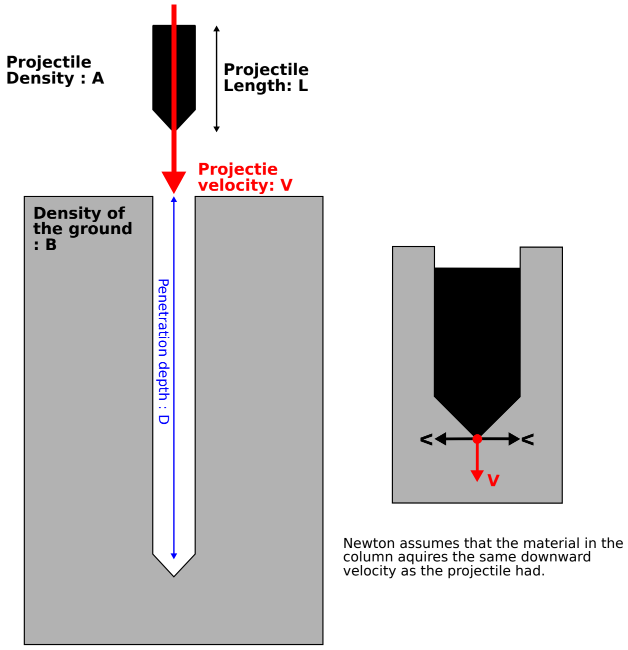

Figure 6: Diagram of a bullet pushing into the ground. Left: The bullet carves out a prismatically shaped column. Right: Newton assumed that all the material moved out of the way of the bullet gained a downwards velocity v equal to the initial velocity of the bullet.

Exercise 18. *Using Newton's approximation, and conservation of momentum, show that the bullet penetrates the ground to a distance $D \approx L\frac{A}{B}$. What is surprising about this result?.*

Exercise 19. *During the Pascal-B nuclear test in 1957, a 900-kilogram steel plate cap (a piece of armor plate) was blasted off the top of a test shaft at a speed of more than 66 kilometers per second. This probably makes it the fastest ever man-made object. Since this is much higher than the Earth's escape velocity, it is sometimes said that this cap was also the first man-made object to be put into space. Assuming the cap was 4 inches thick, use Newton's law of impact to estimate whether it did make it into space. Density of steel* $= 7750\text{kgm}^{-3}$, *density of air* $= 1\text{kgm}^{-3}$.

5 Mechanical oscillations

The more physics you do, the more you realize that almost everything is really oscillator. Indeed the great particle physicist Sidney Coleman once said in an undergraduate lecture *"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."* So, let's make a start!

The simplest possible oscillator is a mass on a spring, as sketched in fig. 7. The total

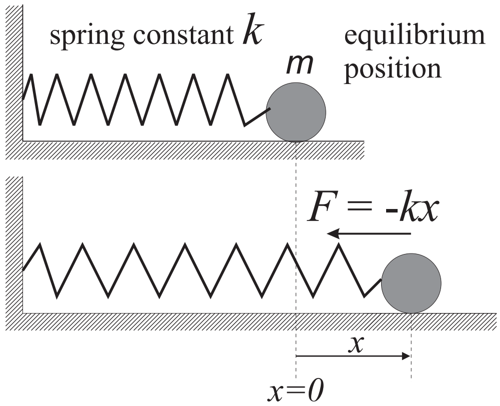

Figure 7: A mass on a horizontal spring. Top: The mass in it's equilibrium position with the spring unstretched. Bottom: The mass is displaced from equilibrium by an amount x, leading the spring to exert a force $-kx$ pulling it back towards the equilibrium point.

force on the mass is simply $F = -kx$, where the minus sign indicates that the spring always pulls (or pushes) the mass back towards the middle. Applying Newton's second law to the mass, we have

$$-kx = \dot{p} = ma. \tag{13}$$

Solving this equation, to find the motion of the mass, $x(t)$, requires calculus and thus is beyond the scope of this course (though if you do know some calculus, you should try) but we don't need to solve the equation to understand what the solution must look like. If we imagine the mass is drawn back a distance a_0 then released at rest then applying Newton's second law qualitatively to the mass on a spring gives the sequence of events shown in fig. 8. Initially the mass is at rest but the spring is pulling it back towards the equilibrium point, causing it to accelerate towards the equilibrium point. As the mass moves towards the equilibrium point the spring remains stretched, so it continues to pull towards the equilibrium point, and the mass continues to accelerate. When the spring reaches the equilibrium point the spring is no longer stretched so there is no force and the mass is moving at its maximum speed. It passes straight through the equilibrium point putting the spring into compression. This causes the spring to again push the mass back towards the equilibrium point, but now, this force is opposite to velocity, so it slows the mass down. Eventually the mass becomes stationary, but the mass is now far from the equilibrium point and the spring is deep in compression, pushing the mass back towards the equilibrium point, and the whole process starts again. The mass oscillates on the spring.

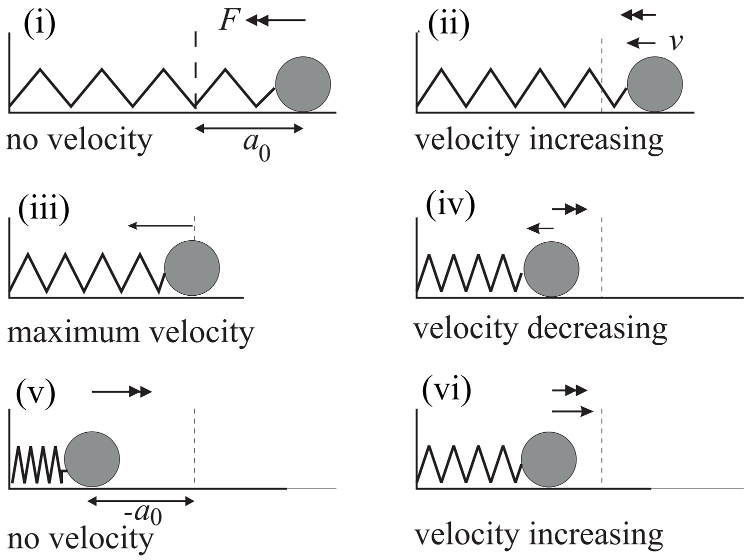

Figure 8: Snapshots of a mass oscillating on a spring. The mass is drawn back to $x = a_0$ then released, leading to oscillations. Double headed arrow indicates force, single headed arrow indicates velocity. Further description in the main text.

As the mass oscillates, it moves fastest as it passes through the middle, $x = 0$, despite the fact that at this point there is no force on the mass. It was this observation that first convinced Galileo that we needed to abandon the old idea of velocity proportional to force, which would have predicted that the mass should have zero velocity at the middle, since that is when the applied force is zero, and instead introduce the idea of momentum. Allegedly he first had this idea watching a lamp, shown on the right, swinging from the ceiling of Pisa Cathedral while he was meant to be praying.

Figure 9: The “lamp of Galileo” in Pisa Cathedral.

Exercise 20. *The mass is drawn back to a_0 and released, at rest. Draw graphs of the force, momentum, velocity, position and acceleration of the mass as a function of time. Axes are provided in fig. 21, though you may wish to sketch on scrap paper first. Where is the mass when the force is maximum? Where is the mass when the force is zero? Where is the mass when the velocity is maximum? Where is the mass when the velocity is zero?*

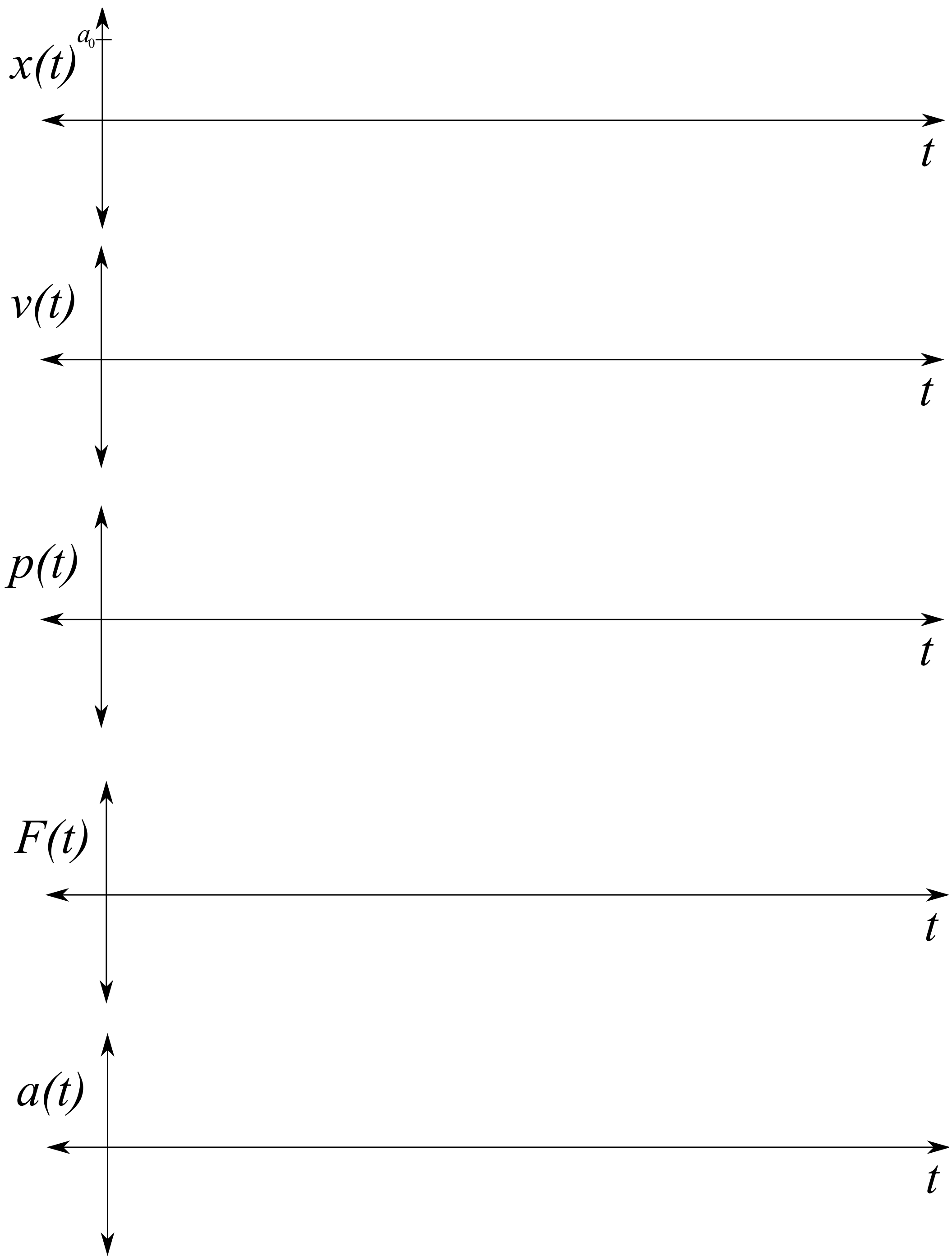

Figure 10: Sketch graphs of the mass's momentum, velocity, position and acceleration on the above axes.

6 Chain fountain

Consider a long chain sitting in a beaker. You hold the beaker above your head, pull the end of the chain out and release it. What happens next? The answer, which was only recently discovered by BBC science presenter Steve Mould, is remarkable. Firstly, the chain flows down in a stream to a pile on the floor. Secondly, not only does the chain flow down to the floor, it jumps up in an arc above the beaker, as shown in fig. 11. The question we wish to address is, why does the chain go up?

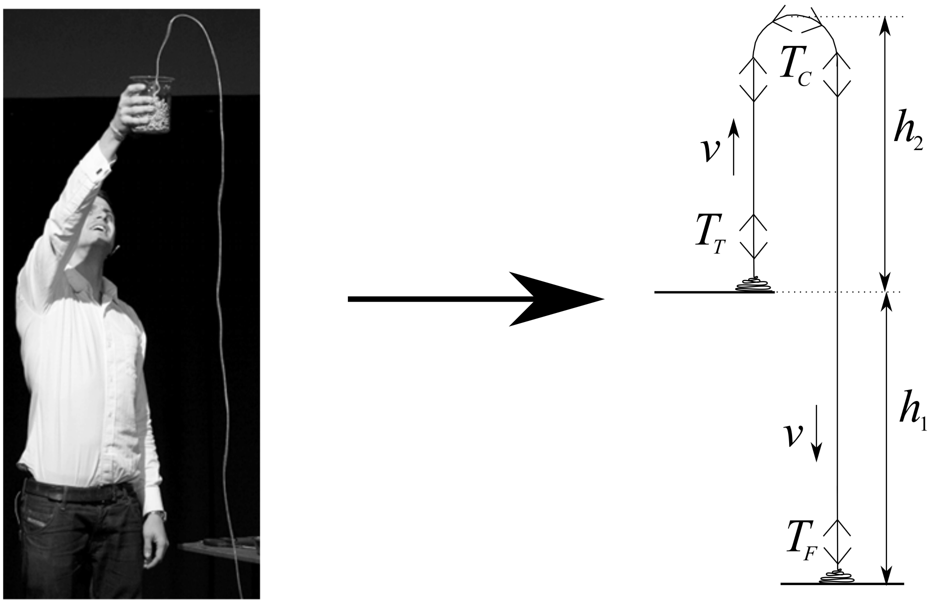

Figure 11: Photo and diagram of a chain fountain

6.1 Aside on tension

To make progress, we need to understand tension in chains, strings and ropes. Imagine two people having a tug-of-war, as sketched in fig. g 12. The two people both pull on the rope with a force F, but in opposite directions. By Newton's third law, the rope must also pull back on the people with a force F.

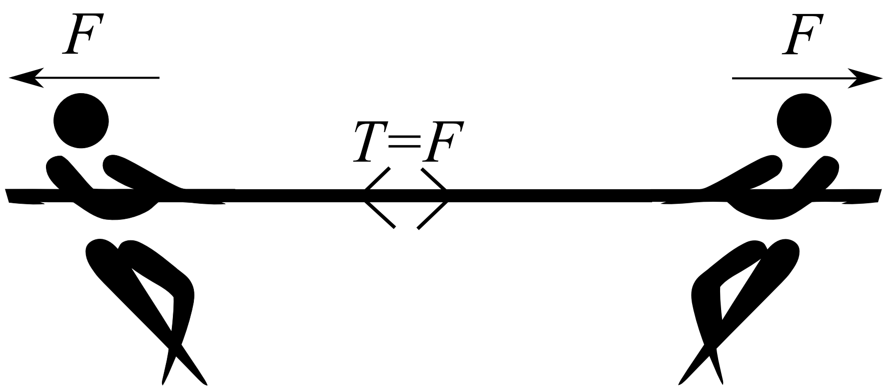

Figure 12: Two people having a tug of war. Both pull on the rope with a force F, but in opposite directions. The rope therefore carries a tension $T = F$.

Exercise 21. *Assume that the tug-of-war is in equilibrium — i.e. both people pull with the same force, and no-one is moving. Draw separate diagrams of the two people and the rope, and and indicate all the forces acting on the subject of the diagram. N.B. everything is in equilibrium, so the forces on each body must add up to zero.*

Exercise 22. *Imagine cutting the rope in the middle. How hard would you have to pull on the new end of the rope to keep everything in equilibrium?*

If you imagine cutting a rope somewhere along its length, in general, as in the tug of war example, the two halves of the rope will fly apart unless you pull back on the ends with the correct amount of force. This amount of force is called the tension at that point in the rope. It describes how much force the rope is transmitting. We can always find the tension at any point in a rope by imagining cutting it at that point, and asking how much force we need to apply at the cut to keep everything in equilibrium.

Exercise 23. *Imagine a rope with mass per unit length λ hanging from a peg under gravity. Use the imaginary cut technique to find the tension in the rope a distance s below the peg?*

6.2 Analyzing the chain fountain

To analyze the chain fountain, it is best to consider the case when it has risen to a steady height h_2, and is still flowing, but not changing shape.

Exercise 24. *Consider the journey of a link of chain through the through the fountain. Identify where in the fountain the link's momentum changes. Identify in each case which force causes the change of momentum.*

Exercise 25. *Consider the portions of the chain where there is no change in momentum. What is the total force on these segments of chain? Use this to find a relationship between T_C and T_T, and a second between T_F and T_C.*

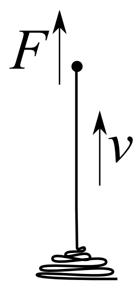

Figure 13: A chain with mass per unit length λ is picked up at a speed v by a force F applied at its end.

Exercise 26. *Imagine you have a pile of chain on a table, and you pick it up with a speed v by pulling upwards on its end, as shown in fig. 13. Draw two diagrams of this process , one at the start and one a time t later. How much momentum has the chain gained in this process? Apply Newton's second law to calculate how big the force you are applying must be.*

Exercise 27. *Now imagine a portion of chain flowing around a $\cap$ shape, as shown in fig. 14. Apply the same method as above to work out how much force must be being applied to the chain for it to flow in this way.*

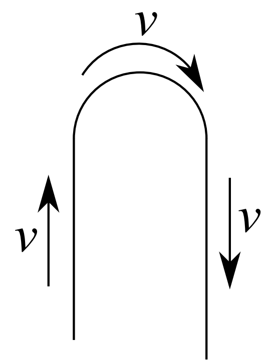

Figure 14: A chain with mass per unit length λ travels with speed v in a $\cap$ shape.

Exercise 28. *Use the results of the previous two exercises to work out the value of T_T and T_C in the chain fountain, assuming the fountain flows at speed v and has mass per unit-length λ. Then put these results into the answers from exercise 25 to find h_2, the height of the fountain.*

Exercise 29. *You should have just discovered that the fountain has zero height. How could we fix this? Repeat the calculation, assuming that the chain is picked up both by its own tension, and by an additional push R from the beaker. Does this produce a fountain? Where could such a force come from?*

Exercise 30. *We have some lengths of chain. Investigate, how the height of the fountain changes with the height of the drop, and plot a graph of your results on the graph paper below. Can you think of a way of investigating the existence and magnitude of R?*

7 What is energy

Energy is an even trickier concept to define than momentum. A common starting point is that energy is the capacity to do work, but really it is best to just start with some examples. We are all familiar with many different types of energy. A moving object has kinetic energy, a hot object has thermal energy, a ball on top of a hill has gravitational potential energy and a can of petrol contains a good deal of chemical energy. However, none of this helps us much with the question "what is energy?". The truth is that we don't have a good answer to this question, the best that can be said is that energy is a quantity that can be transferred between objects and between different forms, but never created or destroyed. Although this sounds rather abstract, it is incredibly helpful when solving problems in physics, as often all you have to do is add up how much energy there is at the beginning, how much energy there is at the end, and, since it can neither be created or destroyed, demand that the two are equal.

7.1 Kinetic energy

The most straightforward form of energy is kinetic energy, the energy of a moving object. If an object with mass m moves at velocity v it has kinetic energy

$$E = \frac{1}{2}mv^2. \tag{14}$$

Since the momentum of such a particle is $p = mv$, we can rewrite this as

$$E = \frac{p^2}{2m}. \tag{15}$$

If we have many particles, each with a different mass and a different speed, the total kinetic energy is simply the sum of the individual kinetic energies.

$$E = \sum_i \frac{1}{2}m_i v_i^2. \tag{16}$$

For objects that are not rigid, or which can rotate, calculating the kinetic energy can be rather tricky since not all parts of the object are moving at the same speed. However, we essentially apply the same basic idea — we imagine breaking the system into small particles, each of which is moving at a single speed, then sum the kinetic energy of the constituent particles to find the kinetic energy of the whole object.

Exercise 31. *What are the units of energy?*

Exercise 32. *Plot a graph of the kinetic energy of the particle against its speed. What happens to the energy if you double the speed?*

Exercise 33. *Which has more kinetic energy, a speeding bullet or a car on a highway?*

7.2 Potential energy

7.2.1 Gravitational energy

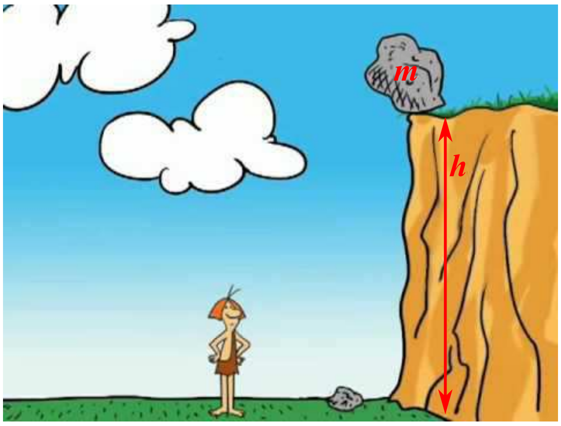

Figure 15: A rock with a lot of potential energy.

Imagine a rock on a cliff-top, as shown in fig. 15. If the rock is pushed over the edge, it will fall to the ground. As it falls it speeds up and gains kinetic energy. This energy appears to come out of nowhere, but since energy is conserved we know it must have come from somewhere. Of course, the rock speeds up because the force of gravity is pulling it down. This suggests that the energy is coming from gravity. To make sense of this, we introduce the concept of gravitational potential energy. When an object falls through a distance h it releases an amount of gravitational potential energy given by:

$$E = mgh. \tag{17}$$

Similarly, if we raise an object through a height h, it gains gravitational energy mgh.

Gravitational potential energy is an example of potential energy. These are types of energy particles store because of their position, unlike kinetic energy which is associated with a particles motion. These types of energy are called potential or stored energy since you can't see the energy, but it has the potential to be released as a more interesting sort of energy which you can see, such as kinetic energy. A ball at the top of the cliff stores lots of potential energy, but we can't see it until the ball falls to the bottom of the cliff, when the ball has converted its potential energy to kinetic energy.

7.2.2 Spring Energy

A second type of potential energy is that stored in a spring when you stretch or compress it. Imagine a mass on a spring, that you extend by a distance x. The mass is not moving, but you know it has potential energy because, if you release it, it will start to move as the spring pulls it back towards the middle. The energy stored in a spring with spring-constant k and stretched by an amount x is

$$E = \frac{1}{2}kx^2. \tag{18}$$

Exercise 34. *I have two identical particles, one in a gravitational field, and one attached to a spring. I move them both up a distance x. Draw graphs of the potential energy I have given the particle as a function of x in both cases. Which is larger for small x? Which is larger for large x? Why?*

7.3 Work

Work is the transfer of energy from one object or type to another object or type. When a ball falls a distance h in a gravitational field, mgh of gravitational energy is converted to kinetic energy, so we say mgh of work has been done. Work thus has the same units as energy, but is the amount of energy transferred rather than the total amount of energy.

However, there is a deep relationship between work and force. When you push an object with a force F through a distance d you do work

$$W = Fd, \tag{19}$$

which means that you have transferred $W = Fd$ of your energy to the object you pushed.

Exercise 35. *A ball of mass m drops a distance h. Show that the gravitational potential energy released by the ball is equal to the work done by gravity as it falls.*

Exercise 36. *A strong man pushes against a brick wall, which does not budge. How much work does the man do?*

7.4 Conservation of energy

There are many other types of energy found in physics, such light energy, sound energy, thermal energy, chemical energy and nuclear energy, but they lie beyond the scope of this course. However, the key thing about energy is that, although it can be transferred from one form to another, as long as you keep track of all the different forms it can move between, you discover it is never created or destroyed. This is a fundamental law of the Universe, known as the first law of thermodynamics. It is essentially an experimental observation. For a long time now scientists have been conducting experiments, and they have never yet found one where the total amount energy changes.

However, in many mechanics problems, the only types of energy that matter are kinetic energy and potential energy. This allows us to solve many problems in Newtonian mechanics without having to worry about forces and accelerations. Here are a few practice examples:

Exercise 37. *I drop a* 1kg *ball from* 10m *high. How fast is it going when it hits the ground?*

Exercise 38. *I have a mass on a spring. It starts at the origin, but with velocity v. How far does it get before it turns around?*

Exercise 39. *A cyclist is traveling at* 18ms^{-1}. *At the foot of a* 15m *hill she stops pedaling. Can she free-wheel over the hill?*

Exercise 40. *A mass is once again oscillating on a spring, as in exercise 20. Sketch graphs of its position, its kinetic energy, its potential energy and its total energy as a function of time. What do you notice about the frequency of oscillation of the kinetic energy compared to the frequency of oscillation of the position?*

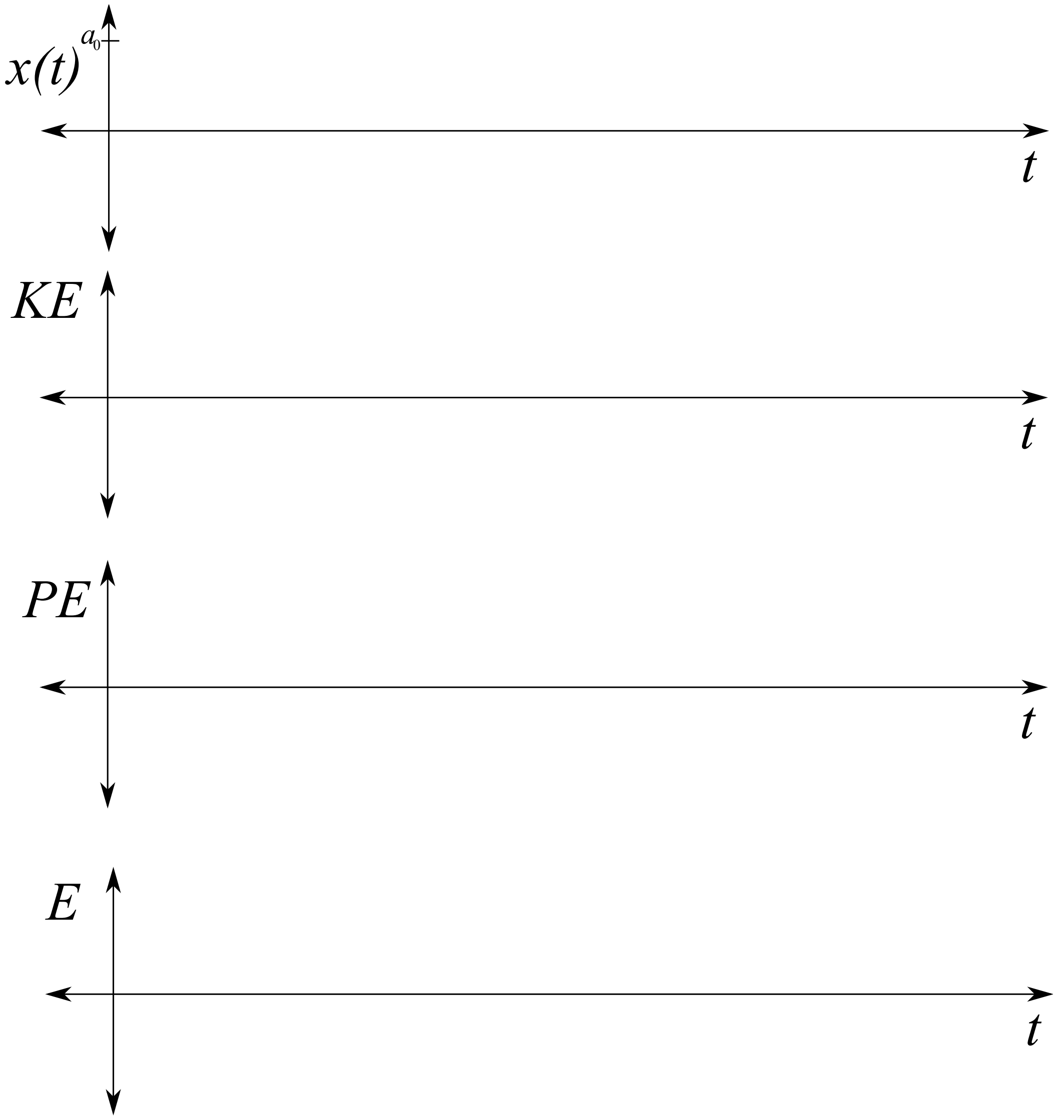

Figure 16: Sketch graphs of the mass's position, kinetic energy potential energy and total energy on the above axes.

8 Completely elastic collisions

All collisions conserve momentum. In this section we consider collisions that also completely conserve energy. This is typically a good model for collisions between rigid objects, or collisions between highly elastic objects such as bouncy balls.

8.1 Snooker and Pool

Exercise 41. *A snooker player strikes a ball directly at a cushion at a speed u. How does the ball rebound? How is this consistent with conservation of energy and momentum?*

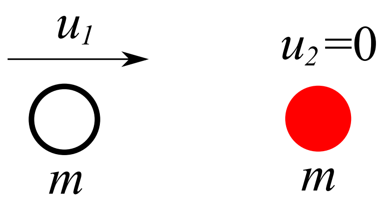

Figure 17: A ball of mass m strikes another, initially stationary, ball, also with mass m.

Exercise 42. *A snooker player strikes a ball of mass m at a velocity u_1, towards a second ball, also with mass m but stationary, as shown in fig. 17. The first ball hits the second in a perfect head on elastic collision. Conserve energy and momentum to calculate the speeds of the ball after the collision. Is this realistic? Can you find other motions of the ball after collision that would be compatible with just momentum conservation, or just energy conservation? Can you find any other solutions compatible with both?*

8.2 Newton's Cradle

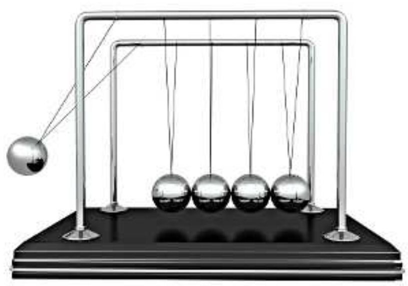

Figure 18: Diagram of a Newton's cradle.

Exercise 43. *A Newton's cradle is shown in fig. 18. One ball is pulled back from the pack and released, so it swings back in and collides with the pack with velocity u. Let us suppose that this causes the other four balls to move off with equal velocity v. Calculate v using energy conservation. Now calculate v using momentum conservation. Is it possible for both energy and momentum to be conserved like this? What must happen for both energy and momentum to be conserved?*

Exercise 44. *The Newton's cradle is used again, but this time two balls are drawn back and released. What is the motion of the balls after this collision?*

8.3 Advanced: General energy conserving collisions

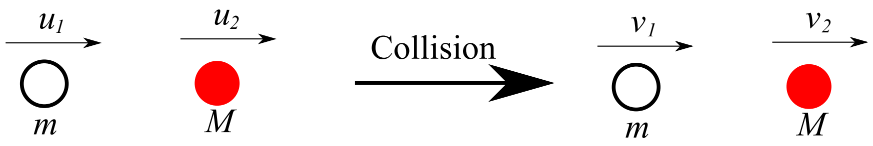

Figure 19: Before and after a general energy conserving collision.

Exercise 45. *A general scenario for a collision is shown in fig. 19, in which two balls with different mass and different initial velocities collide. Use conservation of energy and momentum to show the velocities of the particles after collision are*

$$v_1 = \frac{mu_1 - Mu_1 + 2Mu_2}{m+M} \qquad v_2 = \frac{2mu_1 - mu_2 + Mu_2}{m+M}. \tag{20}$$

Hint: think about completing the square.

Exercise 46. *Consider what happens in the above formulas if $m \ll M$. A train approaches me at speed v, and I throw a tennis ball towards the train at speed u. Show the ball bounces off the train with velocity $2v + u$. Reanalyze this problem in the frame of reference of the train and show that the answer is actually intuitive.*

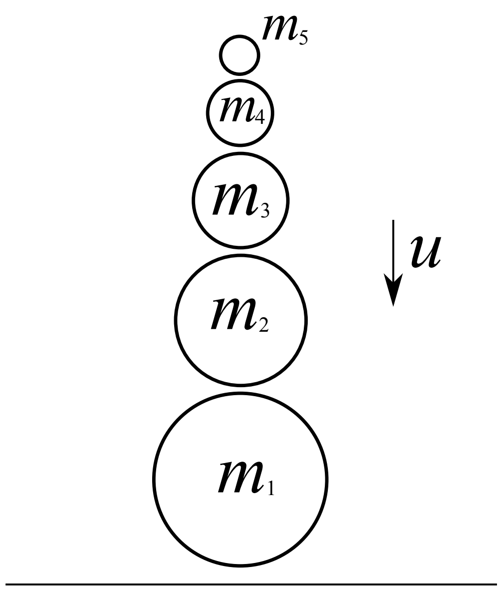

Figure 20: A falling stack of balls.

Exercise 47. *Imagine a stack of five balls, with masses $m_1 \gg m_2 \gg m_3 \gg m_4 \gg m_5$, which are all dropped from a height so the stack hits the ground with downward velocity u, as shown in fig. 20. By imagining the balls as very slightly separated, consider the sequence of collisions after the bottom one hits the ground, and thus calculate the speed the smallest ball bounces up with. If the balls were dropped from a height of 1m, show the smallest ball will bounce back by 3969m.*

9 Heat and Friction

Imagine I have a ball sliding along a surface at a speed v. At first it has momentum mv and kinetic energy $\frac{1}{2}mv^2$. If no external forces act, then momentum is conserved, so the ball will slide along for ever. Similarly, this means the kinetic energy of the ball will be conserved. However, in reality we know this isn't true, rather the ball will be slowed down by friction until it stops.

Exercise 48. *Sliding friction on a rough surface is well modeled as a constant force that acts in the opposite direction to the velocity. For the above sliding ball, the size of this force is F. Use energy conservation to work out how far the ball slides.*

This stopping of the ball isn't a violation of momentum conservation since friction is a force, and force can result in a change of momentum. However, it does seem to be a violation of energy conservation. The ball starts with some kinetic energy, but has none left at the end. Where has the energy gone? The answer is clearly that the ball's kinetic energy has been transferred away from it by the friction. At first sight, this looks rather like the situation where a ball is thrown into the air and is stopped by gravity. In that case, we said the kinetic energy was transferred to gravitational potential energy. Has the energy in the sliding case been transferred to friction potential energy? Clearly not, since there is no way to get the friction to start the ball moving again, so the energy is in no sense stored potential energy. What has actually happened is that the ball's energy has been dissipated as another kind of energy, heat energy: the friction causes the surroundings to heat up.

We might ask what the difference is between friction and gravity. How would we know that work done by gravity or a spring is stored as potential energy, while work done by friction is lost as heat? The key difference is that the force from gravity or a spring depends on position — gravity always pulls down, and a spring always pulls towards its center — whereas friction type forces depend on velocity. To be precise, friction type forces, always point opposite to velocity, acting to slow the particle down. When you throw a ball into the air gravity slows it down, but then, as the ball comes down, gravity still points down so it speeds it back up again. In contrast, when you slide against friction from a to b, friction slows you down and you have to work against it, but if you slide back again from b to a, friction still tries to slow you down. Friction never helps speed you up, so you never get any energy back from it.

9.0.1 Heat energy

In general, it requires an energy

$$E = C\Delta T \tag{21}$$

to raise an objects temperature by ΔT, where C is the objects heat capacity. The heat capacity of an object is proportional to how much stuff it contains, so we normally quote the heat capacity of a material as the heat capacity per gram, c. The actual heat capacity of an object is the heat capacity per gram of the material the object is made from multiplied by the object's weight in grams. The most important heat capacity to know is that of water, $c_w = 4.12$ J/(gm K).

Exercise 49. *Estimate how many Joules of energy are required to heat a kettle of water from room temperature to boiling.*

Exercise 50. *I try to boil the water in my kettle with kinetic energy by firing bullets into it. Estimate how many bullets do I require?*

9.0.2 Heat energy and using energy conservation

In general, if a mechanical process is subject to friction (or air-resistance, or drag) then the total of the kinetic and potential energy will not be conserved, as these forms of energy will be turned into heat energy, from which it cannot be recovered. Rather, the total of the kinetic and potential energy will decreases as time goes on. This means that you cannot use energy conservation to analyze the motion of systems where friction is important.

Exercise 51. *A mass oscillates on a spring, while subject to light air resistance. Sketch a graph of the position of the mass as a function of time. Also sketch graphs of the kinetic, potential and total energy as a function of time.*

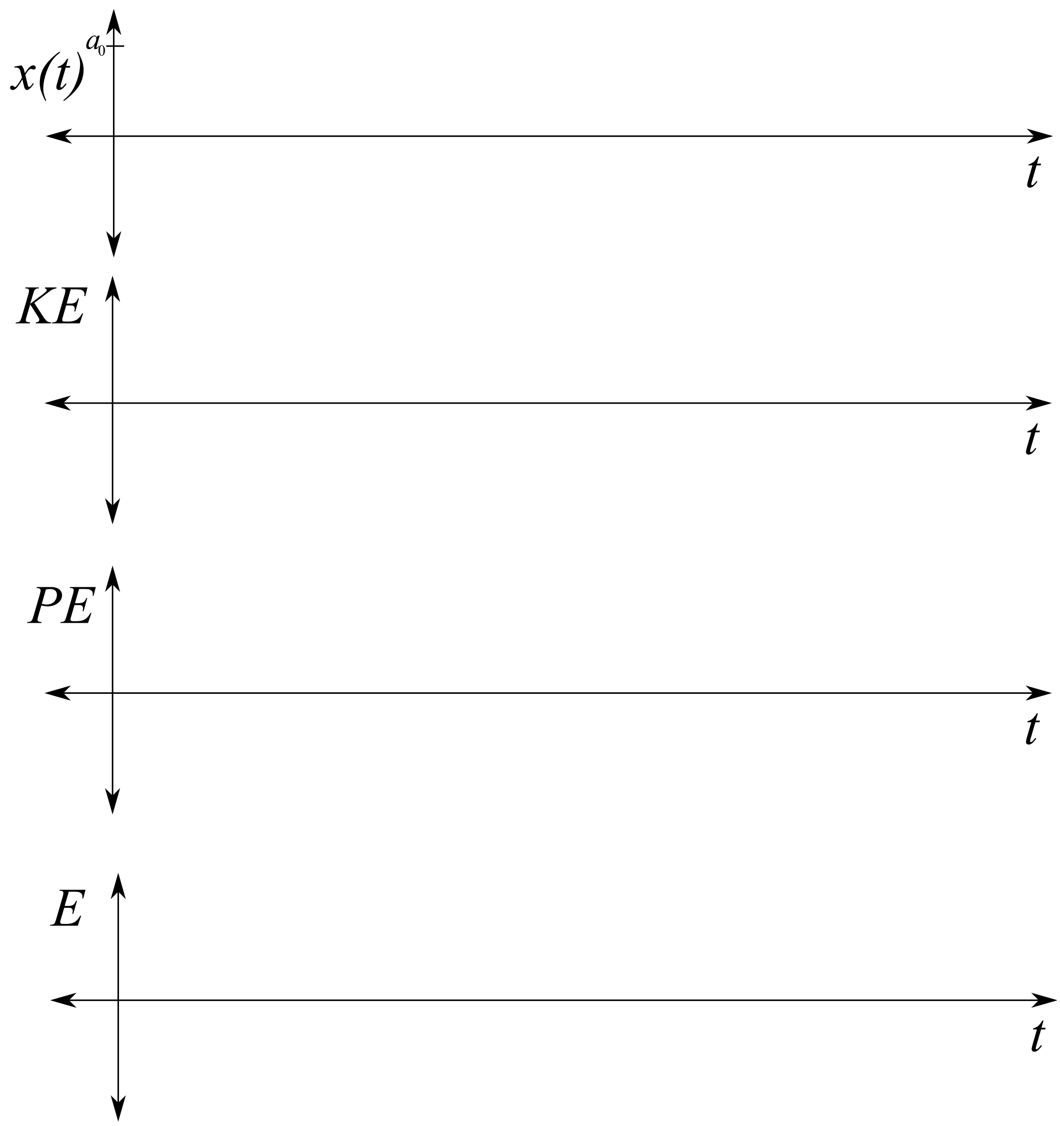

Figure 21: Sketch graphs of the mass's position, kinetic energy potential energy and total energy on the above axes.

10 Rockets

Rockets work in a very simple way. They start stationary, but full of fuel. They then push the fuel out of the back of the rocket at a high speed, which, by momentum conservation, requires the rocket itself to go forwards, as sketched in fig. 22.

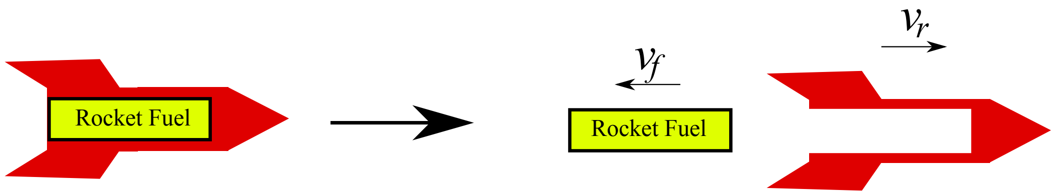

Figure 22: A rocket expels fuels backwards to push itself forwards.

Exercise 52. *A rocket is initially stationary, and weighs, including its fuel, m_0. If it expels a mass m of fuel at a speed v_f (relative to the air) how fast is the rocket itself going?*

Exercise 53. *Very advanced, only try this is you are fluent in calculus and logs! A more realistic model is that the fuel is expelled in a continuous stream with each bit of fuel expelled with a velocity v_f relative to the rocket. Will this correction make the rocket slower or faster? Calculate the more accurate speed of the rocket.*

A rocket also requires energy to expel its fuel in the first place. Traditionally this is done by burning a liquid fuel to make a gas. During the burning the fuel expands violently (since the gas is much bigger than the liquid fuel) and is pushed out of the back of the rocket. The energy for this process comes from the chemical potential energy released when the fuel is burnt. Of course, if you want your rocket fuel to burn in space, it needs to be able to burn without adding any air, since space is a vacuum. Consequently traditional rocket fuel is a mixture of liquid hydrogen and oxygen, which will react together to make water.

Unfortunately, health and safety prevents us from building hydrogen and oxygen fueled rockets at school, but we can study the same physics using a different type of water rocket, depicted in fig. 23. These rockets consist of a soda bottle partially filled with water, and partially filled with compressed air. The compressed air expands, spraying the water out of the bottle, so the bottle takes off.

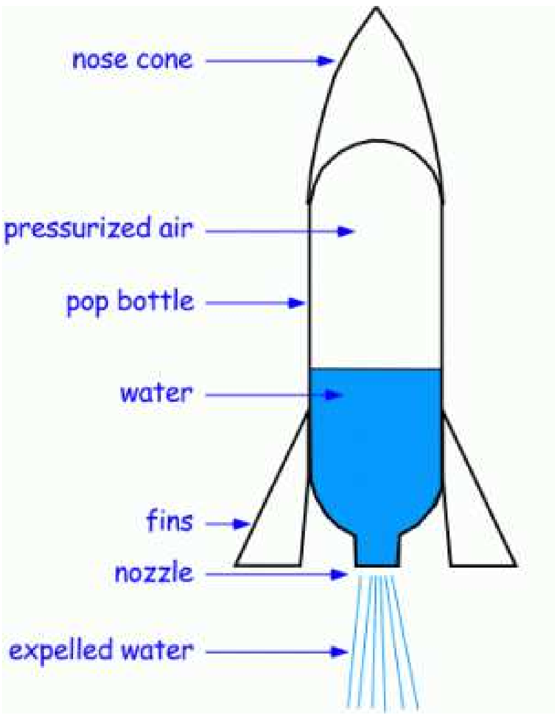

Figure 23: A rocket consisting of a soda bottle filled with water and compressed air.

Exercise 54. *In our water rockets, we pressurize the air with a bicycle pump, and the rocket always takes off when the air reaches a certain pressure (about* 150kPa*). If we fill a fraction f of the volume of the water bottle with water, how fast will the rocket go? What filling fraction, f_{max}, leads to the fastest rocket?*

Exercise 55. *Launch some rockets with different filling fractions, and time how long the fly for. Plot a graph of f vs time on the chart below. How does your experimental f_{max} result for the longest flight match your theoretical prediction for the fastest rocket?*