Chapter 6

Tripos Questions

The following problems are taken from recent Part III Natural Science Tripos papers. Note that the questions are each intended to take around 45 minutes to complete.

6.1 Questions

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1. According to the Holstein-Primakoff transformation, a quantum mechanical spin S can be expressed in terms of bosonic creation $a^\dagger$ and annihilation a operators as

$$\hat{S}^- = (2S)^{1/2} a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2}, \qquad \hat{S}^+ = (\hat{S}^-)^\dagger, \qquad \hat{S}^z = S - a^\dagger a.$$

Show that this definition is consistent with the commutation relations for spin: $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$. [7]

The one-dimensional spin S Heisenberg Quantum Ferromagnetic is specified by the Hamiltonian

$$\hat{H} = -J \sum_{n=1}^{N} \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1},$$

where $J > 0$, and periodic boundary conditions are imposed such that $\hat{\mathbf{S}}_{N+1} \equiv \hat{\mathbf{S}}_1$. Specify one of the possible ferromagnetic ground states. Applying the Holstein-Primakoff transformation, and taking the spin to be large $S \gg 1$, expand the Hamiltonian to quadratic order in the boson operators around the ground state.[9]

Diagonalising the Hamiltonian, show that the spin-wave spectrum takes the form[9]

$$\hbar\omega_k = 2JS(1 - \cos k), \qquad k = 2\pi n/N, \quad n = 1, \cdots N.$$

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2. Write brief notes on **two** of the following: [25]

   (a) the classical and quantum field theory of the harmonic elastic chain;

   (b) electron-phonon coupling and the BCS Interaction;

   (c) the Feynman path integral formulation of quantum mechanical tunneling in the double well potential;

   (d) relativistic covariance of the Dirac equation and the property of spin;

   (e) the free particle solution of the Dirac equation, negative energy states, and the Klein Paradox.

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3. In the Lagrangian formulation, the Feynman path integral describing the transition amplitude for a particle to propagate from a position q_I to a position q_F in a time t is given by

$$\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \int_{q(0)=q_I, q(t)=q_F} Dq(t') \exp\left[\frac{i}{\hbar} S[q(t')]\right],$$

where $S[q(t')]$ denotes the classical action for trajectory $q(t')$. In the semi-classical limit $\hbar \to 0$, explain why the dominant contribution to the path integral derives from the trajectories $\bar{q}(t')$ satisfying the classical equations of motion. [7]

(a) Evaluating the Feynman path integral, show that the propagator associated with the quantum harmonic oscillator Hamillonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2,$$

is given by [9]

$$\langle q_F = 0 | e^{-i\hat{H}t/\hbar} | q_I = 0 \rangle = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega t)}\right)^{1/2} \theta(t).$$

[Note that: $\frac{x}{\sin x} = \prod_{n=1}^{\infty} \left[1 - \frac{x^2}{\pi^2 n^2}\right]^{-1}$.]

(b) More generally, the propagator from $q_I = q$ to $q_F = q$ can be shown to be equal to

$$\langle q | e^{-i\hat{H}t/\hbar} | q \rangle = \langle 0 | e^{-i\hat{H}t/\hbar} | 0 \rangle \exp\left[-\frac{i}{\hbar} m\omega q^2 \tan(\omega t/2)\right].$$

Using this result together with that found in (a) above, obtain an expression for the quantum partition function of the quantum harmonic oscillator [9]

$$\mathcal{Z} = \mathrm{tr} e^{-\beta\hat{H}}.$$

[Note $\int_{-\infty}^{\infty} dq e^{-iaq^2/2} = \sqrt{\frac{2\pi}{ia}}$.]

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4. The phenomenology of a conventional superconductor can be described by the effective Hamiltonian

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k}\sigma} \zeta_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + \Delta c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) + \frac{\Delta^2}{g},$$

where $\zeta_{\mathbf{k}} \equiv \zeta_{-\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$, Δ is a real constant, and the electron field operators obey Fermionic anticommutation relations $c_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}'\sigma'} + c^\dagger_{\mathbf{k}'\sigma'} c_{\mathbf{k}\sigma} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$.

(a) Show that the unitary transformation

$$\begin{pmatrix} \alpha^\dagger_{\mathbf{k}\uparrow} \\ \alpha_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow} \end{pmatrix}$$

preserves the anticommutation relations of the new field operators. [5]

(b) Applying this transformation, rewrite the Hamiltonian in terms of the field operators $\alpha_{\mathbf{k}\sigma}$ and $\alpha^\dagger_{\mathbf{k}\sigma}$. [8]

(c) Show that, by a suitable choice of $\theta_{\mathbf{k}}$, the Hamiltonian takes the diagonal form [7]

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k}\sigma} (\zeta^2_{\mathbf{k}} + \Delta^2)^{1/2} \alpha^\dagger_{\mathbf{k}\sigma}\alpha_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\zeta_k - (\zeta^2_{\mathbf{k}} + \Delta^2)^{1/2} \right) + \frac{\Delta^2}{g}.$$

(d) Explain why the ground state wavefunction takes the form $|\text{g.s.}\rangle = \prod_{\mathbf{k}} \alpha_{\mathbf{k}\uparrow}\alpha_{-\mathbf{k}\downarrow}|0\rangle$.

Comment briefly on nature of the quasi-particle spectrum. [5]

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5. Write detailed notes on **one** of the following: [25]

(a) the spin-wave theory of the Quantum Heisenberg Ferromagnet;

(b) the construction of the Feynman Path Integral from first principles;

(c) instantons and the path integral for the double well;

(d) Lorentz covariance of the Dirac equation;

(e) quantisation of the free scalar field theory.

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6. What is the physical content of the propagator $\langle q_F|e^{-i\hat{H}t/\hbar}|q_I\rangle$? [2]

   (a) Using the *Hamiltonian* formulation (as opposed to the Lagrangian formulation), express the propagator of the one-dimensional harmonic oscillator Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2$$

   as a Feynman path integral over *phase* space coordinates $(p(t), q(t))$. [6]

   (b) Using this result, and setting $\beta = 1/k_B T$, express the quantum partition function $\mathcal{Z} = \mathrm{tr} e^{-\beta\hat{H}}$ of the harmonic oscillator as a Feynman path integral over the same phase space coordinates. [4]

   (c) In the second quantised representation, the Hamiltonian of the one-dimensional harmonic oscillator is given by

$$\hat{H} = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega$$

   where the operators obey the commutation relations $aa^\dagger - a^\dagger a = 1$. In the coherent state representation, the quantum partition function takes the form

$$\mathcal{Z} = e^{-\beta\hbar\omega/2}\int D[\bar{\psi},\psi]\exp\left[-\int_0^\beta d\tau\left(\bar{\psi}\partial_\tau\psi + \hbar\omega\bar{\psi}\psi\right)\right]$$

   where the functional integral runs over the complex field $\psi(\tau)$. What are the boundary conditions on the fields ψ? [3]

   (d) Show that the change of variables $\psi(\tau) = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left[q(\tau) + \frac{ip(\tau)}{m\omega}\right]$, with $q(\tau)$ and $p(\tau)$ real, recovers the Feynman phase space path integral. [10]

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7. In the continuum limit, a classical harmonic chain of length L is described by the Lagrangian density

$$\mathcal{L}[\phi] = \frac{m}{2}\dot{\phi}^2 - \frac{ka^2}{2}(\partial_x\phi)^2$$

   where the displacement field obeys the periodic boundary condition $\phi(x+L) = \phi(x)$.

   (a) Applying the rules of canonical quantisation, show the corresponding quantum Hamiltonian takes the form [8]

$$\hat{H} = \int_0^L dx\left[\frac{\hat{\pi}^2}{2m} + \frac{k_s a^2}{2}(\partial_x\hat{\phi})^2\right]$$

   where the quantum fields obey the canonical commutation relations $[\hat{\pi}(x), \hat{\phi}(x')] = -i\hbar\delta(x - x')$.

(b) Transforming the field operators to a Fourier representation, show that the Hamiltonian takes the form [8]

$$\hat{H} = \sum_k \left[\frac{1}{2m} \hat{\pi}_k \hat{\pi}_{-k} + \frac{k_s a^2}{2} k^2 \hat{\phi}_k \hat{\phi}_{-k} \right]$$

where the field operators obey the commutation relations $[\hat{\pi}_k, \hat{\phi}_{k'}] = -i\hbar\delta_{kk'}$.

(c) By defining

$$a_k \equiv \sqrt{\frac{m\omega_k}{2\hbar}} \left(\hat{\phi}_k + \frac{i}{m\omega_k} \hat{\pi}_{-k} \right)$$

where $\omega_k = a(k_s/m)^{1/2}|k|$, show that the Hamiltonian can be brought to the diagonal form [10]

$$\hat{H} = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right)$$

Confirm that the operators $a_k^\dagger$ and a_k obey bosonic commutation relations.

(d) Comment briefly on the relation between the linear atomic chain and the quantum properties of the relativistic scalar field. [4]

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8. Write notes on **one** of the following topics:

(a) the Holstein-Primakoff transformation and spin-wave theory; [30]

(b) the Feynman path integral and its relation to classical and quantum statistical mechanics; [30]

(c) relativistic covariance of the Dirac equation and quantum mechanical spin;[30]

(d) the Klein paradox. [30]

[Note that credit will be given for a clear demonstration of qualitative understanding. Analytical detail can be kept to a minimum.]

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9. Describe how the density matrix $\rho_{ij} = \langle\theta_i|e^{-\beta\hat{H}}|\theta_j\rangle$, where $\beta = 1/k_B T$, can be expressed as a Feynman path integral. [7]

A quantum particle of mass m confined to a ring of radius a is specified by the Hamiltonian

$$\hat{H} = \frac{\hat{L}_z^2}{2I}$$

where $\hat{L}_z = -i\hbar\frac{\partial}{\partial\theta}$ represents the angular momentum operator and $I = ma^2$.

(a) Setting $\hbar = 1$, show that the density matrix can be cast in the form of a Euclidean time Feynman path integral as [8]

$$\langle\theta_F|e^{-\beta\hat{H}}|\theta_I\rangle = \sum_{n=-\infty}^{\infty}\int D\theta(\tau)\ \exp\left[-\int_0^\beta d\tau\ \frac{I}{2}\dot{\theta}^2\right],$$

where the functional integral over $\theta(\tau)$ must be evaluated subject to the boundary conditions $\theta(0) = \theta_j$ and $\theta(\beta) = \theta_i + 2\pi n$.

(b) With the parameterisation $\theta(\tau) = \theta_{\text{cl.}}(\tau) + \theta_{\text{fl}}(\tau)$, where $\theta_{\text{cl.}}(\tau)$ minimises the action and $\theta_{\text{fl.}}$ references the fluctuations, show that the propagator takes the form [8]

$$\langle\theta_i|e^{-\beta\hat{H}}|\theta_j\rangle = \sqrt{\frac{I}{2\pi\beta}}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{I}{2\beta}(\theta_i - \theta_j + 2\pi n)^2\right].$$

[Hint: in deriving this result, you may make use of the identity for a free particle propagator, $\displaystyle\int_{\theta_{\text{fl.}}(\beta)=\theta_{\text{fl.}}(0)=0} D\theta_{\text{fl.}}(\tau)\exp\left[-\int_0^\beta d\tau\ \frac{I}{2}\dot{\theta}_{\text{fl.}}^2\right] = \sqrt{\frac{I}{2\pi\beta}}$.]

(c) Applying the Poisson summation formula for a general function $f(n)$, [7]

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\phi\ e^{2\pi i m\phi} f(\phi),$$

where the summations run over integers n and m, show that

$$\langle\theta_i|e^{-\beta\hat{H}}|\theta_j\rangle = \sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\theta_j-\theta_i)}\exp\left[-\beta\frac{m^2}{2I}\right].$$

How can this result can be related to the eigenfunctions and eigenvalues of the Hamiltonian?

6.2 Answers

1. Applying the Holstein-Primakoff transformation, the spin commutation relations take the form

$$\begin{aligned}
\frac{1}{2S}[\hat{S}^+, \hat{S}^-] &= \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2} a a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2} - a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right) a \\
&= \left(1 - \frac{a^\dagger a}{2S}\right) + a^\dagger a \left(1 - \frac{a^\dagger a}{2S}\right) - a^\dagger a + \frac{a^\dagger a^\dagger a a}{2S} \\
&= 1 - \frac{a^\dagger a}{S}.
\end{aligned}$$

With $\hat{S}^z = S - a^\dagger a$, we obtain the required commutation relation $[\hat{S}^+, \hat{S}^-] = 2S^z$. [7]

In the semi-classical limit $S \gg 1$, the transformation can be expanded as

$$\hat{S}^z = S - a^\dagger a, \qquad \hat{S}^- \simeq (2S)^{1/2} a^\dagger, \qquad \hat{S}^+ \simeq (2S)^{1/2} a.$$

In this approximation, the Heisenberg Hamiltonian takes the form [9]

$$\begin{aligned}
\hat{H} &= -J \sum_n \left\{ \hat{S}^z_n \hat{S}^z_{n+1} + \frac{1}{2}\left(\hat{S}^+_n \hat{S}^-_{n+1} + \hat{S}^-_n \hat{S}^+_{n+1} \right) \right\} \\
&= -J \sum_n \left\{ S^2 + S\left(a^\dagger_n a_{n+1} + a_n a^\dagger_{n+1} - a^\dagger_n a_n - a^\dagger_{n+1} a_{n+1} \right) + O(S^0) \right\}.
\end{aligned}$$

Keeping fluctuations at leading order in S, the quadratic Hamiltonian can be diagonalised by Fourier transformation. Applying the periodic boundary conditions: $\hat{\mathbf{S}}_{n+N} = \hat{\mathbf{S}}_n$, and $a_{n+N} = a_n$, where N denotes the total number of lattice sites,

$$a_k = \frac{1}{\sqrt{N}} \sum_n e^{ikx_n} a_n, \qquad \left[a_k, a^\dagger_q\right] = \delta_{k,q}.$$

Noting the identity $\sum_n e^{i(k-q)x_n} = N\delta_{k,q}$, the Hamiltonian for a one-dimensional lattice takes the form [9]

$$\hat{H} = -JNS^2 + \sum_k \hbar\omega_k a^\dagger_k a_k + O(S^0),$$

where $\hbar\omega_k = 2JS(1 - \cos k)$ represents the dispersion relation.

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2. (a) **Classical Chain**: The discrete Lagrangian of N-atom chain

$$L = \sum_n^N \left(\frac{m}{2}\dot{\phi}_n^2 - \frac{k_s}{2}\left(\phi_{n+1} - \phi_n\right)^2 \right),$$

where $x_n(t) = \bar{x}_n + \phi_n(t)$. Applying the continuum limit,

$$L[\phi] = \int dx \, \left[\frac{m}{2}\dot{\phi}^2 - \frac{k_s a^2}{2}(\partial_x \phi)^2\right].$$

Minimising L with respect to variations in ϕ, the Euler-Lagrange equations yield the continuum wave equations of motion, $(m\partial_t^2 - k_s a^2 \partial_x^2)\phi = 0$.

Quantum Chain: Applied to the classical Hamiltonian density $\mathcal{H}[\phi, \pi] \equiv \pi\dot{\phi} - \mathcal{L}[\partial_x \phi, \dot{\phi}, t]$ where $\mathcal{L}[\partial_x \phi, \dot{\phi}, t]$ denotes the Lagrangian density, the classical Hamiltonian takes the form

$$H \equiv \int dx \frac{1}{2m}\pi^2 + \frac{k_s a^2}{2}(\partial_x \phi)^2,$$

where $\pi = \delta_{\dot{\phi}}\mathcal{L} = m\dot{\phi}$. The latter is quantised under the transformation $\phi \longrightarrow \hat{\phi}$, $\pi \longrightarrow \hat{\pi}$, subject to the canonical commutation relations $[\hat{\pi}(x), \hat{\phi}(x')] = -i\hbar\delta(x-x')$.

Subjected to a Fourier transform:

$$\hat{\phi}_k \equiv \frac{1}{L^{1/2}} \int dx e^{ikx}\hat{\phi}(x), \qquad \hat{\phi}(x) = \frac{1}{L^{1/2}} \sum_k e^{-ikx}\hat{\phi}_k,$$

(similarly $\hat{\pi}_k$) where $L = Na$ denotes the length of the system and $\sum_k$ is the sum over all quantised momenta $k = 2\pi m/L$, $m \in \mathcal{Z}$, the Hamiltonian density take the form

$$\hat{H} = \sum_k \hat{\mathcal{H}}_k, \qquad \hat{\mathcal{H}}_k = \frac{1}{2m}\hat{\pi}_k\hat{\pi}_{-k} + \frac{k_s a^2}{2}k^2\hat{\phi}_k\hat{\phi}_{-k},$$

and the Fourier transformed equal time commutation relations read $[\hat{\pi}_k, \hat{\phi}_{k'}] = -i\hbar\delta_{k,-k'}$.

In this form, the Hamiltonian can be identified as nothing but a superposition of independent harmonic oscillators. Defining

$$a_k \equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_k}{\hbar}}\hat{\phi}_k + i\frac{1}{\sqrt{m\hbar\omega_k}}\hat{\pi}_k\right),$$

(similarly $a_k^\dagger$) where $\omega_k = a(k_s/m)^{1/2}k = vk$ denotes the spectrum of energies, and v denotes the classical sound wave velocity,

$$\hat{H} = \sum_k \hat{\mathcal{H}}_k, \qquad \hat{\mathcal{H}}_k = \hbar\omega_k\left(a_k^\dagger a_k + \frac{1}{2}\right).$$

Similarly one can show that the a's obey the *bosonic* commutation relations $[a_k, a_{k'}^\dagger] = \delta_{kk'}$.

(b) The simplest model describing the coupling of electron degrees of freedom to a bath of phonons is expressed by through the Hamiltonian

$$\hat{H} = \sum_k \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_q \hbar\omega_q a^\dagger_q a_q + \sum_{k,q} \left(M c^\dagger_{k+q\sigma} c_{k\sigma} a_q + \text{h.c.} \right)$$

where ϵ_k and ω_q represent the dispersion relation of the electrons and phonons respectively, and M represents the scattering amplitude coupling the degrees of freedom.

Treating the coupling as a perturbation $\hat{H}_1$ to the bare non-interacting Hamiltonian $\hat{H}_0$, we can apply the canonical transformation $\hat{H} \longmapsto \hat{H}' = s^{-\hat{S}} \hat{H} e^{\hat{S}}$. Terms 1st order in M can be eliminated by setting $\hat{H}_1 + [\hat{H}_0, \hat{S}] = 0$. At second order, we obtain the effective Hamiltonian

$$\hat{H}' = \hat{H}_0 + \frac{1}{2}\left[\hat{H}_1, \hat{S}\right] + O(M^3).$$

Resolved into the complete set of basis states of the non-interacting Hamiltonian ($|\{n\}\rangle = |\{k\}\rangle \otimes |\{q\}\rangle$, $\{E_n\} = \{\epsilon_k + \hbar\omega_q\}$), we find

$$\langle n|\hat{S}|m\rangle = \frac{\langle n|\hat{H}_1|m\rangle}{E_m - E_n},$$

from which we obtain the effective Hamiltonian

$$\hat{H}' = \hat{H}_0 + \sum_p \langle n|\hat{H}_1|p\rangle\langle p|\hat{H}_1|m\rangle \left[\frac{1}{E_m - E_p} + \frac{1}{E_n - E_p}\right].$$

Written explicitly in terms of the electron operators, the effective Hamiltonian takes the form

$$\hat{H}' = \hat{H}_0 + \sum_{kk'q} \frac{|M|^2}{2} \left\{ \frac{1}{(\epsilon_k - \epsilon_{k-q}) - \hbar\omega_q} + \frac{1}{(\epsilon_{k'+q} - \epsilon_{k'}) - \hbar\omega_q} \right\} c^\dagger_{k'+q\sigma'} c_{k'\sigma'} c^\dagger_{k-q\sigma} c_{k\sigma}.$$

A rearrangement of the momenta summatations gives

$$\hat{H}' = \hat{H}_0 + \sum_{kk'q} |M|^2 \frac{\hbar\omega_q}{(\epsilon_k - \epsilon_{k-q})^2 - \hbar^2\omega_q^2} c^\dagger_{k'+q\sigma'} c_{k'\sigma'} c^\dagger_{k-q\sigma} c_{k\sigma}.$$

The electron-electron interaction is attractive (negative) for excitation energies $|\epsilon_k - \epsilon_{k-q}| < \hbar\omega_q$, and repulsive otherwise. This model, which is known as the BCS Hamiltonian describes an instability of the electron gas towards the formation of a 'paired condensate' and is a paradigm for studies of superconductivity.

(c) To illustrate the instanton method we consider the transition amplitudes

$$G(a, \pm a; \tau) \equiv \langle \pm a|e^{-\tau\hat{H}/\hbar}|a\rangle = G(-a, \mp a; \tau)$$

where $\pm a$ are the two minima of a symmetric double well potential. The Euclidean path integral formulation of the transition amplitudes is given by

$$G(a, \pm a; \tau) = \int_{q(0)=\pm a, q(\tau)=a} Dq e^{-\frac{1}{\hbar}\int_0^\tau d\tau' \left(\frac{m}{2}\dot{q}^2 + V(q)\right)},$$

where the function q now depends on imaginary time. Minimising the action we obtain the stationary phase equations $-m\ddot{q} + V'(q) = 0$. There are three different types of classical solutions which fullfil the condition to be at coordinates $\pm a$ at times 0 and/or τ: (a) The solution wherein the particle rests permanently at a, (b) the corresponding solution staying at $-a$ and (c) the solution in which the particle leaves its initial position at $\pm a$, accelerates through the minimum at 0 and eventually reaches the final position $\mp a$ at τ — the "instanton" trajectory.

Taking into account multi-instanton configurations the transition amplitude is given by

$$G(a, \pm a; \tau) \simeq \sum_{n \text{ even / odd}} A_n,$$

where A_n denotes the amplitude associated with n instantons present, and we have taken into account the fact that in order to connect a with $\pm a$, the number of instantons must be even/odd. The n instanton bounces contributing to each A_i can take place at arbitrary times $\tau_i \in [0, \tau]$, $i = 1, \ldots, n$ and all these possibilities have to be added (integrated). Quantitatively

$$A_n = K^n \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_i A_i(\tau_1, \ldots, \tau_n),$$

where K is a (dimensionful) constant absorbing the temporal dimension $[\text{time}]^n$ introduced by the time integrations, and $A_n(\tau_1, \ldots, \tau_n)$ is the transition amplitude, evaluated within the semiclassical approximation around a configuration of n instanton bounces at times $0 \leq \tau_1 < \tau_2 < \ldots < \tau_n \leq \tau$.

According to the general semiclassical principle, each amplitude $A_n = A_{n,\text{cl.}} A_{n,\text{qu.}}$ factorizes into two parts, a classical contribution $A_{n,\text{cl.}}$ accounting for the action of the instanton configuration, and a quantum contribution $A_{n,\text{qu.}}$ resulting from quadratic fluctuations around the classical path. Focusing initially on $A_{n,\text{cl.}}$, we note that at intermediate times, $\tau_i \ll \tau' \ll \tau_{i+1}$, where the particle rests on top of either of the maxima at $\pm a$, no action accumulates. However, each instanton bounce has a finite action $S_{\text{inst.}}$ and these contributions add up to give the full classical action,

$$A_{n,\text{cl.}}(\tau_1, \ldots, \tau_n) = e^{-nS_{\text{inst.}}/\hbar},$$

which is independent of the time coordinates τ_i. Quantum fluctuations give rise to a further contribution

$$A_{n,\text{qu.}}(\tau_1, \ldots, \tau_n) = \prod_{i=0}^{n} e^{-\omega(\tau_{i+1}-\tau_i)/2} = e^{-\omega\tau/2},$$

again independent of the particular spacing configuration $\{\tau_i\}$. Combining, we find

$$A_n \simeq K^n e^{-nS_{\rm inst.}/\hbar} e^{-\omega\tau/2} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n = e^{-\omega\tau/2} \frac{1}{n!} \left(\tau K e^{-S_{\rm inst.}/\hbar}\right)^n .$$

Substituting this result, we finally obtain

$$\begin{aligned} G(a,a;\tau) &\simeq Ce^{-\omega\tau/2} \cosh\left(\tau K e^{-S_{\rm inst.}/\hbar}\right), \\ G(a,-a;\tau) &\simeq Ce^{-\omega\tau/2} \sinh\left(\tau K e^{-S_{\rm inst.}/\hbar}\right), \end{aligned}$$

where C is some factor that depends in a non-exponential way on the transition time.

[Credit will be given for an accurate discussion of just qualitative elements of the theory. Additional credit will be given for attaching a simple physical interpretation to the form of the solution.]

(d)

(e) For a free particle, the plane wave $\psi(x) = \exp[-ip\cdot x]u(p)$ with energy $E \equiv p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$ will be a solution of the Dirac equation

$$[\gamma^\mu \hat{p}_\mu - m]\,\psi = 0$$

if the components of the spinor $u(p)$ are chosen to satisfy the equation $(\gamma^\mu p_\mu - m)u(p) = 0$. In the Dirac-Pauli block representation,

$$\gamma^\mu p_\mu - m = \begin{pmatrix} p^0 - m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -p^0 - m \end{pmatrix},$$

from which we obtain the spinor elements

$$u^{(r)}(p) = N(p) \begin{pmatrix} \chi_r \\ \frac{\sigma\cdot\mathbf{p}}{p^0+m}\chi_r \end{pmatrix},$$

where χ_r represents a pair of orthogonal two-component vectors, and $N(p)$ is the normalisation.

In the eigenbasis of Helicity, the positive energy states can be written in the form

$$\psi_p^{(\pm)}(x) = N(p)e^{-ip\cdot x} \begin{pmatrix} \chi_\pm \\ \pm\frac{|\mathbf{p}|}{p^0+m}\chi_\pm \end{pmatrix}.$$

Klein Paradox: Suppose a beam of relativistic particles with unit amplitude, energy E, momentum $p\hat{\mathbf{e}}_1$, and spin $\uparrow$ (i.e. $\chi = (1,0)$), is incident upon a potential $V(\mathbf{x}) = V\Theta(x_1)$. At the potential barrier, spin is conserved, while a component r of the beam is reflected (with energy E and momentum $-p\hat{\mathbf{e}}_1$), and a component t is transmitted with energy $E' = E - V$ and momentum $p'\hat{\mathbf{e}}_1$. According to the energy-momentum invariant, $E^2 = p^2 + m^2$ and $E'^2 = p'^2 + m^2$.

Being first order, the boundary conditions on the Dirac equation require only continuity of ψ.

$$\begin{pmatrix} 1 \\ 0 \\ p/(E+m) \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ 0 \\ -p/(E+m) \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ p'/(E'+m) \\ 0 \end{pmatrix}$$

Solving for r and t, we find the ratio of the reflected to transmitted intensity to be

$$r = \frac{1-\zeta}{1+\zeta}, \qquad t = \frac{2}{1+\zeta}, \qquad \zeta = \frac{p'}{p}\frac{(E+m)}{(E'+m)}.$$

Taking each energy interval separately

- $E' \equiv (E-V) > V$: In this case $p'^2 \equiv E'^2 - m^2 > 0$, and (taking $p' > 0$) $|r|^2 < 1$. Unsurprisingly, for sufficient energy of incident particles, a component of the beam is transmitted and the remainder is reflected.
- $-m < E' < m$: $p'^2 < 0$, and $|r|^2 = 1$. In this case, the under barrier solutions are evanescent and quickley decay to the right of the barrier.
- $E' < -m$: $p'^2 > 0$ and, depending on the sign of p', $|r|^2$ can be greater or less than unity. But the solution has the form $e^{-i(p'x - E't)}$. Since we presume the beam to be propagating to the right, we require $E' < 0$ and $p' > 0$. This implies that $|r|^2 > 1$!!

The third regime in which $|r|^2 > 1$ looks strange. However, the solution to this paradox is apparent if we study the current density in the same regime. Applying the definition $\mathbf{j} = \psi^\dagger \alpha \psi$, we obtain the following relations for the ratio of transmitted and reflected current density to the incident:

$$\frac{j_t}{j_i} = \frac{4\zeta}{(1+\zeta)^2}, \qquad \frac{j_r}{j_i} = \left(\frac{1-\zeta}{1+\zeta}\right)^2 = 1 - \frac{j_t}{j_i}.$$

This shows that while current conservation is always satisfied, for $E' < -m$, when $\zeta < 0$, the transmitted current density is negative — it is as if a beam of particles were incident from the right. However, since $\rho < 0$, we should interpret this as a beam of antiparticles (of negative charge) being emitted at $x_1 = 0$ to $x_1 = \infty$.

So the physical interpretation is that the incoming beam stimulates the emission of a beam of particles and antiparticles from the step — the particles moving off towards $x_1 = -\infty$ and the antiparticles towards $x_1 = \infty$. What about energy conservation? One might worry that the energy for these pairs is coming from nowhere. However, the electrostatic energy recovered by the antiparticle when its created is sufficient to outweigh the rest mass energy of the particle and antiparticle pair (remember that a repulsive potential for particles is attractive for antiparticles).

Negative Energy States: With this in mind we should go back to the definition of the free particle plane wave state. In particular, for energies $E < 0$, we should set

$p^0 = +\sqrt(\mathbf{p}^2+m^2)$, and redefine the plane wave solution in the form $\psi(x) = v(p)e^{ip\cdot x}$, where the spinor satisfies the c ondition $(\gamma^\mu p_\mu + m)v(p) = 0$. Accordingly we find,

$$v^{(r)}(p) = N(p)\begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{p^0+m}\chi_r \\ \chi_r \end{pmatrix}.$$

---

3. In the limit $\hbar \to 0$, the path integral is dominated by the stationary minimum of the classical action $S = \int dt' L$ satisfying the Euler-Lagrange equations of motion

$$\partial_t(\partial_{\dot q} L) - \partial_q L = 0.$$

(a) For the harmonic oscillator subject to the boundary conditions $q_I = q_F = 0$ the minimum action is obtained for the classical trajectory $\bar q(t') = 0$, with an action $S[\bar q] = 0$. Parametrising the fluctuations according to $q(t) = \sum_{n=1}^{\infty} q_n \sin(\pi n t/t')$, the action takes the form

$$S[q] = \frac{m}{2}\sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{t'}\right)^2 - \omega^2\right] r_n^2.$$

Integrating over the degrees of freedom r_n, the path integral takes the form

$$\langle 0|e^{-i\hat H t/\hbar}|0\rangle = J\prod_{n=1}^{\infty}\left[\left(\frac{n\pi}{t'}\right)^2 - \omega^2\right]^{-1/2},$$

where the constant J is determined by comparison with the free particle propagator

$$\langle 0|e^{-i\hat H t/\hbar}|0\rangle = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2}$$

Apply this result and the identity given in the question, we obtain the required result.

(b) Using the complete position state basis, the quantum partition function can be expressed as

$$\mathcal{Z} = \int_{-\infty}^{\infty} dq \langle q|e^{-\beta\hat H}|q\rangle.$$

Identifying the inverse temperature with the time $\beta = it/\hbar$, and applying the Gaussian integration to the path integral formula given in the text, we obtain

$$\mathcal{Z} = \frac{1}{2\sin(\beta\hbar\omega)}.$$

Notice that the latter can be compared with the geometric series

$$\mathcal{Z} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\hbar\omega}.$$

---

4. (a) Making use of the definition given in the text

$$\begin{aligned}[][\alpha_{\mathbf{k}\uparrow}, \alpha^\dagger_{\mathbf{k}'\uparrow}]_+ &= [\cos\theta_{\mathbf{k}} c_{\mathbf{k}\uparrow} + \sin\theta_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}, \cos\theta_{\mathbf{k}'} c^\dagger_{\mathbf{k}'\uparrow} + \sin\theta_{\mathbf{k}'} c_{-\mathbf{k}'\downarrow}]_+ \\ &= \left(\cos^2\theta_{\mathbf{k}} + \sin^2\theta_{\mathbf{k}}\right)\delta_{\mathbf{k}\mathbf{k}'} = \delta_{\mathbf{k}\mathbf{k}'}\end{aligned}$$

Similarly

$$\begin{aligned}[][\alpha_{-\mathbf{k}\downarrow}, \alpha^\dagger_{\mathbf{k}'\uparrow}]_+ &= [\sin\theta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} - \cos\theta_{\mathbf{k}} c_{-\mathbf{k}\downarrow}, \cos\theta_{\mathbf{k}'} c^\dagger_{\mathbf{k}'\uparrow} + \sin\theta_{\mathbf{k}'} c_{-\mathbf{k}'\downarrow}]_+ \\ &= (\sin\theta_{\mathbf{k}}\cos\theta_{\mathbf{k}} - \cos\theta_{\mathbf{k}}\sin\theta_{\mathbf{k}})\,\delta_{\mathbf{k}\mathbf{k}'} = 0\end{aligned}$$

(b) The unitary transformation

$$\chi^\dagger_{\mathbf{k}} \equiv \begin{pmatrix} \alpha^\dagger_{\mathbf{k}\uparrow} \\ \alpha_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow} \end{pmatrix} \equiv U\psi^\dagger_{\mathbf{k}}$$

obtains

$$\begin{aligned}\hat{H} - \mu\hat{N} &= \sum_{\mathbf{k}} \chi^\dagger_{\mathbf{k}} \begin{pmatrix} \zeta_{\mathbf{k}}\cos(2\theta_{\mathbf{k}}) - \Delta\sin(2\theta_{\mathbf{k}}) & \zeta_{\mathbf{k}}\sin(2\theta_{\mathbf{k}}) + \Delta\cos(2\theta_{\mathbf{k}}) \\ \zeta_{\mathbf{k}}\sin(2\theta_{\mathbf{k}}) + \Delta\cos(2\theta_{\mathbf{k}}) & -\zeta_{\mathbf{k}}\cos(2\theta_{\mathbf{k}}) + \Delta\sin(2\theta_{\mathbf{k}}) \end{pmatrix} \chi_{\mathbf{k}} \\ &\quad + \sum_{\mathbf{k}} \zeta_{\mathbf{k}} + \frac{\Delta^2}{g}\end{aligned}$$

(c) The latter is diagonalised by the condition $\tan(2\theta_{\mathbf{k}}) = -\Delta/\zeta_{\mathbf{k}}$ where upon

$$\begin{aligned}\hat{H} - \mu\hat{N} &= \sum_{\mathbf{k}} \lambda_{\mathbf{k}} \left(\alpha^\dagger_{\mathbf{k}\uparrow}\alpha_{\mathbf{k}\uparrow} - \alpha_{-\mathbf{k}\downarrow}\alpha^\dagger_{-\mathbf{k}\downarrow}\right) + \sum_{\mathbf{k}} \zeta_{\mathbf{k}} + \frac{\Delta^2}{g} \\ &= \sum_{\mathbf{k}\sigma} \lambda_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}\sigma}\alpha_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} (\zeta_{\mathbf{k}} - \lambda_{\mathbf{k}}) + \frac{\Delta^2}{g}\end{aligned}$$

with $\lambda_{\mathbf{k}} = (\Delta^2 + \zeta_k^2)^{1/2}$

(d) That the wavefunction

$$|\text{g.s.}\rangle = \prod_{\mathbf{k}} \alpha_{\mathbf{k}\uparrow}\alpha_{-\mathbf{k}\downarrow}|\Omega\rangle \sim \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|\Omega\rangle$$

is the ground state is evident because the latter is annihilated by all quasi-particle operators. The Hamiltonian shows that the quasi-particle states have a minimum energy Δ. The spectrum has a gap.

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5. (a) The Hamiltonian of the quantum Heisenberg ferromagnet is given by

$$\begin{aligned}\hat{H} &= -J\sum_n \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1}, \\ &= -J\sum_n \left[S^z_n S^z_{n+1} + \frac{1}{2}\left(S^+_n S^-_{n+1} + S^z_- S^+_{n+1}\right)\right]\end{aligned}$$

where $\hat{\mathbf{S}}_n$ denotes the quantum spin at site n. [4]
The spin operators obey the spin commutation relations $[\hat{S}^i, \hat{S}^j] = i\epsilon^{ijk}\hat{S}^k$. [2]
The ground state spin configuration is ferromagnetic, and multiply degenerate. Choosing the spin polarisation to be along the $\hat{e}_3$ axis, the ground state is $|\Omega\rangle = \otimes_n |S^z = S, S\rangle_n$. [2]
In the Holstein-Primakoff representation, the spin S operators can be represented in terms of bosonic operators [4]

$$\hat{S}^- = (2S)^{1/2} a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2}, \qquad \hat{S}^+ = (\hat{S}^-)^\dagger, \qquad \hat{S}^z = S - a^\dagger a$$

where the operators $a^\dagger$ create spin deviations away from the $\hat{e}_3$ axis. In the semiclassical limit $S \gg 1$, an expansion obtains [3]

$$\hat{S}^- \simeq (2S)^{1/2} a^\dagger, \qquad \hat{S}^+ = (\hat{S}^-)^\dagger, \qquad \hat{S}^z = S - a^\dagger a.$$

In this approximation, the Hamiltonian takes the form [2]

$$\hat{H} = -JNS^2 - JS\sum_n \left[a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n - a_n^\dagger a_n - a_{n+1}^\dagger a_{n+1}\right] + O(S^0).$$

Switching to the Fourier basis

$$a_n = \frac{1}{\sqrt{N}} \sum_k e^{-ikn} a_k, \qquad a_k = \frac{1}{\sqrt{N}} \sum_n e^{ikn} a_n,$$

$[a_k, a_{k'}^\dagger] = \delta_{kk'}$, the Hamiltonian takes the form

$$\hat{H} = -JNS^2 + \sum_k \hbar\omega_k a_k^\dagger a_k + O(S^0)$$

where $\hbar\omega_k = 2JS(1 - \cos k)$. [6]
In the hydrodynamic limit $\hbar\omega_k = JSk^2$. [3]

(b) The basic idea behind Feynman's path integral approach is the discretisation of the time-evolution operator according to

$$e^{-i\hat{H}t/\hbar} = \left[e^{-i\hat{H}\Delta t/\hbar}\right]^N$$

where $\Delta t = t/N$. [4]
Making use of the Baker-Campbell-Hausdorff formula each element can be separated according to

$$e^{-i\hat{H}\Delta t/\hbar} = e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} + O(\Delta t^2),$$

where the Hamiltonian $\hat{H} = \hat{T} + \hat{V}$ is the sum of a kinetic energy $\hat{T} = \hat{p}^2/2m$, and some potential energy operator $\hat{V}$. The key advantage of this factorisation is that the eigenstates of each of the factors $e^{-i\hat{T}\Delta t/\hbar}$ and $e^{-i\hat{V}\Delta t/\hbar}$ are known independently. [3]

To exploit this fact we consider the time evolution operator factorised as a product, [3]

$$\left\langle q_F \left| \left[e^{-i\hat{H}\Delta t/\hbar} \right]^N \right| q_I \right\rangle$$
$$\simeq \left\langle q_F \left| {}_\wedge e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} {}_\wedge e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} {}_\wedge \cdots {}_\wedge e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} \right| q_I \right\rangle$$

and insert at each of the positions indicated by a $\wedge$ the resolution of identity

$$\text{id.} = \int dq_n \int dp_n |q_n\rangle\langle q_n|p_n\rangle\langle p_n|,$$

where $|q_n\rangle$ and $|p_n\rangle$ represent a complete set of position and momentum eigenstates respectively, and $n = 1, \ldots, N$ serves as an index keeping track of the time steps at which the unit operator is inserted. [3]

Using the identity [2]

$$\langle q|p\rangle = \frac{1}{2\pi\hbar} e^{iqp/\hbar},$$

one obtains [3]

$$\left\langle q_F \left| e^{-i\hat{H}t/\hbar} \right| q_I \right\rangle$$
$$= \int_{q_N = q_F, q_0 = q_I} \prod_{n=1}^{N-1} dq_n \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar} e^{-i\frac{\Delta t}{\hbar}\sum_{n=0}^{N-1}\left(V(q_n) + T(p_{n+1}) - p_{n+1}\frac{q_{n+1}-q_n}{\Delta t}\right)}.$$

Sending $N \to \infty$ whilst keeping $t = N\Delta t$ fixed, the formerly discrete set $t_n = n\Delta t,\ n = 1, \ldots, N$ becomes dense on the time interval $[0, t]$, and the set of phase space points $\{\mathbf{x}_n\}$ becomes a continuous curve $\mathbf{x}(t)$. In the same limit we have [3]

$$\Delta t \sum_{n=0}^{N-1} \longrightarrow \int_0^t dt', \qquad \frac{q_{n+1} - q_n}{\Delta t} \longrightarrow \partial_{t'} q\Big|_{t'=t_n} \equiv \dot{q}|_{t'=t_n},$$
$$V(q_n) + T(p_{n+1}) \longrightarrow (T(p|_{t'=t_n}) + V(q|_{t'=t_n})) \equiv H(\mathbf{x}|_{t'=t_n}),$$
$$\lim_{N\to\infty} \int_{q_N = q_F, q_0 = q_I} \prod_{n=1}^{N-1} dq_n \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar} = \int_{\substack{q(t)=q_F \\ q(0)=q_I}} D(q, p).$$

We obtain the final form of the Feynman path integral [4]

$$\left\langle q_F \left| e^{-i\hat{H}t/\hbar} \right| q_I \right\rangle = \int_{\substack{q(t)=q_F \\ q(0)=q_I}} D(q, p) \exp\left[\frac{i}{\hbar} \int_0^t dt' \,(p\dot{q} - H(p, q)) \right]$$

(c) To illustrate the instanton method we consider the transition amplitudes

$$G(a,\pm a;\tau) \equiv \langle \pm a|e^{-\tau \hat{H}/\hbar}|a\rangle = G(-a,\mp a;\tau)$$

where $\pm a$ are the two minima of a symmetric double well potential. [2]
The Euclidean path integral formulation of the transition amplitudes is given by

$$G(a,\pm a;\tau) = \int_{q(0)=\pm a, q(\tau)=a} Dq e^{-\frac{1}{\hbar}\int_0^\tau d\tau' \left(\frac{m}{2}\dot{q}^2 + V(q)\right)},$$

where the function q now depends on imaginary time. [5]
Minimising the action with respect to variations in q, we obtain the saddle-point equation $-m\ddot{q} + V'(q) = 0$ corresponding to the classical equation of motion in an inverted potential. [2]
There are three different types of classical solutions which fullfil the condition to be at coordinates $\pm a$ at times 0 and/or τ: (a) The solution wherein the particle rests permanently at a, (b) the corresponding solution staying at $-a$ and (c) the solution in which the particle leaves its initial position at $\pm a$, accelerates through the minimum at 0 and eventually reaches the final position $\mp a$ at τ — the "instanton" trajectory. [3]
Taking into account multi-instanton configurations the transition amplitude is given by

$$G(a,\pm a;\tau) \simeq \sum_{n \text{ even / odd}} K^n \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_i A_n(\tau_1,\ldots,\tau_n)$$

where the sum involves the Feynman amplitude associated with n instantons present, and we have taken into account the fact that in order to connect a with $\pm a$, the number of instantons must be even/odd. The n instanton bounces contributing to each factor can take place at arbitrary times $\tau_i \in [0,\tau]$, $i = 1,\ldots,n$ and all these possibilities have to be added (integrated). K is a (dimensionful) constant absorbing the temporal dimension $[\text{time}]^n$ introduced by the time integrations, and $A_n(\tau_1,\ldots,\tau_n)$ is the transition amplitude, evaluated within the semiclassical approximation around a configuration of n instanton bounces at times $0 \le \tau_1 < \tau_2 < \ldots < \tau_n \le \tau$. [5]
According to the general semiclassical principle, each amplitude $A_n = A_{n,\text{cl.}} A_{n,\text{qu.}}$ factorizes into two parts, a classical contribution $A_{n,\text{cl.}}$ accounting for the action of the instanton configuration, and a quantum contribution $A_{n,\text{qu.}}$ resulting from quadratic fluctuations around the classical path. Focusing initially on $A_{n,\text{cl.}}$, we note that at intermediate times, $\tau_i \ll \tau' \ll \tau_{i+1}$, where the particle rests on top of either of the maxima at $\pm a$, no action accumulates. However, each instanton bounce has a finite action $S_{\text{inst.}}$ and these contributions add up to give the full classical action,

$$A_{n,\text{cl.}}(\tau_1,\ldots,\tau_n) = e^{-nS_{\text{inst.}}/\hbar},$$

which is independent of the time coordinates τ_i. Quantum fluctuations give rise to a further contribution

$$A_{n,\mathrm{qu.}}(\tau_1,\ldots,\tau_n)=\prod_{i=0}^{n}e^{-\omega(\tau_{i+1}-\tau_i)/2}=e^{-\omega\tau/2},$$

again independent of the particular spacing configuration $\{\tau_i\}$. Combining, we find [4]

$$A_n\simeq K^n e^{-nS_{\mathrm{inst.}}/\hbar}e^{-\omega\tau/2}\int_0^{\tau}d\tau_1\int_0^{\tau_1}d\tau_2\cdots\int_0^{\tau_{n-1}}d\tau_n=e^{-\omega\tau/2}\frac{1}{n!}\left(\tau K e^{-S_{\mathrm{inst.}}/\hbar}\right)^n.$$

Substituting this result, we finally obtain

$$\begin{aligned}G(a,a;\tau)&\simeq Ce^{-\omega\tau/2}\cosh\left(\tau K e^{-S_{\mathrm{inst.}}/\hbar}\right),\\ G(a,-a;\tau)&\simeq Ce^{-\omega\tau/2}\sinh\left(\tau K e^{-S_{\mathrm{inst.}}/\hbar}\right),\end{aligned}$$

where C is some factor that depends in a non-exponential way on the transition time. [4]

(d) If $\psi(x)$ obeys Dirac eqn. its counterpart $\psi'(x')$ in a Lorentz transformed frame $x'=\Lambda x$, must obey

$$\left(i\gamma^\mu\frac{\partial}{\partial x'^\mu}-m\right)\psi'(x')=0.$$

If an observer in the 2nd frame can reconstruct $\psi'(x')$ from $\psi(x)$ there must exist a local (linear) transformation $\psi'(x')=S(\Lambda)\psi(x)$ where $S(\Lambda)$ is a non-singular 4×4 matrix. [4]

Using $\partial x^\nu/\partial x'^\mu=(\Lambda^{-1})^\nu{}_\mu$, and

$$\left(i\gamma^\mu\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}-m\right)S(\Lambda)\psi(x)=0.$$

the latter is compatible with the Dirac equation if [4]

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda)=(\Lambda^{-1})^\mu{}_\nu\gamma^\nu$$

For an infinitesimal proper Lorentz transformation

$$\Lambda^\mu{}_\nu=g^\mu{}_\nu+\omega^\mu{}_\nu,\qquad(\Lambda^{-1})^\mu{}_\nu=g^\mu{}_\nu-\omega^\mu{}_\nu+\cdots,$$

with $\omega_{\mu\nu}$ antisymmetric. Expansion of $S(\Lambda)$ obtains

$$S(\Lambda)=\mathbf{1}-\frac{i}{4}\Sigma_{\mu\nu}\omega^{\mu\nu}+\cdots,\qquad S^{-1}(\Lambda)=\mathbf{1}+\frac{i}{4}\Sigma_{\mu\nu}\omega^{\mu\nu}+\cdots$$

where $\Sigma_{\mu\nu}$ is antisymmetric in $\mu\nu$. [4]

Requiring $S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu$,

$$\begin{aligned}\left(\mathbf{1} - \frac{i}{4}\Sigma_{\mu\nu}\omega^{\mu\nu} + \cdots\right)\gamma^\nu\left(\mathbf{1} - \frac{i}{4}\Sigma_{\mu\nu}\omega^{\mu\nu}\right) &= \gamma^\eta\left(\delta^\nu{}_\eta - \omega^\nu{}_\eta\right) + \cdots \\ \gamma^\nu - \frac{i}{4}\left[\Sigma_{\mu\eta}\omega^{\mu\eta}, \gamma^\nu\right] + \cdots &= \gamma^\nu - \gamma^\eta\omega^\nu{}_\eta + \cdots \\ \left[\Sigma_{\mu\eta}, \gamma^\nu\right]\omega^{\mu\eta} = -4i\gamma_\eta\omega^{\nu\eta} &= -4i\gamma_\eta\delta^\mu{}_\nu\omega^{\mu\eta}\end{aligned}$$

Therefore $[\Sigma_{\mu\eta}, \gamma^\nu]\,\omega^{\mu\eta} = 2i\left(\gamma_\mu\delta^\eta{}_\nu - \gamma_\eta\delta^\mu{}_\nu\right)\omega^{\mu\eta}$. This equation is satisfied by $\Sigma_{\alpha\beta} = \frac{i}{2}\left[\gamma_\alpha, \gamma_\beta\right]$. [4]

For infinitesimal spatial rotation (angle θ about axis $\mathbf{n}$), $\omega^i{}_j = -\epsilon^{ijk}n_k\theta$

$$\Lambda^{ij}x_j = (\delta^{ij} + \omega^{ij})x_j = x^i - \epsilon^{ijk}x_j n_k\theta, \qquad \Lambda^\mu{}_0 = \Lambda^0{}_\mu = 0$$

Applied to argument of wavefunction

$$\begin{aligned}\psi(x') &= \psi(\Lambda^{-1}x) = \psi(x^0, \mathbf{x} + \mathbf{x}\times\mathbf{n}\theta) = (1 - \theta\mathbf{n}\cdot\mathbf{x}\times\nabla)\psi(x) \\ &= (1 - i\theta\mathbf{n}\cdot\hat{\mathbf{L}})\psi(x),\end{aligned}$$

where $\hat{\mathbf{L}} = \hat{\mathbf{x}}\times\hat{\mathbf{p}}$ denotes angular momentum operator. Similarly

$$S(\Lambda) = 1 - \frac{i}{4}\epsilon^{ijk}n_k\Sigma_{ij}\theta = 1 + \frac{1}{8}\epsilon^{ijk}n_k[\gamma_i, \gamma_j]\theta$$

In Dirac/Pauli representation $\Sigma_{ij} = \epsilon_{ijk}\sigma^k$ where σ_k denote Pauli spin matrices

$$[\gamma_i, \gamma_j] = -2i\epsilon_{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$$

i.e. $S(\Lambda) = 1 - i\mathbf{n}\cdot\mathbf{S}\theta$ where $S^k = \sigma^k/2$. [6]

Altogether

$$\psi'(x') = S(\Lambda)\psi(\Lambda^{-1}x) = (1 - i\theta\mathbf{n}\cdot\hat{\mathbf{J}})\psi(x)$$

where $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \mathbf{S}$ represents the total angular momentum. [3]

(e) If ϕ were a classical field, the Klein-Gordon equation would represent the equation of motion associated with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2,$$

Defining the canonical momentum $\pi(x) = \partial\mathcal{L}/\partial\dot{\phi}(x) = \dot{\phi}(x) \equiv \partial_0\phi(x)$, the corresponding Hamiltonian density takes the form [4]

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\pi^2 + (\partial_i\phi)^2 + m^2\phi^2\right].$$

In the Heisenberg representation, the quantisation of the fields is still enforced by promoting the classical fields to operators, $\pi \mapsto \hat{\pi}$ and $\phi \mapsto \hat{\phi}$, and imposing the equal time commutation relations,

$$\begin{aligned} \left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{x}',t)\right] &= i\delta^3(\mathbf{x}-\mathbf{x}'), \\ \left[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{x}',t)\right] &= [\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{x}',t)] = 0, \end{aligned}$$

with $\hat{\pi} = \partial_0 \hat{\phi}$, after which the Hamiltonian takes the form [3]

$$\hat{\mathcal{H}} = \frac{1}{2}\left[\hat{\pi}^2 + (\partial_i \hat{\phi})^2 + m^2 \hat{\phi}^2\right].$$

Applying the Fourier transform the field operators to obtain the normal modes of the Hamiltonian

$$\hat{\phi}(x) = \int \frac{d^4k}{(2\pi)^4} \widetilde{\phi}(k) e^{-ik\cdot x}.$$

Taking into account the Hermiticity of the operators $\hat{\phi}(x)$, one finds that $\widetilde{\phi}^\dagger(k) = \widetilde{\phi}(-k))$. Moreover, for $\phi(k)$ to satisfy the Klein-Gordon equation, $\widetilde{\phi}(k) = 2\pi\delta(k^2 - m^2)(\theta(k^0)a(\mathbf{k}) + \theta(-k^0)a^\dagger(\mathbf{k})$ where $k^0 = \omega_k \equiv +\sqrt{\mathbf{k}^2 + m^2}$, and $a(\mathbf{k})$ represent the operator valued Fourier coefficients. With this choice it is easy to confirm that the Lorentz covariance of

$$\hat{\phi}(x) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k^0)\left[a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right]$$

is guaranteed. [4]

Integrating over k^0, one obtains

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k}\left(a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right).$$

More compactly, making use of the orthonormality of the basis

$$f_k = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ik\cdot x}, \qquad \int f_k^*(x) i \overleftrightarrow{\partial_0} f_{k'}(x) d^3x = \delta^3(\mathbf{k} - \mathbf{k}'),$$

where $A \overleftrightarrow{\partial_0} B = A\partial_t B - (\partial A)B$, we obtain

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_k}}\left[a(\mathbf{k})f_k(x) + a^\dagger(\mathbf{k})f_k^*(x)\right].$$

Similarly, [5]

$$\hat{\pi}(x) \equiv \partial_0 \hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_k}} i\omega_k\left[-a(\mathbf{k})f_k(x) + a^\dagger(\mathbf{k})f_k^*(x)\right].$$

Making use of the orthogonality relations, the latter can be inverted to give[2]

$$\begin{aligned} a(\mathbf{k}) &= \int d^3x \sqrt{(2\pi)^3 2\omega_k} f_k^*(x) i \overleftrightarrow{\partial_0} \hat{\phi}(x), \\ a^\dagger(\mathbf{k}) &= \int d^3x \sqrt{(2\pi)^3 2\omega_k} \hat{\phi}(x) i \overleftrightarrow{\partial_0} f_k(x). \end{aligned}$$

With these results, one obtains the bosonic commutation relations [3]

$$\begin{aligned} \left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] &= (2\pi)^3 2\omega_k \delta^3(\mathbf{k} - \mathbf{k}'), \\ \left[a(\mathbf{k}), a(\mathbf{k}')\right] &= \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = 0. \end{aligned}$$

Associated with these field operators, one can defined the particle number operator

$$\hat{N} = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_k}} a^\dagger(\mathbf{k}) a(\mathbf{k}).$$

Similarly, the total energy-momentum operator for the system is given by [4]

$$\hat{P}^\mu = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_k}} k^\mu a^\dagger(\mathbf{k}) a(\mathbf{k}).$$

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6. The propagator $\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle$ represents the quantum transmission probability amplitude for a particle created at a position q_I to propagate to a position q_F in a time t.

   (a) In the phase space representation, the Feynman path integral for the harmonic oscillator Hamiltonian is given by

   $$\langle q_F | e^{-i\hat{H}t/\hbar} | q_I \rangle = \int DpDq \exp\left[\frac{i}{\hbar} S\right]$$

   where the field $q(t')$ obeys the boundary conditions $q(0) = q_I$ and $q(t) = q_F$, and

   $$S = \int_0^t dt' \left[p\dot{q} - H(p,q)\right]$$

   denotes the classical action.

   (b) Making use of this result, and identifying the Euclidean time integration with β, viz $t = -i\hbar\beta$, one obtains

   $$\langle q | e^{-\beta\hat{H}} | q \rangle = \int DpDq \exp\left[\int_0^\beta d\tau \left(\frac{i}{\hbar} p\dot{q} - H(p,q)\right)\right],$$

where $\dot{q} = \partial q/\partial\tau$. (Marks can be lost here for carelessness with factors of $\hbar$ and signs.) Here the field $q(\tau)$ is subject to the boundary condition $q(\beta) = q(0) = q$. Integrating this expression over q, one obtains the path integral representation of the of the partition function,

$$\mathcal{Z} = \int dq \langle q|e^{-\beta\hat{H}}|q\rangle.$$

(c) In the coherent state formulation, the boundary conditions are periodic $\psi(\beta) = \psi(0)$ appropriate for bosonic degrees of freedom.

(d) Substituting, and making use of the identities

$$\begin{aligned}
\int_0^\beta d\tau q\dot{q} &= \frac{1}{2}q^2|_0^\beta = 0 \\
\int_0^\beta d\tau p\dot{p} &= \frac{1}{2}p^2|_0^\beta = 0 \\
\int_0^\beta d\tau q\dot{p} &= pq|_0^\beta - \int_0^\beta d\tau q\dot{q} = \int_0^\beta d\tau p\dot{q}
\end{aligned}$$

one obtains

$$S = -\int_0^\beta d\tau \left(\frac{i}{\hbar}p\dot{q} - H(p,q)\right).$$

The latter coincides with the expression derived from the Feynman path integral.

---

7. (a) As first step in the quantisation procedure we must obtain the classical Hamiltonian. Defining the canonical momentum as

$$\pi = \partial_{\dot{\phi}}\mathcal{L} = m\dot{\phi},$$

one obtains

$$\begin{aligned}
\hat{H} &= \int_0^L dx \left[\pi\dot{\phi} - \mathcal{L}\right] = \int_0^L dx \left[\frac{\pi^2}{2m} + \frac{k_s a^2}{2}(\partial_x\phi)^2\right] \\
&= \int_0^L dx \left[\frac{\pi^2}{2m} + \frac{k_s a^2}{2}(\partial_x\phi)^2\right]
\end{aligned}$$

Promoting the fields to operators $\pi \mapsto \hat{\pi}$ and $\phi \mapsto \hat{\phi}$ where the latter obey the commutation relations $[\hat{\pi}(x), \hat{\phi}] = -i\hbar\delta(x - x')$, one obtains the Hamiltonian as given.

(b) Transforming to Fourier space,

$$\hat{\phi}(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \hat{\phi}_k, \qquad \hat{\pi}(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \hat{\pi}_k,$$

with the inverse transformation

$$\hat{\phi}_k = \frac{1}{\sqrt{L}} \int_0^L dx\ e^{-ikx} \hat{\phi}(x), \qquad \hat{\pi}_k = \frac{1}{\sqrt{L}} \int_0^L dx\ e^{ikx} \hat{\pi}(x),$$

one finds the commutation relations

$$[\hat{\pi}_k, \hat{\phi}_{k'}] = \frac{1}{L} \int_0^L dx \int_0^L dx'\ e^{-ikx+ik'x'} [\hat{\pi}(x), \hat{\phi}(x')]$$

$$= -\frac{i\hbar}{L} \int_0^L dx \int_0^L dx'\ e^{-ikx+ik'x'} \delta(x-x') = -i\hbar \overbrace{\frac{1}{L} \int_0^L dx\ e^{-i(k-k')x'}}^{\delta_{kk'}} = -i\hbar \delta_{kk'}.$$

To fulfill the periodic boundary conditions, the Fourier components take the values $k = 2\pi m/L$ where m is integer. Using the definition above, the kinetic component of the Hamiltonian takes the form

$$\int_0^L dx\ \frac{\hat{\pi}^2}{2m} = \sum_{kk'} \overbrace{\frac{1}{L} \int_0^L dx\ e^{i(k+k')x}}^{\delta_{k+k',0}} \frac{1}{2m} \hat{\pi}_k \hat{\pi}_{k'} = \sum_k \frac{1}{2m} \hat{\pi}_k \hat{\pi}_{-k}.$$

Similarly, applied to the potential component one obtains the Hamiltonian as advertised.

(c) Inverting the expression for the field operators, one finds

$$\hat{\phi}_k = \left(\frac{\hbar}{2m\omega_k}\right)^{1/2} \left(a_k + a_{-k}^\dagger\right), \qquad \hat{\pi}_k = i \left(\frac{m\hbar\omega_k}{2}\right)^{1/2} \left(a_k^\dagger - a_{-k}\right).$$

When substituted into the expression for the Hamiltonian, and expanding, one obtains

$$\hat{H} = \sum_k \left[\frac{1}{2m} \hat{\pi}_k \hat{\pi}_{-k} + \frac{k_s a^2}{2} k^2 \hat{\phi}_k \hat{\phi}_{-k}\right] = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right).$$

Using the commutation relations derived above, one finds

$$[a_k, a_{k'}^\dagger] = \frac{i}{2\hbar} \left([\hat{\phi}_{-k}, \hat{\pi}_{-k'}] - [\hat{\pi}_k, \hat{\phi}_{k'}]\right) = \delta_{kk'}.$$

(d) The Lagrangian density of the linear chain coincides with that of a free relativistic scalar field. A variation of the Lagrangian density with respect to ϕ one obtains the relativistic wave equation, the Klein-Gordon equation

$$\left(m\partial_t^2 - k_s a^2 \partial_x^2\right) \phi = 0$$

The quantisation of the linear chain coincides with the quantisation of the relativistic scalar field in the Schrödinger representation.

---

8. (a) According to the Holstein-Primakoff transformation, a quantum mechanical spin S can be expressed in terms of bosonic creation $a^\dagger$ and annihilation a operators as [8]

$$\hat{S}^- = (2S)^{1/2} a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2}, \qquad \hat{S}^+ = (\hat{S}^-)^\dagger, \qquad \hat{S}^z = S - a^\dagger a.$$

This definition is compatible with the commutation relations: $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$. [3]

To develop spin-wave theory, we introduce the one-dimensional spin S Heisenberg Quantum Ferromagnet [3]

$$\hat{H} = -J \sum_{n=1}^{N} \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1},$$

where $J > 0$, and periodic boundary conditions are imposed such that $\hat{\mathbf{S}}_{N+1} \equiv \hat{\mathbf{S}}_1$. In the semi-classical limit $S \gg 1$, the transformation can be expanded as [3]

$$\hat{S}^z = S - a^\dagger a, \qquad \hat{S}^- \simeq (2S)^{1/2} a^\dagger, \qquad \hat{S}^+ \simeq (2S)^{1/2} a.$$

In this approximation, the Heisenberg Hamiltonian takes the form [3]

$$\begin{aligned} \hat{H} &= -J \sum_n \left\{ \hat{S}^z_n \hat{S}^z_{n+1} + \frac{1}{2}\left(\hat{S}^+_n \hat{S}^-_{n+1} + \hat{S}^-_n \hat{S}^+_{n+1} \right) \right\} \\ &= -J \sum_n \left\{ S^2 + S\left(a^\dagger_n a_{n+1} + a_n a^\dagger_{n+1} - a^\dagger_n a_n - a^\dagger_{n+1} a_{n+1} \right) + O(S^0) \right\}. \end{aligned}$$

Keeping fluctuations at leading order in S, the quadratic Hamiltonian can be diagonalised by Fourier transformation. Applying the periodic boundary conditions: $\hat{\mathbf{S}}_{n+N} = \hat{\mathbf{S}}_n$, and $a_{n+N} = a_n$, where N denotes the total number of lattice sites, [5]

$$a_k = \frac{1}{\sqrt{N}} \sum_n e^{ikx_n} a_n, \qquad \left[a_k, a^\dagger_q\right] = \delta_{k,q}.$$

Noting the identity $\sum_n e^{i(k-q)x_n} = N\delta_{k,q}$, the Hamiltonian for a one-dimensional lattice takes the form

$$\hat{H} = -JNS^2 + \sum_k \hbar\omega_k a^\dagger_k a_k + O(S^0),$$

where $\hbar\omega_k = 2JS(1 - \cos k)$ represents the dispersion relation.

(b) In the real space representation, the quantum mechanical time-evolution operator or propagator [4]

$$U(q_F, q_I; t) = \langle q_F | \exp\left[-\frac{i}{\hbar}\hat{H}t\right] |q_I\rangle$$

is represented by the Feynman path integral [6]

$$U(q_F, q_I; t) = \int D(q(t), p(t)) \exp\left[\frac{i}{\hbar}S(p,q)\right], \qquad S(p,q) = \int_0^t dt' \left[p\dot{q} - H(p,q)\right],$$

where the sum is over classical phase space trajectories subject to the boundary condition $q(t) = q_F$ and $q(0) = q_I$ and weighted by the classical action.

If the kinetic part of the Hamiltonian is free, the path integral over p can be performed explicitly. In this case, the path integral reduces to the Lagrangian formulation, [4]

$$U(q_F, q_I; t) = \int Dq(t) \exp\left[\frac{i}{\hbar}S(q)\right], \qquad S(q) = \int_0^t dt' \left[\frac{m}{2}\dot{q}^2 - V(q)\right].$$

To establish an analogy with statistical mechanics we have to consider propagation in *imaginary* or *Euclidean time* T. In this way, we obtain [8]

$$U(q, q; t = -iT) = \int Dq(\tau) \exp\left[-\frac{1}{\hbar}S(q)\right],$$

where

$$S(q) = \int_0^T d\tau \left[\frac{m}{2}\left(\frac{\partial q}{\partial \tau}\right)^2 + V(q(\tau))\right].$$

Interpreting the action as a classical free energy functional, and the path integral as a classical partition function, one has the analogy: the transition amplitude for a quantum particle for the time $(-iT)$ is equal to the classical partition function for an elastic string of length T confined to a gutter-like potential $V(q)$ computed at the inverse temperature $\beta = 1/\hbar$. More generally, the quantum propagator of a d-dimensional system is equivalent to the classical statistical mechanics of a $d+1$-dimensional system.

A second analogy follows from the fact that the *quantum* partition function for the particle is given by $\mathcal{Z}_{\rm qu} = \mathrm{tr}\exp[-\beta H]$ and hence, [8]

$$\mathcal{Z}_{\rm qu} = \int dq \langle q | e^{-\beta H} | q\rangle = \int dq U(q, q; t = -i\beta\hbar).$$

Therefore, in quantum statistical mechanics, the inverse temperature plays the role of an imaginary time.

(c) Relativistic covariance: if $\psi(x)$ obeys the Dirac equation its counterpart $\psi'(x')$ in a Lorentz transformed frame $x' = \Lambda x$, must obey

$$\left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)\psi'(x') = 0.$$

If an observer can reconstruct $\psi'(x')$ from $\psi(x)$ there must exist a local (linear) transformation $\psi'(x') = S(\Lambda)\psi(x)$ where $S(\Lambda)$ is a non-singular 4×4 matrix. [5]

Using $\partial x^\nu / \partial x'^\mu = (\Lambda^{-1})^\nu{}_\mu$, and

$$\left(i\gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} - m\right)S(\Lambda)\psi(x) = 0.$$

the latter is compatible with the Dirac equation if [6]

$$S(\Lambda)\gamma^\nu S^{-1}(\Lambda) = (\Lambda^{-1})^\nu{}_\mu \gamma^\mu$$

For an infinitesimal proper Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad (\Lambda^{-1})^\mu{}_\nu = \delta^\mu{}_\nu - \omega^\mu{}_\nu + \cdots,$$

with $\omega_{\mu\nu}$ antisymmetric. Expansion of $S(\Lambda)$ obtains

$$S(\Lambda) = \mathbf{1} - \frac{i}{4}\Sigma_{\mu\nu}\omega^{\mu\nu} + \cdots, \qquad S^{-1}(\Lambda) = \mathbf{1} + \frac{i}{4}\Sigma_{\mu\nu}\omega^{\mu\nu} + \cdots$$

where $\Sigma_{\mu\nu}$ is antisymmetric in $\mu\nu$. [5]

Requiring $S(\Lambda)\gamma^\nu S^{-1}(\Lambda) = (\Lambda^{-1})^\nu{}_\mu \gamma^\mu$, after some algebra (not required as part of the question) one finds that $\Sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$. [5]

For infinitesimal spatial rotation (angle θ about axis $\mathbf{n}$), $\omega^i{}_j = -\epsilon_{ijk} n_k \theta$,

$$\psi(x) = \psi(\Lambda^{-1}x') = (1 - i\theta\mathbf{n}\cdot\hat{\mathbf{L}})\psi(x'),$$

where $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$ denotes angular momentum operator. Similarly

$$S(\Lambda) = 1 - \frac{i}{4}\epsilon_{ijk} n_k \Sigma_{ij}\theta = 1 + \frac{1}{8}\epsilon_{ijk} n_k [\gamma_i, \gamma_j]\theta$$

In Dirac/Pauli representation $\Sigma_{ij} = \epsilon_{ijk}\sigma^k$ where σ_k denote Pauli spin matrices

$$[\gamma_i, \gamma_j] = -2i\epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

i.e. $S(\Lambda) = 1 - i\mathbf{n}\cdot\mathbf{S}\theta$ where $S_k = \sigma_k/2$. [6]

Altogether

$$\psi'(x') = S(\Lambda)\psi(\Lambda^{-1}x) = (1 - i\theta\mathbf{n}\cdot\hat{\mathbf{J}})\psi(x')$$

where $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \mathbf{S}$ represents the total angular momentum. [3]

(d) For a free particle, the plane wave $\psi(x) = \exp[-ip\cdot x]u(p)$ with energy $E \equiv p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}$ will be a solution of the Dirac equation

$$[\gamma^\mu \hat{p}_\mu - m]\,\psi = 0$$

if the components of the spinor $u(p)$ are chosen to satisfy the equation $(\gamma^\mu p_\mu - m)u(p) = 0$. In the Dirac-Pauli block representation,

$$\gamma^\mu p_\mu - m = \begin{pmatrix} p_0 - m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -p_0 - m \end{pmatrix},$$

from which we obtain the spinor elements

$$u^{(r)}(p) = N(p)\begin{pmatrix} \chi^{(r)} \\ \frac{\sigma\cdot\mathbf{p}}{p_0+m}\chi^{(r)} \end{pmatrix},$$

where $\chi^{(r)}$ represents a pair of orthogonal two-component vectors, and $N(p)$ is the normalisation. In the eigenbasis of Helicity, the positive energy states can be written in the form [8]

$$\psi_p^{(\pm)}(x) = N(p)e^{-ip\cdot x}\begin{pmatrix} \chi^{(\pm)} \\ \pm\frac{|\mathbf{p}|}{p_0+m}\chi^{(\pm)} \end{pmatrix}.$$

Klein Paradox: now suppose a beam of relativistic particles with unit amplitude, energy E, momentum $p\hat{\mathbf{e}}_1$, and spin $\uparrow$ (i.e. $\chi = (1,0)$), is incident upon a potential $V(\mathbf{x}) = V\Theta(x_1)$. At the potential barrier, spin is conserved, while a component r of the beam is reflected (with energy E and momentum $-p\hat{\mathbf{e}}_1$), and a component t is transmitted with energy $E' = E - V$ and momentum $p'\hat{\mathbf{e}}_1$. According to the energy-momentum invariant, $E^2 = p^2 + m^2$ and $E'^2 = p'^2 + m^2$. [5]

Being first order, the boundary conditions on the Dirac equation require only continuity of ψ.

$$\begin{pmatrix} 1 \\ 0 \\ p/(E+m) \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ -p/(E+m) \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ p'/(E'+m) \\ 0 \end{pmatrix}$$

Solving for r and t, we find the ratio of the reflected to transmitted intensity to be [5]

$$r = \frac{1-\zeta}{1+\zeta}, \qquad t = \frac{2}{1+\zeta}, \qquad \zeta = \frac{p'}{p}\frac{(E+m)}{(E'+m)}.$$

Taking each energy interval separately [8]

- $E' \equiv (E - V) > V$: In this case $p'^2 \equiv E'^2 - m^2 > 0$, and (taking $p' > 0$) $|r|^2 < 1$ — a component of the beam is transmitted and the remainder is reflected.

- $-m < E' < m$: $p'^2 < 0$, and $|r|^2 = 1$. In this case, the under barrier solutions are evanescent and decay to the right of the barrier.
- $E' < -m$: $p'^2 > 0$ and, depending on the sign of p', $|r|^2$ can be greater or less than unity. But the solution has the form $e^{-i(p'x-E't)}$. Since we presume the beam to be propagating to the right, we require $E' < 0$ and $p' > 0$. This implies that $|r|^2 > 1$

Although the third regime seems to violate particle conservation, one can confirm that current is conserved at the barrier (an explicit calculation is not necessary for this question). By calculating the current, one finds that for $E' < -m$, when $\zeta < 0$, the transmitted current density is negative — it is as if a beam of particles were incident from the right. However, since $\rho < 0$, we should interpret this as a beam of antiparticles (of negative charge) being emitted at $x_1 = 0$ to $x_1 = \infty$. Physically, the incoming beam stimulates the emission of a beam of particles and antiparticles from the step — the particles moving off towards $x_1 = -\infty$ and the antiparticles towards $x_1 = \infty$. Notice that the electrostatic energy recovered by the antiparticle when it is created is sufficient to outweigh the rest mass energy of the particle and antiparticle pair. [4]

---

9. According to the Feynman path integral, the density matrix can can be cast in the form

$$\langle\theta_i|e^{-\beta\hat{H}}|\theta_j\rangle = \int_{\theta(0)=\theta_j,\theta(\tau)=\theta_i} D\theta(\tau)e^{-S[\theta]/\hbar},$$

where $S[\theta]$ represents the Euclidean action corresponding to the classical Hamiltonian H.

(a) For the classical Hamiltonian, the corresponding classical Euclidean action takes the form

$$S[\theta] = \int_0^\beta d\tau\ \frac{I}{2}\dot{\theta}^2.$$

Being periodic in the field θ, the path integral must be evaluated allowing for integer windings of the field by 2π. Putting these elements together we obtain the path integral given in the question.

(b) Minimising the action with respect to fluctuations of θ, one obtains the 'classical equation of motion' $I\ddot{\theta}_{\text{cl.}} = 0$, which admits the solution $\theta_{\text{cl.}}(\tau) = \theta_0 + v\tau$. Respecting the boundary conditions, one obtains $\theta_0 = \theta_j$ and $v = (\theta_i - \theta_j + 2\pi n)/\beta$. Substituting this parameterisation into the path integral, one obtains

$$\langle\theta_i|e^{-\beta\hat{H}}|\theta_j\rangle = \int_{\theta_{\text{fl.}}(\beta)=\theta_{\text{fl.}}(0)=0} D\theta_{\text{fl.}}(\tau)\exp\left[-\int_0^\beta d\tau\ \frac{I}{2}\dot{\theta}_{\text{fl.}}^2\right]$$
$$\times\sum_{n=-\infty}^{\infty}\exp\left[-\frac{I}{2\beta}(\theta_i-\theta_j+2\pi n)^2\right].$$

Using the hint, this result coincides with the formula given in the question.

(c) A direct application of the Poisson summation formula obtains

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty} \exp\left[-\frac{I}{2\beta}(\theta_i-\theta_j+2\pi n)^2\right] \\
&\quad = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\ \exp\left[-\frac{I}{2\beta}(\theta_i-\theta_j+2\pi\phi)^2\right] e^{2\pi i m\phi} \\
&\quad = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\ \exp\left[-\frac{I}{2\beta}(2\pi\phi)^2 + im(2\pi\phi)\right] e^{im(\theta_j-\theta_i)} \\
&\quad = \sqrt{\frac{\beta}{2\pi I}} \sum_{m=-\infty}^{\infty} e^{im(\theta_j-\theta_i)} \exp\left[-\beta\frac{m^2}{2I}\right]
\end{aligned}
$$

Noting that the eigenfunctions of the angular momentum operator are given by $|m\rangle = e^{im\phi}/\sqrt{2\pi}$ with eigenvalues $m^2/2I$, the result is compatible:

$$
\langle\theta_i|e^{-\beta\hat{H}}|\theta_j\rangle = \sum_{m=-\infty}^{\infty} \langle\theta_i|m\rangle\ e^{-\beta m^2/2I}\ \langle m|\theta_j\rangle
$$