Tale 45
Spin and Oscillator

An oscillator as a Classical Spin

Consider first a classical particle in a two-dimensional quadratic potential well.

$$H = \frac{p^2 + q^2 + x^2 + y^2}{2},. \tag{1}$$

where p and q momentums conjugated to coordinates x and y respectively. Introduction of the coplex amplitudes

$$a = \frac{x + ip}{\sqrt{2}},\ \bar{a} = \frac{x - ip}{\sqrt{2}}, \tag{2}$$

$$b = \frac{y + iq}{\sqrt{2}},\ \bar{b} = \frac{y - iq}{\sqrt{2}} \tag{3}$$

allows to re-write the Hamiltonian in the form

$$H = \bar{a}a + \bar{b}b = \bar{\alpha}\alpha, \tag{4}$$

where the spinor notations

$$\alpha = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \bar{\alpha} = \begin{pmatrix} \bar{a}, & \bar{b} \end{pmatrix} \tag{5}$$

is introduced. Next stage is to parametrize the spinor α by the mean of spherical angles θ and χ:

$$\alpha = \sqrt{E} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} e^{i\chi/2}, \tag{6}$$

where E is the total energy. Using the the fact that the pairs x and p and y and q are canonically conjugated, one can find out that the complex amplitudes are conjugated in the sence of canonic equations

$$\dot{a} = -i\frac{\partial H}{\partial \bar{a}} = -ia, \quad \dot{\bar{a}} = i\frac{\partial H}{\partial a} = i\bar{a}, \tag{7}$$

$$\dot{b} = -i\frac{\partial H}{\partial \bar{b}} = -ib, \quad \dot{\bar{b}} = i\frac{\partial H}{\partial b} = i\bar{b}, \tag{8}$$

or, in the spinor notations,

$$\dot{\alpha} = -i\frac{\partial H}{\partial \bar{\alpha}} = -i\alpha, \quad \dot{\bar{\alpha}} = i\frac{\partial H}{\partial \alpha} = i\bar{\alpha}. \tag{9}$$

Making the Legendre transformation, find the lagrangian

$$\text{Ł}(\dot{\alpha}, \alpha) = -i\dot{\alpha}\bar{\alpha} - H(\alpha, \bar{\alpha}) \tag{10}$$

An oscillator as Spin

Consider first a quantum particle in a two-dimensional quadratic potential well.

$$H = \frac{p^2 + q^2 + x^2 + y^2}{2}; \qquad [x, p] = i; \; [y, q] = i. \tag{11}$$

This Hamiltonian can be diagonalised by introducing the creation-destruction operators

$$b = \frac{x + ip}{\sqrt{2}}, \; b^+ = \frac{x - ip}{\sqrt{2}}, \; a = \frac{y + iq}{\sqrt{2}}, \; a^+ = \frac{y - iq}{\sqrt{2}}; \tag{12}$$

$$H = \frac{b^+b + bb^+ + a^+a + aa^+}{2}, \quad E(n_1, n_2) = n_1 + n_2 + 1. \tag{13}$$

The Hamiltonian (11) possesses axial symmetry and, as a result, the operator of angular momentum

$$\hat{l} = py - qx = i(b^+a - ba^+) \tag{14}$$

commutes with the Hamiltonian. But this is not the only symmetry of the Hamiltonian (11). Since the variables x, p and y, q are separable, the Hamiltonians for the motion along x and y axis commutes with the total Hamiltonian. As a result, the operator

$$\hat{s} = \frac{p^2 - q^2 + x^2 - y^2}{2} = b^+b - a^+a \tag{15}$$

also commutes with the Hamiltonian. The commutator of $\hat{l}$ and $\hat{s}$ is

$$[\hat{l}, \hat{s}] = 2i\hat{h}, \tag{16}$$

where the Hermitian operator

$$\hat{h} = b^+ a + b a^+ = pq + xy \tag{17}$$

has non-zero matrix elements for following transitions

$$n_1 \to n_1 + 1, \quad n_2 \to n_2 - 1,$$

which does not change the energy. So, it is not surprising that $\hat{h}$ commutes with the Hamiltonian as well. Three operators $\hat{h}$, $\hat{l}$ and $\hat{s}$ form a closed algebra with the commutation relations:

$$[\hat{h}, \hat{l}] = 2i\hat{s}; \quad [\hat{s}, \hat{h}] = 2i\hat{l}; \quad [\hat{l}, \hat{s}] = 2i\hat{h}. \tag{18}$$

These relations strongly remind the commutation relation for spin operators. Introducing

$$\hat{s} = 2\hat{j}_x; \quad \hat{h} = 2\hat{j}_y; \quad \hat{l} = 2\hat{j}_z, \tag{19}$$

we find that $\hat{j}_{x,y,z}$ commute exactly like the spin components. Thus, the Hamiltonian (11) commutes with all three generators of the $SU(2)$ group, which implies that each energy level can be characterized by the eigenvalues $j(j+1)$ of the Casimir operator

$$J^2 = j_x^2 + j_y^2 + j_z^2 = j(j+1). \tag{20}$$

The Casimir quantum number j can be equal to any positive integer or half-integer or zero. It follows directly from definitions (14) and (15) that

$$4J^2 = 4j^2 + 4j = H^2 - 1, \tag{21}$$

which gives the following values of the energy

$$E_j = 2j + 1, \tag{22}$$

where j is either zero, or a positive integer or half-integer. Each energy level is $(2j+1)$-fold degenerate. The orbital momentum l is equal to the doubled value of the projection j_z of $\mathbf{J}$ on the z-axis, j_z being equal to

$$j_z = -j, -j+1, ...j.$$

Therefore, if j is an integer (or zero), then l is an even integer

$$0 \leq l \leq 2j.$$

If j is a half-integer, then l is an odd integer

$$1 \leq l \leq 2j.$$

This separation of the even and odd values of the angular momentum is the result of the commutation of the parity operator

$$P: \quad x \to -x, \; y \to -y, \; p \to -p, \; q \to -q \tag{23}$$

with the Hamiltonian (11) and all operators l, s and h of the algebra. The commutation implies that the eigenstates of the Hamiltonian have either even or odd parity. This means, in particular, that, unlike the hydrogen atom, a linear oscillator has no linear Stark-effect.

Complex Coordinates

To find the wave functions of a two-dimensional oscillator it is useful to switch from the Cartesian coordinates x and y to the complex coordinates z and $\bar{z}$

$$z = \frac{x + iy}{\sqrt{2}}, \qquad \bar{z} = \frac{x - iy}{\sqrt{2}}, \tag{24}$$

$$\partial = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \qquad \bar{\partial} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \tag{25}$$

$$dz \wedge d\bar{z} = -i \cdot dx \wedge dy. \tag{26}$$

Instead of the creation-destruction operators a, a^+, b, b^+, we introduce the operators

$$\phi = \frac{b + ia}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(z + \bar{\partial} \right), \tag{27}$$

$$\bar{\phi} = \frac{b - ia}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\bar{z} + \partial \right), \tag{28}$$

$$\phi^+ = \frac{b^+ - ia^+}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\bar{z} - \partial \right), \tag{29}$$

$$\bar{\phi}^+ = \frac{b^+ + ia^+}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(z - \bar{\partial} \right). \tag{30}$$

The operators $\phi,\ \bar{\phi}$ and $\phi^+,\ \bar{\phi}^+$ commute exactly like $a,\ b$ and $a^+,\ b^+$

$$[\phi, \bar{\phi}] = [\phi^+, \bar{\phi}^+] = [\phi, \bar{\phi}^+] = 0, \quad [\phi, \phi^+] = [\bar{\phi}, \bar{\phi}^+] = 1, \tag{31}$$

and the Hamiltonian can be rewritten as

$$H = \{\phi^+ \phi + \phi\ \phi^+ + \bar{\phi}^+ \bar{\phi} + \bar{\phi}\ \bar{\phi}^+\}. \tag{32}$$

The ground state $|0>$ corresponds to the wave function $\Phi_0(z,\bar{z})$, which obeys two conditions:

$$\phi\ \Psi_0(z,\bar{z}) \propto \left(z+\bar{\partial}\right)\Psi_0(z,\bar{z}) = 0 \qquad (33)$$
$$\bar{\phi}\ \Psi_0(z,\bar{z}) \propto (\bar{z}+\partial)\,\Psi_0(z,\bar{z}) = 0. \qquad (34)$$

The solution of both Eqs (33) and (34) has the form:

$$\Phi_0(z,\bar{z}) = \exp\left[-z\bar{z}\right] \qquad (35)$$

The first excited states $\Phi_1(z,\bar{z})$ can be obtained by acting by operators ϕ^+ and $\bar{\phi}^+$ on Ψ_0:

$$\Psi_{1,1}(z,\bar{z}) = \phi^+ \exp\left[-z\bar{z}\right] = \sqrt{2}\,\bar{z}\ \exp\left[-z\bar{z}\right], \qquad (36)$$
$$\Psi_{1,-1}(z,\bar{z}) = \bar{\phi}^+ \exp\left[-z\bar{z}\right] = \sqrt{2}\,z\ \exp\left[-z\bar{z}\right]. \qquad (37)$$

Acting on $\Phi_0(z,\bar{z})$ n_+ times by operator ϕ^+ and n_- times by operator $\bar{\phi}^+$, where $n_+ + n_- = n$, we obtain the basis of $2n+1$ functions of the n-th excited state. All wave functions for these states are polynomials of joint order n in z and $\bar{z}$, multiplied by the exponential Eq (35).

Spin as an Oscillator

We found that the wave functions, which are forming the multiplets of a two-dimensional oscillator, form, at the same time, representation of appropriate degeneracy of the *su(2)* algebra with generators λ, σ and η. This means that the oscillator's creation and annihilation operators form a representation (14,15,17,19) of the spin operators (the Schwinger representation):

$$j_y = \frac{i(a^+b - b^+a)}{2}, \quad j_x = \frac{a^+b + b^+a}{2}, \quad j_z = \frac{a^+a - b^+b}{2}. \tag{38}$$

The Schwinger construction of angular momentum via creation and annihilation operators of the oscillator consists of several stages:

- Take the groud state $|0, 0>$ of the oscillator and create an excited state

$$|n_1, n_2> = \frac{(a^+)^{n_1}(b^+)^{n_2}}{\sqrt{n_1!\ n_2!}}\ |0.0>; \tag{39}$$

- introduce the rising and lowering operators $j_\pm$ and the operator j_z

$$j_+ = j_x + ij_y = a^+b; \qquad j_- = j_x - ij_y = b^+a; \tag{40}$$

$$j_z = \frac{a^+a - b^+b}{2}; \tag{41}$$

$$< (n_1+1, n_2-1|j_+|n_1, n_2> = \sqrt{(n_1+1)n_2}; \tag{42}$$

$$< (n_1-1, n_2+1|j_-|n_1, n_2> = \sqrt{n_1(n_2+1)} \tag{43}$$

$$< (n_1, n_2|j_z|n_1, n_2> = \frac{n_1 - n_2}{2}. \tag{44}$$

- assume $n_1 + n_2 = 2j$ and $n_1 - n_2 = 2m$, from which follows

$$n_1 = j + m;\ n_2 = j - m;\ < j, m|j_z|j, m> = m; \tag{45}$$

$$< j, m+1|j_+|j, m> = \sqrt{(j+m+1)(j-m)}; \tag{46}$$

$$< j, m-1|j_-|j, m> = \sqrt{(j+m)(j-m+1)}. \tag{47}$$

- this makes

$$|j,m>=\frac{(a^+)^{j+m}(b^+)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\;|0,0>; \qquad (48)$$

$$|j,j>=\frac{(a^+)^{2j}}{\sqrt{(2j)!}}\;|0,0>\;. \qquad (49)$$

- the square of total momentum $\mathbf{j}^2 = j_z^2 + (j_+ j_- + j_- j_+)/2$ is equal $\mathbf{j}^2 = j(j+1)$.

Schwinger managed to obtain from this representation the explicite expression for rotation matrices, the Clebsch-Gordon coefficients and many other things.

Appendix. Two-dimensional electron in magnetic field

Complex coordinateds are very convenient for solving the Schrödinger equation in magnetic field. First of all, the Laplacian ∇^2 can be written in very simple form:

$$\nabla^2 = 2\,\bar\partial\,\partial.$$

It follows from curl $\mathbf{A} = \mathbf{B}$ that

$$B_z = \partial_x A_y - \partial_y A_x = i\,(\partial\,A - \bar\partial\,\bar A); \quad A = \frac{A_x - iA_y}{\sqrt{2}},\ \bar A = \frac{A_x - iA_y}{\sqrt{2}}. \quad (50)$$

In the uniform magnetic field $\mathbf{B}$ directed along z-axis, the vector potential $(A, \bar A)$ has the following form:

$$A = -\frac{i\,z\,B}{2} + f(\bar z); \qquad \bar A = \frac{i\,\bar z\,B}{2} + \bar f(z). \quad (51)$$

The gauge-fixing condition (axial gauge)

$$\bar\partial A = \partial \bar A = 0 \quad (52)$$

gives

$$A = -\frac{i\,z\,B}{2}; \qquad \bar A = \frac{i\,\bar z\,B}{2}. \quad (53)$$

The Schrödinger Hamiltonian H in magnetic field may be written as

$$\hat H = -\nabla^2 = -(\bar\partial - i\,e\,\bar A)(\partial - i\,e\,A) - (\partial - i\,e\,A)(\bar\partial - i\,e\,\bar A) = \quad (54)$$

Measuring coordinates z and $\bar z$ in the units of magnetic length and introducing operators $\phi, \bar\phi, \phi^+$ and $\bar\phi^+$ in these new units by the means of Eqs (27, 28, 29, 30), obtain

$$\hat H = e\,B\,\left(\phi^+\phi + \phi\,\phi^+\right) = e\,B\,(2n+1). \quad (55)$$

The fact that the Hamiltoian (54) contains only operators ϕ and ϕ^+ and does not contain (and, therefore, commutes with) operators $\bar\phi$ and $\bar\phi^+$, means that:

- the ground state still has the form

$$\Psi_{0,0} \propto \exp[-\bar{z}z\ ] \tag{56}$$

- all states

$$\Psi_{n,0} = \left(\phi^+\right)^n \Psi_{0,0} \propto \bar{z}^n\ \exp[-\bar{z}z\ ] \tag{57}$$

  correspond to the states at n-th Landau level with the energies $E_n = e\ B\ (2n+1)$.

- all states

$$\Psi_{n,m} = \left(\bar{\phi}^+\right)^m \Psi_{n,0} \tag{58}$$

  correspond to the states at n-th Landau level with the value m of angular momentum and the energie $E_n = e\ B\ (2n+1)$.

- in paricular, the states

$$\Psi_{0,m} = \left(\bar{\phi}^+\right)^m \Psi_{0,0} \propto z^m \exp[-\bar{z}z\ ] \tag{59}$$

  correspond to the states at lowest Landau level.

An example of a state at n-th Landau level is

$$\Psi_{n,1} \propto (2\bar{z}z - n)\ \bar{z}^{n-1} \exp[-\bar{z}z\ ] \tag{60}$$