4.4 Questions on the Field Integral Method

1. **Exercises on Fermion Coherent States:** To practice the coherent state method, we begin with a few simple exercises on the fermionic coherent state which complements the structures discussed in the main text.

   Considering a fermionic coherent state $|\eta\rangle$, verify the following identities: (a) $\langle\eta|a_i^\dagger = \langle\eta|\bar{\eta}_i$, (b) $a_i^\dagger|\eta\rangle = -\partial_{\eta_i}|\eta\rangle$ and $\langle\eta|a_i = \partial_{\bar{\eta}_i}\langle\eta|$, (c) $\langle\eta|\nu\rangle = \exp[\sum_i \bar{\eta}_i\nu_i]$, and (d) $\int d(\bar{\eta},\eta)\, d\eta_i\, e^{-\sum_i \bar{\eta}_i\eta_i}|\eta\rangle\langle\eta| = \mathbf{1}_{\mathcal{F}}$, where $d(\bar{\eta},\eta) \equiv \prod_i d\bar{\eta}_i d\eta_i$. Finally, (e) show that $\langle n|\psi\rangle\langle\psi|n\rangle = \langle\zeta\psi|n\rangle\langle n|\psi\rangle$, where $|n\rangle$ is an n-particle state in Fock space while $|\psi\rangle$ is a coherent state.

2. **Feynman path integral from the Functional Field Integral**

   The abstraction of the coherent state representation betrays the close similarity between the Feynman and coherent state path integrals. To help elucidate the connection, the goal of the present problem is to confirm that the Feynman path integral of the quantum harmonic oscillator follows from the coherent state path integral.

   Consider the simplest bosonic many-body Hamiltonian, $\hat{H} = \hbar\omega(a^\dagger a + \frac{1}{2})$, where $a^\dagger$ creates 'structureless' particles, i.e. states in a one–dimensional Hilbert space. $\hat{H}$ can be interpreted as the Hamiltonian of a single oscillator degree of freedom. Show that the field integral for the partition function $\mathcal{Z} = \mathrm{tr}\,[\exp(-\beta\hat{H})]$ can be mapped onto the (imaginary–time) path integral of an harmonic oscillator by a suitable variable transformation. [Hint: Let yourself be guided by the fact that the conjugate operator pair $(a, a^\dagger)$ is related to the momentum and coordinate operators $(\hat{p}, \hat{q})$ through a canonical transformation.]

3. **Quantum Partition Function of the Harmonic Oscillator** The following involves a practice exercise on elementary field integral manipulations, and infinite products.

   Compute the partition function of the harmonic oscillator Hamiltonian in the field integral formulation. To evaluate the resulting infinite product over Matsubara frequencies apply the formula $x/\sin x = \prod_{n=1}^{\infty}(1 - x^2/(\pi n)^2)^{-1}$. [Hint: The normalization of the result can be fixed by demanding that, in the zero temperature limit, the oscillator occupies its ground state.] Finally, compute the partition function by elementary means and check your result. As an additional exercise, repeat the same steps for the 'fermionic oscillator', i.e. with $a, a^\dagger$ fermion operators. Here you will need the auxiliary identity $\cos x = \prod_{n=1}^{\infty}(1 - \frac{x^2}{(\pi(n+1/2))^2})$.