UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

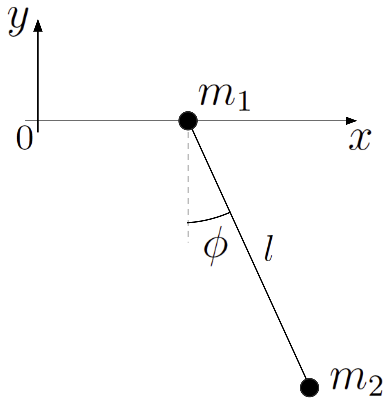
Wednesday 18 January 2012 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 A simple pendulum of mass m_2 is free to oscillate in the vertical plane $x-y$. At its point of support the pendulum is attached to a mass m_1 which is free to move along the line $y=0$.



(a) Show that the Lagrangian for this system is

$$L = \frac{1}{2}\left(m_1+m_2\right)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi\right) + m_2 g l\cos\phi,$$

where ϕ is the angular displacement of the pendulum and x is the horizontal position of the mass m_1, as shown in the figure. [8]

(b) Deduce the canonical momenta p_x and p_ϕ conjugate to the generalised coordinates x and ϕ and show that p_x is a conserved quantity. [6]

(c) Show that the path of m_2 is the arc of an ellipse if $p_x = 0$. [10]

(d) For the case considered in (c) derive an expression for the energy E of the system and use it to show that the time t taken for the pendulum to move from angle ϕ_1 to ϕ_2 within a single oscillation is given by

$$t = l\sqrt{\frac{m_2}{2(m_2+m_1)}}\int_{\phi_1}^{\phi_2} \mathrm{d}\phi\sqrt{\frac{m_1+m_2\sin^2\phi}{E+m_2 g l\cos\phi}}.$$ [9]

2 A harmonic oscillator is weakly perturbed by a cubic potential λx^3 so that its Hamiltonian has the form

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x^3,$$

where λ is small.

(a) Find constraints on the parameters α_i, β_i which make the coordinate transformation

$$\begin{aligned} x &= X + \alpha_1 X^2 + 2\alpha_2 XP + \alpha_3 P^2 \\ p &= P + \beta_1 X^2 + 2\beta_2 XP + \beta_3 P^2 \end{aligned}$$

canonical to first order in α_i and β_i. [8]

(b) Carry out the canonical transformation from part (a) on the Hamiltonian H and find values for the parameters α_i, β_i in terms of m, ω and λ which make the transformed Hamiltonian $K(X, P)$ harmonic to first order in α_i and β_i, i.e.

$$K(X,P) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 + O(\alpha_i^2, \beta_i^2),$$

and state the resulting canonical transformations. [10]

(c) Use Hamilton's equations for K to find expressions for $X(t)$ and $P(t)$ to first order in α_i and β_i. [6]

(d) Use your answers to parts (b) and (c) to find expressions for $x(t)$ and $p(t)$ and comment on the effect of the perturbation. [9]

3 Show explicitly that the Lagrangian

$$L = \frac{1}{2}mv^2 + e\boldsymbol{v}\cdot\boldsymbol{A} - e\phi$$

yields the correct equation of motion for a particle of (positive) charge e and mass m moving in an electromagnetic field:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \quad \text{and} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A},$$

where $\boldsymbol{A}$ and ϕ are the usual electromagnetic potential functions. [7]

Explain what is meant by *gauge invariance* in this context. [3]

In terms of cylindrical coordinates (r, θ, z), the potential functions are $\phi = \lambda z^2$ and $\boldsymbol{A} = (0, \mu r, 0)$, where λ and μ are positive constants.

(a) Use the Euler-Lagrange equations to derive the (three) equations of motion of the particle. [7]

(b) Determine the total energy of the particle and show that it is a constant of the motion. [5]

(c) Show that the Euler-Lagrange equation for $\theta(t)$ gives rise to a second constant of the motion. [3]

(d) Describe the motion of the particle given that r is constant, $r = R$, and the angular velocity is non-zero, $\dot{\theta} \neq 0$. [4]

(e) Explain the significance of the special case $\lambda = (2e\mu^2/m)n^2$, where n is an integer. [4]

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4 The Klein-Gordon Lagrangian density for a real scalar field $\varphi(\boldsymbol{x}, t)$ is

$$\mathcal{L}_{\mathrm{KG}}[\varphi] = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2}m^2\varphi^2,$$

where ∂^μ represents the differential operator $(\partial/\partial t, -\nabla)$. Use the Euler-Lagrange equations to derive the equation of motion: [5]

$$\partial^\mu \partial_\mu \varphi + m^2 \varphi = 0.$$

The Fourier transformed field $\tilde{\varphi}(\boldsymbol{k}, t)$ is defined by

$$\varphi(\boldsymbol{x}, t) = \int \mathrm{d}^3\boldsymbol{k}\, \tilde{\varphi}(\boldsymbol{k}, t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Find and solve the equation of motion satisfied by $\tilde{\varphi}(\boldsymbol{k}, t)$. [5]

A dynamical system is described by two real scalar fields, φ_1 and φ_2, with Lagrangian density

$$\mathcal{L} = \mathcal{L}_{\mathrm{KG}}[\varphi_1] + \mathcal{L}_{\mathrm{KG}}[\varphi_2] + \mathcal{L}_{\mathrm{int}},$$

where the interaction term is $\mathcal{L}_{\mathrm{int}} = g\varphi_1\varphi_2$, with g a real constant, $0 < g < m^2$.

Derive the (two) coupled equations of motion for the system. [5]

Solve these equations to obtain general solutions in terms of the Fourier transformed fields $\tilde{\varphi}_i(\boldsymbol{k}, t)$. [9]

At time $t = 0$, the system is in a state corresponding to

$$\varphi_1 = A\sin(\boldsymbol{q}\cdot\boldsymbol{x}), \qquad \frac{\partial\varphi_1}{\partial t} = \frac{\partial\varphi_2}{\partial t} = \varphi_2 = 0,$$

with A and $\boldsymbol{q}$ a constant scalar and vector respectively. Find φ_1 and φ_2 for $t > 0$. [9]

5 State *Noether's theorem* and explain its significance. [5]

A Lagrangian density $\mathcal{L}$ is a functional of a scalar field $\varphi(x, t)$. If the Lagrangian is invariant under an infinitessimal field transformation of the form

$$\varphi \to \tilde{\varphi} = \varphi + \delta\varphi,$$

show that there is a continuity equation

$$\frac{\partial J_x}{\partial x} + \frac{\partial \rho}{\partial t} = 0,$$

where

$$\rho = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\delta\varphi, \qquad J_x = \frac{\partial \mathcal{L}}{\partial \varphi'}\delta\varphi,$$

and $\dot{\varphi}$ and φ' denote partial differentiation with respect to t and x respectively. [10]
Generalising to 3 spatial dimensions, and using covariant notation, show that this corresponds to conservation of the Noether current J^μ, i.e. $\partial_\mu J^\mu = 0$, where [3]

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi.$$

The Lagrangian density for a scalar field in n space-time dimensions, $\varphi(t, x_1, x_2, ..., x_{n-1})$, is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \lambda \varphi^4,$$

where $\partial^\mu = (\partial/\partial t, -\partial/\partial x_1, -\partial/\partial x_2, ..., -\partial/\partial x_{n-1})$ and hence $\partial_\mu x^\mu = n$. Use the Euler-Lagrange equations to derive the equation of motion [5]

$$\partial^\mu \partial_\mu \varphi + 4\lambda \phi^3 = 0.$$

A current J^μ is defined by

$$J^\mu = (\varphi + x^\nu \partial_\nu \varphi)\partial^\mu \varphi - x^\mu \mathcal{L}.$$

Show that

$$\partial_\mu J^\mu = (n-4)\mathcal{L},$$

and hence that J^μ is a conserved current only in 4 space-time dimensions. [10]

6 An infinite one-dimensional system has a temperature distribution $T(x,t)$ given by the heat transmission equation

$$-\frac{\partial^2 T}{\partial x^2} + 2\alpha \frac{\partial T}{\partial t} + \frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} = s(x,t),$$

where $s(x,t)$ is a heat source, and α and c are positive constants.

(a) Use Fourier methods to show that the Green's function

$$G(k; t-t') = \int_{-\infty}^{\infty} e^{-ik(x-x')} G(x, x'; t, t') \mathrm{d}x$$

for this heat equation has the form

$$\begin{aligned} G(k, t-t') &= 0 \qquad t < t' \\ &= \frac{1}{\sqrt{\alpha^2 - k^2/c^2}} e^{-\alpha c^2 (t-t')} \sinh \sqrt{\alpha^2 c^4 - k^2 c^2}(t-t') \qquad t > t' \end{aligned}$$

and comment on the result. [19]

(b) Find the temperature $T(x,t)$ of the system if $s(x,t) = \cos(px)\delta(t-t_0)$ and $T(x, t < t_0) = 0$, for the two cases $\alpha > p/c$ and $\alpha < p/c$ and discuss your results. [14]

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