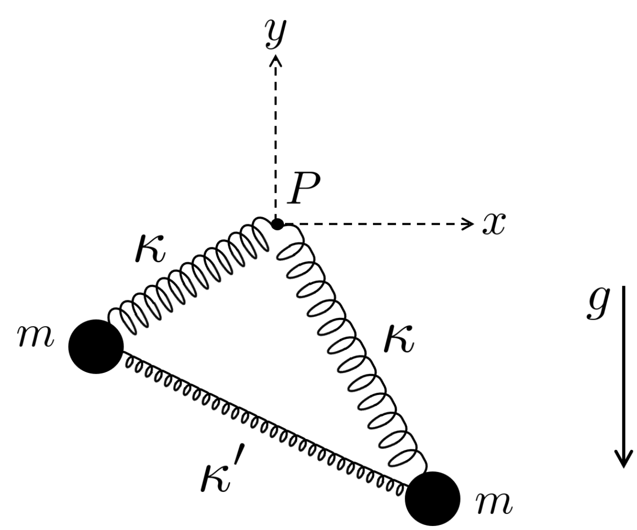
NATURAL SCIENCES TRIPOS Part II

Wednesday 15 January 2020 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **all** *questions to the best of your abilities. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 Two identical beads of mass m are each attached to a pivot point P by a light spring of constant κ and unstretched length $\ell = 0$, in the presence of a gravitational acceleration g. They are further connected to one another by a spring of constant κ' and unstretched length $l' > 0$. The centres of the two beads are confined to move within a vertical plane through P, as sketched below.



(a) Show that the Lagrangian of the system can be written as

$$L = m\left(\dot{X}^2 + \dot{Y}^2\right) - 2mg\,(Y-\alpha) - \kappa\left[X^2 + (Y-\alpha)^2\right]$$
$$+m\left(\dot{x}^2+\dot{y}^2\right) - \kappa\left(x^2+y^2\right) - 2\kappa'\left(\sqrt{x^2+y^2} - \frac{\ell'}{2}\right)^2 ,$$

where $X = (x_1 + x_2)/2$, $Y = (y_1 + y_2)/2 + \alpha$, $x = (x_1 - x_2)/2$, $y = (y_1 - y_2)/2$. Here, (x_1, y_1) and (x_2, y_2) denote the coordinates of the centres of the two beads in the Cartesian reference frame given by the dashed axes in the figure, and α is a generic constant. Find the Euler-Lagrange equations of motion of the system. [7]

(b) Find the equilibrium positions of the beads and show that, for an appropriate choice of α, they satisfy [6]

$$X = Y = 0\,, \qquad\qquad x^2 + y^2 = \left(\frac{\kappa'\ell'}{\kappa + 2\kappa'}\right)^2 .$$

(c) For the value of α chosen in (b), find the normal modes and the corresponding frequencies of small oscillations about the equilibrium positions. [8]

(d) Discuss the continuous symmetries of the Lagrangian and find the corresponding conserved quantities.
[Up to 3 bonus marks]. [4]

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2 Consider a 2-component real vector field $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ and its relativistic Lagrangian density in 3+1 space-time dimensions,

$$\mathcal{L} = (\partial_\mu \boldsymbol{\phi})^T M (\partial^\mu \boldsymbol{\phi}) - \boldsymbol{\phi}^T M \boldsymbol{\phi} - \lambda \left(\boldsymbol{\phi}^T M \boldsymbol{\phi}\right)^2 ,$$

where M is a real symmetric 2×2 matrix and $\lambda > 0$.

(a) Derive the Euler-Lagrange equations of motion for the fields ϕ_1 and ϕ_2. [5]

(b) State Noether's theorem and write a general expression for the conserved current in the case of a multi-component field. [3]

(c) Consider the transformation $\boldsymbol{\phi} \to D\boldsymbol{\phi}$ where $D = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}$ with $\alpha > 0$. Find the conditions that one must impose on the elements of the matrix M so that this transformation is a symmetry of the system. [5]

(d) Show that, under such conditions, the symmetry of the system with respect to the transformation $\boldsymbol{\phi} \to D\boldsymbol{\phi}$ leads to a conserved current of the form [5]

$$J^\mu \propto \phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1 .$$

(e) Using the Fourier representation for each component of the field,

$$\phi_i = \int \mathrm{d}^3k \; N(\boldsymbol{k}) \left[a_i(\boldsymbol{k}) e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}} + a_i^*(\boldsymbol{k}) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \right] , \qquad i = 1, 2 ,$$

where $N(\boldsymbol{k}) = [(2\pi)^3 2\omega]^{-1}$, express the conserved charge associated with the current $J^\mu = \phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1$ in terms of the relevant Fourier modes $a_i(\boldsymbol{k})$. [7]

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3 Consider a real scalar field $\phi(t, x)$ in $1+1$ space-time dimensions, with action $S = \int \mathrm{d}t\, \mathrm{d}x\, \mathcal{L}$ and Lagrangian density

$$\mathcal{L} = \dot{\phi}^2 - \gamma\, \phi'^2 + \alpha\, \phi^2 - \frac{\beta}{2}\, \phi^4\,,$$

where $\dot{\phi} = \frac{\partial \phi}{\partial t}$ and $\phi' = \frac{\partial \phi}{\partial x}$, and α, β, γ are real and positive constants.

(a) Find the units of α, β and γ, and use them to obtain a characteristic length scale and an energy scale for the system. [7]

(b) Derive the components of the stress-energy tensor, and discuss its conservation. Use it to define the total energy E of the system. [8]

(c) Consider a field that interfaces between the two constant values:

$$\lim_{x \to -\infty} \phi(x, t) = -\sqrt{\frac{\alpha}{\beta}} \qquad \text{and} \qquad \lim_{x \to +\infty} \phi(x, t) = \sqrt{\frac{\alpha}{\beta}}\,.$$

Using an appropriate variational calculation ($\phi \to \phi + \delta\phi$), or otherwise, show that the field $\phi(x, t)$ which minimises the total energy E for the above boundary conditions takes the form

$$\phi(x) = \sqrt{\frac{\alpha}{\beta}} \tanh\left[\sqrt{\frac{\alpha}{2\gamma}}(x - x_0)\right]\,,$$

where x_0 is an arbitrary constant. (It is sufficient to show that $\delta E = 0$ and you do not need to demonstrate that it is an actual minimum.) [10]

(d) Define the energy E_I of the interface as the difference between the energy of the field discussed in part (c), namely with the interface present, and the energy of a uniform field, $\phi = \sqrt{\alpha/\beta}$. Either by direct computation or by an appropriate scaling analysis, determine how E_I depends on the parameters α, β and γ.

[Up to 5 bonus marks.]

4 Consider the following Lagrangian density for a complex relativistic scalar field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) + \frac{m^2}{2} (\phi^* \phi)^2 - \frac{\lambda}{3} (\phi^* \phi)^3 \,,$$

where m and λ are real positive constants.

(a) Derive the minimal energy state(s) of the field ϕ and obtain the Lagrangian density for small fluctuations χ about (one of) the minimum energy state(s), ϕ_0, up to quadratic order in χ. Discuss briefly the behaviour of the real and imaginary components of χ. [8]

(b) Consider coupling the complex scalar field ϕ in this question to an electromagnetic field via the covariant derivative. Discuss briefly what happens to the fluctuating complex field χ as a result. [3]

(c) Write the Lagrangian density of such a coupled electromagnetic field when the complex scalar field is exactly at (one of) its minimum energy state(s). If needed, you may ignore irrelevant constant terms. Find the corresponding Euler-Lagrange equations of motion for the 4-vector potential, and show that (by an appropriate choice of gauge or otherwise) they can be written as [6]

$$\left(\partial_t^2 - \partial_x^2 + \frac{2e^2 m^2}{\lambda} \right) A^\nu = 0 \,,$$

in 1+1 space-time dimensions and natural units. You may use, without deriving it, the result:

$$\frac{\partial}{\partial(\partial_\mu A_\nu)} \left(F_{\alpha\beta} F^{\alpha\beta} \right) = 4F^{\mu\nu} \qquad \text{where} \qquad F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \,.$$

(d) Using the Fourier transform conventions:

$$\tilde{A}^\nu(k,t) = \int \hat{A}^\nu(k,\omega)\, e^{-i\omega t} \,\frac{\mathrm{d}\omega}{2\pi} \qquad A^\nu(x,t) = \iint \hat{A}^\nu(k,\omega)\, e^{-ikx - i\omega t} \,\frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2} \,,$$

and defining the constant $M^2 = 2e^2 m^2/\lambda$ for convenience, derive the Green's function $\tilde{A}^\nu(k,t)$ for the equations of motion in (c). (This may require shifting poles or deforming the integration contour, according to the physical expectation in a relativistic system.) [8]

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