Chapter 5

Broken Symmetry and Collective Phenomena

Previously, we have seen how the field integral method can be deployed in many-particle theories. In the following chapter, we will learn how elements of perturbation theory can be formulated efficiently by staying firmly within the framework of the field integral. In doing so, we will see how the field integral provides a method for identifying and exploring non-trivial reference ground states — 'mean–fields'. A fusion of perturbative and mean–field methods will provide us with analytical machinery powerful enough to address a spectrum of rich applications.

Historically, the effects of interaction on many-body systems are typically dealt with within the framework of "diagrammatic" perturbation theory, a series expansion in the interaction strength. However, in the following, we will search for a different methodology. Our motivations are two-fold: Firstly, the structures that typically appear from perturbative expansions can be assimilated more straightforwardly. But, more importantly, the development of phase instabilities typically reflect the appearance of non-perturbative structures. Thus, what we would like to develop is a theoretical framework that is capable of (a) detecting the 'right' reference states or 'mean-fields' of a system, (b) that enables us to efficiently apply perturbative methods around these states and, finally, (c) to do this in a language that draws upon the 'physical' rather than the plain microscopic degrees of freedom as the fundamental units.

To this end, in the following sections we will develop a functional integral based approach that meets these criteria. In contrast to the previous chapters, the discussion here will be decidedly biased towards concrete application to physically motivated problems. After the formulation of the general strategy of field integral based mean-field methods, the next section will address the problem of the weakly interacting electron gas. The exemplification of the new concepts on "familiar territory" will enable us to better understand the intimate connection between the mean-field approach and straightforward perturbation theory. In subsequent sections we will then turn to the discussion of problems which lie firmly outside the range of direct perturbative summation, viz. the phenomena of superfluidity and superconductivity.

5.1 Mean-Field Theory

Roughly speaking, the functional approach to problems with a large parameter proceeds according to the following programme:

1. In the first place, one must identify the relevant structural units of the theory. (This part of the programme *can* be efficiently carried out by the straightforward methods discussed earlier.)

2. Secondly, it is necessary to introduce a new field — let us call it ϕ for concreteness — that encapsulates the relevant degrees of freedom of the low energy theory.

3. With this in hand, one can then trade integration over the 'microscopic fields' for an integration over ϕ, a step often effected by an operation known as the Hubbard–Stratonovich transformation.

4. The low–energy content of the theory can often be explored by subjecting the resulting action $S[\phi]$ to a stationary phase analysis. (The justification for applying stationary phase methods is provided by the existence of a large parameter $N \gg 1$.) Often, at this stage, instabilities in the theory show up — an indication of a physically interesting problem!

5. Finally, the nature of the elementary (collective) excitations above the ground state can by explored by expanding the functional integral around the solution of the stationary phase equations — the 'mean-field'. From this low-energy effective action, one can compute physical observables.

In the next section, we will illustrate how such a programme can be implemented on a specific example which can also be studied by perturbative means:

5.2 Plasma Theory of the Interacting Electron Gas

In first quantised form, the *weakly* interacting electron gas is described by the many-body Hamiltonian,

$$\boxed{\hat{H} = \sum_{i=1}^{N} \frac{\hat{\mathbf{p}}_i^2}{2m} + \sum_{i<j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}} \tag{5.1}$$

For simplicity, we have chosen to neglect to presence of any underlying lattice potential. Formally, the corresponding quantum partition function is obtained as $\mathcal{Z} = \mathrm{tr} e^{-\beta(\hat{H}-\mu\hat{N})}$ where the trace runs over a complete basis of many-body states.

▷ INFO. Before plunging into the computation of the quantum partition function from the coherent state path integral we should first try to understand in what limit Coulomb interactions may be thought of as *weak*. In fact, the control parameter is the average density of the electron gas: $n \equiv 1/r_0^3$ where r_0 denotes the average interelectron spacing. Coulomb interaction is weak when the average potential energy (measured in units of e^2/r_0) is small as compared to the typical kinetic energy (measured in units of $\hbar^2/mr_0^2$). The ratio of energy scales defines the **dimensionless density parameter**

$$\frac{e^2}{r_0}\frac{mr_0^2}{\hbar} = \frac{r_0}{a_0} \equiv r_s,$$

where $a_0 = \hbar/e^2 m$ denotes the Bohr radius. Physically, r_s is the radius of the spherical volume containing one electron on average; the denser the electron gas, the smaller r_s.

Below, we will be concerned with the limit of high density $r_s \ll 1$, in which the effects of Coulomb interaction can be treated perturbatively. In the opposite limit, $r_s \gg 1$, properties become increasingly dominated by the Coulomb interaction. Ultimately, for sufficiently large r_s (or low density) it is believed that the electron gas undergoes a (first order) phase transition to a condensed or 'solid' phase known as a **Wigner crystal**. (Indeed, this phenomenon is the continuum counterpart of the Mott-Hubbard transition described in section 2.2.3.) Although Wigner[1] crystals have never been unambiguously observed, several experiments performed on low density electron gases are consistent with a Wigner crystal ground state. Quantum Monte-Carlo simulation suggests that Wigner crystallisation may occur for densities $r_s > 37$. (Note that this scenario relies crucially on being at low temperatures, and the long-range nature of the Coulomb interaction. In particular, if the Coulomb interaction is screened $V(r) \sim e^{-r/\lambda}$, $r_s \sim (r_0/a_0)e^{-r_0/\lambda}$ and the influence of Coulomb interaction at high densities becomes diminished.)

For $r_s \sim O(1)$, the potential and kinetic energies are comparable. This regime of intermediate coupling is notoriously difficult to describe quantitatively. Yet most metals lie in a regime of intermediate coupling $2 < r_s < 6$. Fortunately, there is overwhelming evidence to suggest that a *weak coupling* description holds even well outside the regime over which microscopic theory can be justified. The phenomenology of the intermediate coupling regime is the realm of **Landau's Fermi Liquid Theory**.

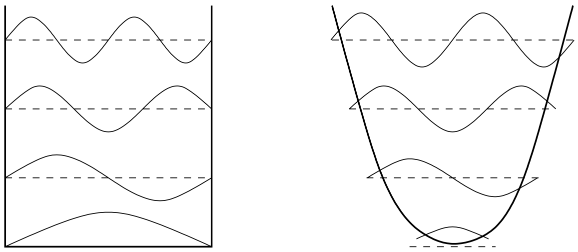


Figure 5.1: Sketch showing the adiabatic continuity of the eigenstates in a one-dimensional potential well.

The fundamental principle underlying the Fermi liquid theory is one of "adiabatic continuity" [5]: In the absence of an electronic phase transition (such as Wigner crystallisation or

[1] Eugene P. Wigner 1902-1995; 1963 Nobel Laureate in Physics for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles.

the Mott transition), a non-interacting ground state evolves smoothly or adiabatically into the interacting ground state as the strength of interaction is increased.[2] An elementary excitation of the non-interacting system represents an "approximate excitation" of the interacting system (i.e. the 'lifetime' of an elementary excitation is long). Excitations are quasi-particles (and quasi-holes) above a sharply defined Fermi surface. The remarkable success (as well as the few notorious failures) of Landau Fermi liquid theory[3] make the subject an important area of modern condensed matter physics but one which we will not have time to explore. Instead, motivated in part by the success of Fermi liquid theory, we will proceed to explore the quantum partition function of the *weakly* interacting electron gas, $r_s \ll 1$.

To prepare for a discussion of the field integral, we must first recast the Hamiltonian in second quantised form as

$$\hat{H} = \int d^3r c^\dagger_\sigma \frac{\hat{\mathbf{p}}^2}{2m} c_\sigma + \frac{1}{2}\int d^3r\, d^3r' c^\dagger_\sigma(\mathbf{r}) c^\dagger_{\sigma'}(\mathbf{r}') \frac{e^2}{|\mathbf{r}-\mathbf{r}'|} c_{\sigma'}(\mathbf{r}') c_\sigma(\mathbf{r}),$$

where the sum over repeated spin indicies σ is here implied. When cast as functional field integral, the corresponding quantum partition function takes the form $\mathcal{Z} \equiv \mathrm{tr}(e^{-\beta(\hat{H}-\mu\hat{N})}) = \int D(\bar{\psi},\psi) e^{-S}$, where

$$\begin{aligned} S[\psi,\bar{\psi}] = &\int_0^\beta d\tau \int d^3r \bar{\psi}_\sigma \left(\partial_\tau + \frac{\hat{\mathbf{p}}^2}{2m} - \mu\right)\psi_\sigma \\ &+\frac{1}{2}\int_0^\beta d\tau \int d^3r\, d^3r'\, \bar{\psi}_\sigma(\mathbf{r})\bar{\psi}_{\sigma'}(\mathbf{r}') \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\psi_{\sigma'}(\mathbf{r}')\psi_\sigma(\mathbf{r}), \end{aligned}$$

Employing the "four-vector" shorthand $x = (\tau, \mathbf{r})$ and $q = (\omega_m, \mathbf{q})$, where ω_m denotes a bosonic Matsubara frequency (exercise: think why), we may define the "density field" $\rho_q = \frac{1}{\sqrt{\beta}}\int dx e^{iq\cdot x}\,\bar{\psi}_\sigma(x)\psi_\sigma(x) = \frac{1}{\sqrt{\beta}}\sum_p \bar{\psi}_{p\sigma}\psi_{p+q\sigma}$ (with $\int dx \equiv \int_0^\beta d\tau \int d^3r$ and $q\cdot x \equiv \omega_m\tau - \mathbf{q}\cdot\mathbf{r}$). Then, expressed in the Fourier basis, the action takes the form (exercise)

$$S[\psi,\bar{\psi}] = \sum_p \bar{\psi}_{p\sigma}\left(-i\omega_n + \frac{\mathbf{p}^2}{2m} - \mu\right)\psi_{p\sigma} + \frac{1}{2L^3}\sum_q{}' \rho_q V(\mathbf{q})\rho_{-q},$$

[2] As a simple non-interacting example, consider the adiabatic evolution of the bound states of a quantum particle as the confining potential is changed from a box to a harmonic potential well (see Fig. 5.1). While the wavefunctions and energies evolve, the topological characteristics of the wavefunctions, i.e. the number of the nodes, and therefore the assignment of the corresponding quantum numbers remains unchanged.

[3] L. D. Landau, Sov. Phys. JETP **3**, 920 (1956); *ibid.* **5**, 101 (1957).

Lev D. Landau 1908-1968, 1962 Nobel Laureate in Physics for his pioneering theories for condensed matter, especially liquid helium.

where $V(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}) = 4\pi e^2/\mathbf{q}^2$.[4] Here, the prime on the q summation denotes the exclusion of the $\mathbf{q}=0$ contribution and reflects the presence of a neutralising background charge.

Now, being quartic in the fields ψ_σ, the Coulomb interaction prevents an explicit computation of the functional ψ–integral. However, it is actually a straightforward matter to reduce, or 'decouple' the interaction operator bringing it to a form quadratic in the fields ψ. Let us multiply the functional integral by the 'fat unity'

$$1 \equiv \mathcal{N} \int D\phi \, \exp\left[-\frac{e^2}{2} \sum_q \phi_q V^{-1}(\mathbf{q}) \phi_{-q} \right] ,$$

where ϕ represents a complex bosonic field variable, and a normalization constant has been absorbed in the definition of the functional measuere $D\phi$. Employing the variable shift $e\phi_q \mapsto e\phi_q + \frac{i}{L^{3/2}} V(\mathbf{q})\rho_q$, one obtains

$$1 = \int D\phi \, \exp\left[\sum_q \left(-\frac{e^2}{2} \phi_q V^{-1}(\mathbf{q}) \phi_{-q} - \frac{i}{L^{3/2}} e\rho_q \phi_{-q} + \frac{1}{2L^3} \rho_q V(\mathbf{q}) \rho_{-q} \right) \right] .$$

The rational behind this exercise can be seen in the last contribution to the exponent: This term is equivalent to the quartic interaction contribution to the fermionic path integral, albeit with opposite sign. Therefore, multiplication of $\mathcal{Z}$ by our unity leads to the field integral $\mathcal{Z} = \int D\phi \int D(\bar\psi_\sigma, \psi_\sigma) e^{-S}$, where

$$S = \frac{1}{8\pi} \sum_q \phi_q \mathbf{q}^2 \phi_{-q} + \sum_{pp'} \bar\psi_{p\sigma} \left[\left(-i\omega_n + \frac{\mathbf{p}^2}{2m} - \mu \right) \delta_{pp'} + \frac{i}{\sqrt{\beta L^3}} e\phi(p'-p) \right] \psi_{p'\sigma} , \quad (5.2)$$

denotes the action, i.e. an expression that is free of quartic field interactions of ψ_σ. Before proceeding, to acquire some intuition for the nature of the action, it is helpful to rewrite S in a real space/time representation. With $\phi_q = \frac{1}{\sqrt{\beta L^3}} \int dx \, e^{-iq\cdot x} \phi(x)$ (four-vector notation as above), one may confirm that,

$$S[\phi, \bar\psi, \psi] = \int_0^\beta d\tau \int d^3r \left\{ \frac{1}{8\pi} (\partial\phi)^2 + \bar\psi_\sigma \left[\partial_\tau - \frac{\partial^2}{2m} - \mu + ie\phi \right] \psi_\sigma \right\} .$$

Physically, ϕ couples to the electron degrees of freedom as a space/time dependent (imaginary) scaler potential, while the first term reflects the Lagrangian energy density associated with the electric component of the electromagnetic (alias the photon) field. Said differently, the field ϕ represents the gauge particle — in this case, the photon — that mediates the Coulomb interaction between electrons. Before proceeding, let us now step back and discuss the general philosophy of the manipulations that led from the original partition function to the two-field representation (5.2).

▷ INFO. The sequence of manipulations developed above, i.e. the 'decoupling' of a quartic interaction through an auxiliary field, is known more generally as a **Hubbard–Stratonovich**

[4] By representing Poisson's equation for a point charge in the Fourier space, one may confirm that $\frac{1}{L^3} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{q}) = \frac{e^2}{|\mathbf{r}|}$.

transformation. The essence of the transformation is a straightforward manipulation of a Gaussian integral. To make this point more transparent, let us reformulate the Hubbard-Stratonovich transformation in a notation that is not burdened by the presence of model–specific constants. Consider an interaction operator of the form $S_{\rm int} = V_{\alpha\beta\gamma\delta}\,\bar\psi_\alpha\psi_\beta\bar\psi_\gamma\psi_\delta$ (summation convention implied), where $\bar\psi$ and ψ may be either bosonic or fermionic field variables, the indices $\alpha, \beta, \ldots$ refer to an unspecified set of quantum numbers, Matsubara frequencies, etc., and $V_{\alpha\beta\gamma\delta}$ is an interaction matrix element. Now, let us introduce composite operators $\hat\rho_{\alpha\beta} \equiv \bar\psi_\alpha\psi_\beta$ to rewrite the interaction as $S_{\rm int} = V_{\alpha\beta\gamma\delta}\,\hat\rho_{\alpha\beta}\hat\rho_{\gamma\delta}$. The notation can be compactified still further by introducing composite indices $m \equiv (\alpha\beta)$, $n \equiv (\gamma\delta)$, whereupon the action $S_{\rm int} = \hat\rho_m V_{mn}\hat\rho_n$ acquires the structure of a generalized bilinear form. To reduce the action to a form quadratic in the ψs one may simply multiply the exponentiated action by unity, viz.

$$e^{-\hat\rho_m V_{mn}\hat\rho_n} = \underbrace{\int D\phi\, e^{-\frac{1}{4}\phi_m V^{-1}_{mn}\phi_n}}_{1}\, e^{-\hat\rho_m V_{mn}\hat\rho_n}\,,$$

where ϕ is bosonic. (Notice that here V^{-1}_{mn} represents the matrix elements of the inverse and not the inverse $(V_{mn})^{-1}$ of individual matrix elements.) Finally, applying the variable change $\phi_m \to \phi_m + 2i(V\hat\rho)_m$ where the notation $(V\hat\rho)$ is shorthand for $V_{mn}\hat\rho_n$, one obtains

$$\boxed{\exp\left[-\hat\rho_m V_{mn}\hat\rho_n\right] = \int D\phi\, \exp\left[-\frac{1}{4}\phi_m V^{-1}_{mn}\phi_n - i\phi_m\hat\rho_n\right]}$$

I.e. the term quadratic in $\hat\rho$ is cancelled.[5] This completes the formulation of the Hubbard-Stratonovich transformation. The interaction operator has been traded for (a) an integration over an auxiliary field coupled (b) to a ψ-bilinear (the operator $\phi_m\hat\rho_m$).

▷ In essence, the Hubbard-Stratonovich transformation is tantamount to Gaussian integral identity (3.13) but read in reverse. An exponentiated square is removed in exchange for a linear coupling. (In (3.13) we showed how terms linear in the integration variable can be removed.)

▷ To make the skeleton outlined above a well defined prescription, one has to be more specific about the meaning of the Gaussian integration over the kernel $\phi_m V^{-1}_{mn}\phi_n$, i.e. the integration variables can be real or complex, and V must be a positive matrix (which is usually the physical situation).

▷ There is some freedom as to the choice of the integration variable. For example, the factor of $1/4$ in front of the Gaussan weight $\phi_m V^{-1}_{mn}\phi_n$ was introduced for mere convenience (viz. to generate a coupling $\phi_m\hat\rho_m$ free of numerical factors). If one does not like to invert the matrix kernel V_{mn}, one can scale $\phi_m \to (V\phi)_m$, whereupon the key formula reads

$$e^{-\hat\rho_m V_{mn}\hat\rho_n} = \int D\phi\, e^{-\frac{1}{4}\phi_m V_{mn}\phi_n - i\phi_m V_{mn}\hat\rho_n}\,.$$

▷ EXERCISE. Show that the passage from the Lagrangian to the Hamiltonian formulation of the Feynman path integral can be interpreted as a Hubbard–Stratonovich transformation.

[5]Here we have assumed that the matrix V is symmetric. If it is not, we can apply the relation $\hat\rho_m V_{mn}\hat\rho_n \equiv \hat\rho^T V\hat\rho = \frac{1}{2}\left[\hat\rho^T(V+V^T)\hat\rho\right]$ to symmetrize the interaction.

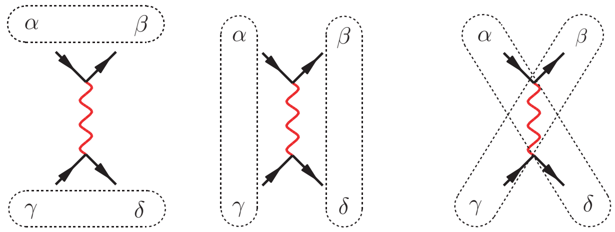


Figure 5.2: On the different channels of decoupling an interaction by Hubbard-Stratonovich transformation. Left: decoupling in the 'density' channel; middle: decoupling in the 'pairing' or 'Cooper' channel; and right: decoupling in the 'exchange' channel.

As defined, the Hubbard-Stratonovich transformation is exact. However, to make it a meaningful operation, it must be motivated by some physical considerations. In our discussion above, we split up the interaction by choosing $\hat{\rho}_{\alpha\beta}$ as a composite operator. However, there is clearly some arbitrariness with this choice. Why not, for example, pair the fermion–bilinears according to $(\bar{\psi}_\alpha\psi_\delta)(\bar{\psi}_\gamma\psi_\delta)$, or otherwise? The three inequivalent choices of pairing up operators are shown in Fig. 5.2 where, as usual, the wavy line with attached field vertices represents the interaction, and the dashed ovals indicate how the field operators are paired.

The version of the transformation discussed above corresponds to the left diagram of the figure. That type of pairing is sometimes referred to as decoupling in the **direct channel**. The denotation becomes more transparent if we consider the example of the spinfull electron-electron interaction,

$$S_{\rm int} = \frac{1}{2}\int d\tau \int d^3r\, d^3r'\, \bar{\psi}_\sigma(\mathbf{r},\tau)\bar{\psi}_{\sigma'}(\mathbf{r}',\tau)V(\mathbf{r}-\mathbf{r}')\psi_{\sigma'}(\mathbf{r}',\tau)\psi_\sigma(\mathbf{r},\tau),$$

i.e. here $\alpha = \beta = (\mathbf{r},\tau,\sigma)$, $\gamma = \delta = (\mathbf{r}',\tau,\sigma')$, and $V_{\alpha\beta\gamma\delta} = V(\mathbf{r}-\mathbf{r}')$. The 'direct' decoupling proceeds via the most obvious choice, i.e. the density operator $\hat{\rho}(\mathbf{r},\tau) = \bar{\psi}_\sigma(\mathbf{r},\tau)\psi_\sigma(\mathbf{r},\tau)$. One speeks about decoupling in a 'channel' because, as will be elucidated below, the propagator of the decoupling field can be interpreted in terms of two Green function lines tied together by multiple interactions, a sequential object reminiscent of a 'channel'.

However, more important than the terminology is the fact that there are other choices for ρ. Decoupling in the **exchange channel** is generated by the choice $\rho_{\alpha\gamma} \sim \bar{\psi}_\alpha\psi_\delta$ where, in the context of the Coulomb interaction, the reversed pairing of field operators is reminiscent of the exchange contraction generating Fock–type contributions. Finally, one may decouple in the **Cooper channel**, $\hat{\rho} = \bar{\psi}_\alpha\bar{\psi}_\gamma$, $\rho_{\beta\gamma} = \rho^\dagger_{\gamma\beta}$. Here, the pairing field is conjugate to two creation operators. Below we will see that this type of decoupling is tailored to problems involving superconductivity.

The remarks above may convey the impression of a certain arbitrariness inherent in the Hubbard-Stratonovich scheme. Indeed, the 'correct' choice of decoupling can only be motivated by physical reasoning, not by plain mathematics. Put differently, the transformation as such is exact, no matter what channel we choose. However, later, we will want to derive an effective low energy theory based on the decoupling field. In cases where one has accidentally decoupled in an 'unphysical' channel, it will be difficult, if not impossible to distill a meaningful low energy theory for the field ϕ conjugate to ρ. Although the initial model still contains the full microscopic information (by virtue of the exactness of the transformation) it is not amenable to further approximation schemes.

In fact, one is frequently confronted with situations where more than one Hubbard-Stratonovich field is needed to capture the full physics of the problem. To appreciate this point, consider the

Coulomb interaction in momentum space.

$$S_{\text{int}}[\bar{\psi},\psi] = \frac{1}{2}\sum_{p_1,\dots,p_4} \bar{\psi}_{\sigma p_1}\bar{\psi}_{\sigma' p_3} V(\mathbf{p}_1-\mathbf{p}_2)\psi_{\sigma' p_4}\psi_{\sigma p_2}\,\delta_{p_1-p_2+p_3-p_4}\,. \tag{5.3}$$

In principle, we can decouple this interaction in any one of the three channels discussed above. However, 'interesting' physics is usually generated by processes where one of the three unbound momenta entering the interaction vertex is small. Only these interaction processes have a chance to accumulate an overall collective excitation of low energy (cf. many of the examples to follow). It may be instructive to imagine the situation geometrically: In the three dimensional cartesian space of free momentum coordinates (p_1, p_2, p_3) entering the vertex, there are three thin layers where one of the momenta is small, (q, p_2, p_3), (p_1, q, p_3), (p_1, p_2, q), $|q| \ll |p_i|$. (Why not make all momenta small? Because that would be in conflict with the condition that the Green functions connecting to the vertex be close to the Fermi surface.) One will thus often choose to break down the full momentum summation to a restricted summation over the small–momentum sublayers:

$$S_{\text{int}}[\bar{\psi},\psi] \simeq \frac{1}{2}\sum_{p,p',q}\Big(\bar{\psi}_{\sigma p}\psi_{\sigma p+q}V(\mathbf{q})\bar{\psi}_{\sigma' p'}\psi_{\sigma' p'-q} - \bar{\psi}_{\sigma p}\psi_{\sigma' p+q}V(\mathbf{p}'-\mathbf{p})\bar{\psi}_{\sigma' p'+q}\psi_{\sigma p'} - \\ \bar{\psi}_{\sigma p}\bar{\psi}_{\sigma' -p+q}V(\mathbf{p}'-\mathbf{p})\psi_{\sigma p'}\psi_{\sigma' -p'+q}\Big)\,.$$

Now, each of these three contributions has its own predestined choice of a slow decoupling field. The first term should be decoupled in the direct channel $\hat{\rho}_{\mathrm{d},q} \sim \sum_p \bar{\psi}_{\sigma p}\psi_{\sigma p+q}$, the second in the exchange channel $\hat{\rho}_{\mathrm{x},\sigma\sigma' q} \sim \sum_p \bar{\psi}_{\sigma p}\psi_{\sigma' p+q}$, and the third in the Cooper channel $\hat{\rho}_{\mathrm{c},\sigma\sigma' q} \sim \sum_p \bar{\psi}_{\sigma p}\bar{\psi}_{\sigma' -p+q}$. One thus winds up with an effective theory that contains three independent slow Hubbard–Stratonovich fields. (Notice that the decoupling fields in the exchange and in the Cooper channel explicitly carry a spin–structure.)

After this digression on the principles of the Hubbard–Stratonovich transformation, let us now return to the discussion of the electron gas.

At the expense of introducing a second field, the Hubbard-Stratonovich transformation provides an action quadratic in the fermion fields. In this case, the fermion integration can be undertaken exactly. Making use of the Gaussian functional integral (4.19), one obtains

$$\mathcal{Z} = \int D\phi\, e^{-\frac{1}{8\pi}\sum_q \phi_q \mathbf{q}^2 \phi_{-q}} \det\left[\partial_\tau + \frac{\hat{\mathbf{p}}^2}{2m} - \mu + ie\phi\right]^2,$$

where the factor of two accounts for the spin degeneracy.

The standard procedure to deal with the determinants generated at intermediate stages of the manipulation of a field integral is to simply re–exponentiate them. This is achieved by virtue of the identity,

$$\boxed{\ln\det\hat{A} = \operatorname{tr}\ln\hat{A}} \tag{5.4}$$

valid for arbitrary (non–singular) operators $\hat{A}$.[6] Thus, the quantum partition function

[6] Eq. (5.4) is readily established by switching to an eigenbasis whereupon one obtains $\ln\det\hat{A} = \sum_a \ln\epsilon_a = \operatorname{tr}\ln\hat{A}$, where ϵ_a are the eigenvalues of $\hat{A}$ and we have used the fact that the eigenvalues of $\ln\hat{A}$ are $\ln\epsilon_a$.

takes the form $\mathcal{Z} = \int D\phi\, e^{-S[\phi]}$, where

$$S[\phi] = \frac{1}{8\pi}\sum_q \phi_q \mathbf{q}^2 \phi_{-q} - 2\mathrm{tr}\,\ln\left[\partial_\tau + \frac{\hat{\mathbf{p}}^2}{2m} - \mu + ie\phi\right]. \tag{5.5}$$

This is as far as purely formal exact manipulations can carry us. We have managed to trade the integration over the interacting Grassmann field ψ_σ for an integration over an auxiliary field ϕ; a field that we believe encapsulates the relevant degrees of freedom of the model. This completes steps 1, 2, and 3 of the general programme outlined above.

Ordinarily, the next step in the programme is to subject the action to a stationary phase analysis, i.e. to seek solutions of the set of saddle-point equations such that

$$\forall q = (\mathbf{q} \neq 0, \omega): \qquad \frac{\delta S[\phi]}{\delta \phi_q} \stackrel{!}{=} 0\,.$$

Such a solution $\phi(\mathbf{x}, t) \leftrightarrow \phi_q$ is commonly referred to as a **mean-field**. This terminology can be understood by inspection of the argument of the 'tr ln' above. The structure $\hat{\mathbf{p}}^2/2m - \mu + ie\phi$, where ϕ is a fixed configuration (to be determined by solving the saddle–point equations), resembles the Hamiltonian operator of particles subject to some background potential, or 'mean' field. The notation on the left hand side of the saddle-point equations indicates that our original interaction $V(\mathbf{q})$ and, therefore, the decoupling field ϕ do not possess a zero momentum mode (a consequence of charge neutrality).

However, in the present case, since the interaction is considered weak, we may anticipate that the solution to the saddle-point variation is the trivial one, viz. $\phi = 0$ — an assumption that we may check self-consistently. In this case, step 4 of the general programme may be considered as achieved and we may turn to explore fluctuations around $\phi = 0$. Since the mean-field solution vanishes, it makes no sense to introduce new notation, i.e. we will denote the fluctuations again by the symbol ϕ.

As regards the first term in the action (5.5), it has already a quadratic form. The logarithmic contribution can be expanded as if we were dealing with a function (again, a consequence of the trace), i.e. setting

$$\hat{G}^{-1} \equiv \hat{G}_0^{-1} + ie\phi,$$

$\hat{G}_0^{-1} \equiv \partial_\tau + \frac{\hat{\mathbf{p}}^2}{2m} - \mu$ is the momentum and frequency diagonal operator whose matrix elements give the free Green function of the electron gas, we may express

$$\begin{aligned}\mathrm{tr}\ln\hat{G}^{-1} &= \mathrm{tr}\ln(\hat{G}_0^{-1} + ie\phi) = \mathrm{tr}\ln\hat{G}_0^{-1} + \mathrm{tr}\ln(1 + ie\hat{G}_0\phi)\\ &= \mathrm{tr}\,\ln\hat{G}_0^{-1} + ie\mathrm{tr}\,(\hat{G}_0\phi) + \frac{e^2}{2}\mathrm{tr}\,(\hat{G}_0\phi\hat{G}_0\phi) + \ldots.\end{aligned}$$

Being ϕ–independent, the first term generates an overall constant multiplying the functional integral, viz. a constant that must describe the non–interacting content of the theory. Indeed, one may note that $e^{2\mathrm{tr}\,\ln\hat{G}_0^{-1}} = e^{-2\mathrm{tr}\,\ln\hat{G}_0} = (\det\hat{G}_0^{-1})^2 \equiv \mathcal{Z}_0$ is just the partition function of the non-interacting electron gas. Linear in $\hat{\phi}$, the second term of the expansion must, by virtue of the mean-field analysis, vanish. (Afterall, we are expanding around an extremum! To this end, one may note that $\mathrm{tr}\,(\hat{G}_0\phi) = \sum_q G_0(q)\phi_{q=0} = 0$.)

The third term is the interesting one. Remembering that ϕ couples to the theory as a potential, this term describes how potential flucutations are affected by the presence of the electron gas, i.e. it must encode the *screening* of the electromagnetic field by the electron degrees of freedom.

To resolve this connection, let us make the momentum dependence of the second–order term explicit (exercise[7]):

$$\frac{e^2}{2}\mathrm{tr}\,(\hat{G}_0\phi\hat{G}_0\phi) = \frac{e^2}{2\beta L^3}\sum_{p,q} G_0(p+q)\phi_q G_0(p)\phi_{-q} = \frac{e^2}{4}\sum_q \Pi(q)\phi_q\phi_{-q}\,,$$

where, setting $\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu$,

$$\boxed{\Pi(q) = \sum_p G_0(p)G_0(p+q) = \frac{2}{\beta L^3}\sum_{\omega_n,\mathbf{p}} \frac{1}{-i\omega_n + \xi_{\mathbf{p}}}\;\frac{1}{-i\omega_n - i\omega_m + \xi_{\mathbf{p}+\mathbf{q}}}}$$

Collecting together the bare photon action with this expansion, to leading order in e, the quantum partition function takes the form

$$\mathcal{Z} = \mathcal{Z}_0 \int D\phi\, e^{-S[\phi]},$$

where the effective action for the electromagnetic field ϕ is given by

$$\boxed{S[\phi] = \frac{1}{2}{\sum_q}' \frac{1}{D(q)}|\phi_q|^2 + O(e^4)}$$

This result has a clear physical interpretation: the interaction of the electron gas with the electromagnetic field induces a modified or **screened Coulomb interaction** (see Fig. 5.3),

$$\boxed{D(\omega_m,\mathbf{q}) = \frac{1}{\epsilon(\omega_m,\mathbf{q})}\frac{4\pi}{\mathbf{q}^2}, \qquad \epsilon(\omega_n,\mathbf{q}) = 1 - \frac{4\pi e^2}{\mathbf{q}^2}\Pi(\omega_m,\mathbf{q})}$$

where $\epsilon(\omega_m,\mathbf{q})$ is the energy and momentum dependent effective **dielectric function**. This perturbative result, which is known in the literature as the **Random Phase Approximation (RPA)**, amounts to treating the long-range part of the Coulomb interaction as an "external" polarisation field, and the correction to the dielectric function, $(4\pi e^2/\mathbf{q}^2)\Pi(\omega_m,\mathbf{q})$, is known as the **screened polarisability**. To explore the dielectric properties of the interacting electron gas it is necessary to understand the frequency and momentum dependence of the density-density response function (5.6). To do so we must first learn how to perform Matsubara frequency summations.

▷ INFO. Frequently, in working with imaginary time field integrals one often needs to perform **Matsubara frequency summations**. At low temperatures it can be argued that

[7]Hint: Consider the strategic incorporation of the resolution of identity $1 = \sum_q |q\rangle\langle q|$ underneath the trace.

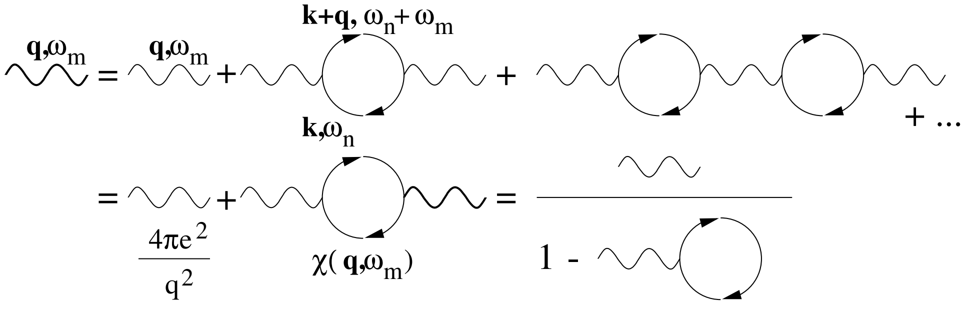


Figure 5.3: The modified screened Coulomb interaction, $D(\omega_m, \mathbf{q})$ can be viewed as the summation of an infinite 'diagrammatic' series expansion in the interaction: The bare Coulomb interaction vertex is 'dressed' by repeated particle-hole excitations of the electron gas. The corresponding summation of the geometric series (known as a **Dyson equation**) is shown schematically and can be compared to Eq. (5.6).

the spacing $2\pi/\beta$ between neighbouring Matsubara frequencies scales to zero which legitimates changing the summations into integrals. However, as we have seen previously, there exists an alternative way of evaluating the sums which not only keeps the finite temperature content of the theory but is also more efficient *even* in the limit of zero temperature.

Referring to our ealier discussion for details, the basic idea behind the standard scheme of evaluating frequency summations is to introduce a complex auxiliary function $g(z)$ that has simple poles at $z = i\omega_n$. The sum $\sum_n f(i\omega_n)$ one wishes to compute then emerges as the sum of residues obtained by integrating the product gf along a suitably chosen path C in the complex plane. The choice of both the function g and the integration contour C depend on the structure of the sum in which one is interested (convergence and analyticity properties of f, etc.). In the present case, the screened polarisability (5.6),

$$g(z) = \frac{\beta}{\exp(\beta z) + 1}, \qquad f(z) \equiv \frac{1}{z - \mu + \epsilon} \, \frac{1}{z - \mu + \epsilon'},$$

and C is a circular contour of infinite radius in the complex plane (see Fig. 5.4). We are therefore led to consider the integral

$$I \equiv \int_C dz \frac{\beta}{\exp(\beta z) + 1} \frac{1}{z - \mu + \epsilon} \, \frac{1}{z - \mu + \epsilon'}. \tag{5.6}$$

Two important observations can be made without explicit computation: (i) The integral exists (the integrand decays sufficiently fast in all directions for $|z| \to \infty$), and (ii) $I = 0$ (the reason being that for $|z| \to \infty$, the product $fg < z^{-\gamma}$, where $\gamma > 1$). Thus the integral of fg over a circle (radius $\sim z$) scales to zero as the radius $\to \infty$. On the other hand, the integral along C gives the sum over the residues of all enclosed poles. The function fg has poles at the Matsubara frequencies $i\omega_n$ and poles on the real axis at $z = \mu - \epsilon$ and $z = \mu - \epsilon'$. Hence,

$$0 = \int_C dz (fg)(z) = 2\pi i \left(\sum_n \text{ Res } (fg)(i\omega_n) + \text{ Res } (fg)(\mu - \epsilon) + \text{ Res } (fg)(\mu - \epsilon') \right).$$

At the Matsubara frequencies, f is analytic and g has, by construction, unit residue. Thus

$$\text{Res } (fg)(i\omega_n) = -f(i\omega_n).$$

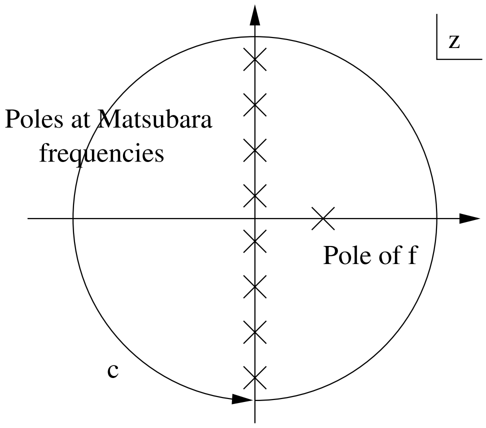


Figure 5.4: Complex integration contour employed in calculating the sum (5.6).

At $z = \mu - \epsilon$, f has a simple pole, whereas g is analytic, i.e.

$$\text{Res}\,(fg)(\mu - \epsilon) = \frac{g(\mu - \epsilon)}{(\epsilon' - \epsilon)}.$$

Combined with the pole at $z = \mu - \epsilon'$, and making use of the identity $e^{i\beta\omega_n} = 1$, we obtain

$$\Pi(\omega_m, \mathbf{q}) = \frac{2}{L^3} \sum_{\mathbf{k}} \frac{n_{\rm F}(\xi_{\mathbf{k}}) - n_{\rm F}(\xi_{\mathbf{k}+\mathbf{q}})}{i\omega_m + \xi_{\mathbf{k}} - \xi_{\mathbf{k}+\mathbf{q}}} \tag{5.7}$$

where we have made use of the fact that, on the real axis, the auxiliary function g is proportional to the Fermi-Dirac distribution function,

$$n_{\rm F}(\xi_{\mathbf{k}}) = \frac{1}{e^{\beta(\mu - \epsilon_{\mathbf{k}})} + 1}. \tag{5.8}$$

To analyse the screened Coulomb interaction $D(\omega_n, \mathbf{q})$ we consider Eq. (5.7) for the density-density response function, and divide our consideration into two limits:

▷ **Static Limit** $(|\omega_n| \ll k_F|\mathbf{q}|/m)$

For frequencies small as compared to the momentum transfer (and temperatures $T \ll \mu$), the response function converges to the static limit

$$\frac{\Pi(0, \mathbf{q})}{2} \simeq \frac{}{L^3} \sum_{\mathbf{k}} \frac{\mathbf{q} \cdot \partial_{\mathbf{k}} n_{\rm F}(\xi_{\mathbf{k}})}{\mathbf{q} \cdot \partial_{\mathbf{k}} \xi_{\mathbf{k}}} \simeq \int \frac{d^3k}{(2\pi)^3} \partial_{\xi_{\mathbf{k}}} n_{\rm F}(\xi_{\mathbf{k}}) = \int_0^\infty d\epsilon \nu(\epsilon) \partial_\epsilon n_{\rm F}(\epsilon - \mu) \simeq -\nu(\mu),$$

where here we have made use of the fact that $\xi_{\mathbf{k}}$ depends only on $|\mathbf{k}|$, and we have deployed the continuum limit, $\frac{1}{L^3} \sum_{\mathbf{k}} \mapsto \int \frac{d^3k}{(2\pi)^3} = \int_0^\infty d\epsilon \nu(\epsilon)$ with $\nu(\epsilon) \equiv 1/|\nabla_{\mathbf{k}} \xi_{\mathbf{k}}| = 2mk/(2\pi)^2$ the density of states. From this result, one obtains the screened Coulomb interaction

$$D(q) = \frac{1}{\frac{\mathbf{q}^2}{4\pi} + 2e^2 \nu(\mu)}$$

When transformed back to real space, one obtains the effective static screened Coulomb interaction

$$\boxed{D(\mathbf{r}) = \frac{e^2}{|\mathbf{r}|} e^{-|\mathbf{r}|/\lambda_{\mathrm{TF}}}}$$

where $\lambda_{\mathrm{TF}} = (8\pi e^2 \nu(\mu))^{1/2}$ defines the **Thomas-Fermi screening length**. Physically, the bare Coulomb interaction is screened by the collective fluctuations of the electron gas.

▷ **High Frequency Limit** $(|\omega_n| \gg k_F|\mathbf{q}|/m)$

By contrast, in the high frequency limit, the density-density response function takes the form

$$\Pi(\omega_m, \mathbf{q}) \simeq -\frac{2}{L^3} \sum_{\mathbf{k}} \frac{\mathbf{q} \cdot \partial_{\mathbf{k}} n_{\mathrm{F}}(\xi_{\mathbf{k}})}{i\omega_m - \mathbf{q} \cdot \partial_{\mathbf{k}} \xi_{\mathbf{k}}} \simeq -\int \frac{d^3k}{(2\pi)^3} \frac{2}{i\omega_m} \left(1 + \frac{\mathbf{q} \cdot \mathbf{k}}{im\omega_m}\right) \mathbf{q} \cdot \partial_{\mathbf{k}} n_{\mathrm{F}}(\xi_{\mathbf{k}})$$

$$\stackrel{\text{by parts}}{=} -\int \frac{d^3k}{(2\pi)^3} \frac{2\mathbf{q}^2}{m\omega_m^2} n_{\mathrm{F}}(\xi_{\mathbf{k}}) = -\frac{1}{(2\pi)^3} \frac{4}{3}\pi k_F^3 \frac{2\mathbf{q}^2}{m\omega_m^2} = -\frac{N}{2L^3} \frac{2\mathbf{q}^2}{m\omega_m^2} = -\frac{n\mathbf{q}^2}{m\omega_m^2},$$

where $n = N/L^d$ denotes the total number density. Applying the analytic continuation to real frequencies, $i\omega_n \to \omega + i0$, we obtain

$$\lim_{m\omega/k_F|\mathbf{q}| \to \infty} D(\omega, \mathbf{q}) = \frac{4\pi e^2}{\mathbf{q}^2} \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{-1},$$

where $\omega_p = 4\pi e^2 n/m$ represents the **Plasma frequency**. At high frequencies the dielectric response of the system is sensitive to the plasma oscillations of the background electron charge. In particular, for $\omega = \omega_p$, the collective excitations become undamped.

▷ Ground State Energy

Finally, from the partition function, it is possible to obtain an estimate of the ground state energy of the interacting electron gas.

$$\lim_{\beta \to \infty} \mathcal{Z} \sim e^{-\beta E_{\mathrm{g.s.}}}.$$

In the RPA approximation, performing the functional integral over the Gaussian action in ϕ, we obtain

$$E_{\mathrm{g.s.}} = E_{\mathrm{g.s.}}(e = 0) - \frac{1}{2\beta} \sum_{\omega_n, \mathbf{q}} \ln D(\omega_n, \mathbf{q}).$$

where $E_{\mathrm{g.s.}} \equiv -\ln \mathcal{Z}_0/\beta = 3n\mu/5$ is the ground state energy of the free electron gas. This was the formula derived by Gell-Mann and Brückner from which, after some extensive algebra (exercise!), one obtains the high density $(r_s \ll 1)$ expansion

$$E_{\mathrm{g.s.}} = n \left(\frac{2.21}{r_s^2} - \frac{0.916}{r_s} + 0.622 \ln r_s - 0.142\right) \text{Ryd.}$$

This concludes our preliminary analysis of the screening properties of the weakly interacting electron gas. By employing the coherent state path integral, we were (implicitly) able to establish the stability of the non-interacting ground state, determine the modified screened Coulomb interaction, and identify plasma oscillations. In the following section, we will use the path integral to study the weakly interacting Bose gas and the phenomenon of superfluidity.

5.3 Bose–Einstein Condensation and Superfluidity

Previously, we have considered the influence of weak Coulomb interaction on the properties of the electron gas. In the following, our goal will be to consider the phases realised by a weakly interacting Bose gas. To this end, let us introduce the quantum partition function $\mathcal{Z} = \int D(\bar{\psi}, \psi)\, e^{-S[\bar{\psi},\psi]}$, where

$$S[\bar{\psi}, \psi] = \int d^d r \int d\tau \left[\bar{\psi}(\mathbf{r}, \tau)(\partial_\tau + \hat{H}_0 - \mu)\psi(\mathbf{r}, \tau) + \frac{g}{2}(\bar{\psi}(\mathbf{r}, \tau)\psi(\mathbf{r}, \tau))^2 \right]. \qquad (5.9)$$

Here ψ represents a complex field subject to the periodic boundary condition $\psi(\mathbf{r}, \beta) = \psi(\mathbf{r}, 0)$. The functional integral $\mathcal{Z}$ describes the physics of a system of bosonic particles in d–dimensions subject to a repulsive contact interaction of strength $g > 0$. For the moment the specific structure of the one–body operator $\hat{H}_0$ need not be specified. The most remarkable phenomena displayed by systems of this type are Bose–Einstein condensation and superfluidity. However, contrary to a widespread belief, these two effects do not mutually depend on each other: Superfluidity can arise without condensation and *vice versa*. We begin our discussion with the more elementary of the two phenomena.

5.3.1 Bose–Einstein Condensation

As may be recalled from elementary statistical mechanics, at sufficiently low temperatures, the ground state of a bosonic system can involve the condensation of a macroscopic fraction of particles into a single state. This phenomenon, predicted in a celebrated work by Einstein[8] is known as Bose–Einstein condensation. To see how this phenomenon is born out of the functional integral formalism, let us temporarily switch off the interaction and turn to the basis in which the one–particle Hamiltonian is diagonal. Expressed in the frequency representation, the partition function of the non–interacting system is given by,

$$\mathcal{Z}_0 \equiv \mathcal{Z}\Big|_{g=0} = \int D(\bar{\psi}, \psi)\, \exp\left[-\sum_{an} \bar{\psi}_{an}\,(-i\omega_n + \epsilon_a - \mu)\,\psi_{an} \right].$$

---

[8] **Albert Einstein** 1879–1955: 1921 Nobel Laureate in Physics "for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect". His work on the low temperature behaviour of the bosonic quantum gas is published in A. Einstein, *Quantentheorie des einatomigen idealen Gases*, Sitzungsber. Preuss. Akad. Wiss. **1925**, 14 (1925).

Without loss of generality, we may assume that the eigenvalues $\epsilon_a \geq 0$ are positive with a ground state $\epsilon_0 = 0$.[9] (In contrast to the fermionic systems discussed above, we should not have in mind low energy excitations superimposed on high energy microscopic degrees of freedom. Here, everything will take place in the vicinity of the ground state of the microscopic single–particle Hamiltonian.) Furthermore, we note that, to ensure stability, the chemical potential determining the number of particles in the system must be negative for, otherwise, the Gaussain weight corresponding to the low–lying states $\epsilon_a < -\mu$ would change sign resulting in an ill–defined theory.

From our discussion of section 4.2.1 we recall that the number of particles in the system is set by the relation ($k_B = 1$)

$$N(\mu) = -\frac{\partial F}{\partial \mu} = T\frac{\partial}{\partial \mu}\ln \mathcal{Z} = T\sum_{na}\frac{1}{i\omega_n - \epsilon_a + \mu} = \sum_a n_{\rm B}(\epsilon_a)\,,$$

where, as usual, $n_{\rm B}(\epsilon) = (e^{\beta(\epsilon-\mu)} - 1)^{-1}$ denotes the Bose distribution. For a given number of particles, this equation determines the temperature dependence of the chemical potential, $\mu(T)$. As the temperature is reduced, the distribution function controlling the population of individual states decreases. Since the number of particles must be kept constant, this scaling must be counter–balanced by a corresponding increase in the chemical potential.

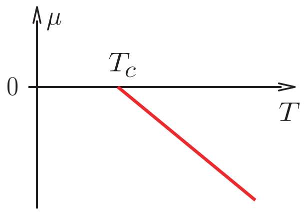


Figure 5.5: Schematic plot showing the variation of the chemical potential as a function of temperature. Note that, for $T < T_c$, the chemical potential remains pinned at zero.

Below a certain critical temperature T_c, even the maximum value of the chemical potential, $\mu = 0$, will not suffice to keep the distribution function $n_{\rm B}(\epsilon_{a\neq 0})$ large enough to accomodate all particles in the states of non–vanishing energy, viz. $\sum_{a>0} n_{\rm B}(\epsilon_a)|_{\mu=0} \overset{T<T_c}{\equiv} N_1 < N$. I.e. below the critical temperature, the chemical potential stays constant at $\mu = 0$ (see the figure). As a result, a macroscopic number of particles $N - N_1$ must accumulate in the single–particle ground state: **Bose–Einstein condensation**.

▷ Exercise. For a three–dimensional free particle spectrum, $\epsilon_k = \hbar^2 k^2/2m$, show that the critical temperature is set by $T_c = \frac{c_0\hbar^2}{ma^2}$, where $a = \rho^{-1/3}$ is the average interparticle spacing, and c_0 is a constant of order unity. Show that for temperatures $T < T_c$, the density of particles in the condensate ($\mathbf{k} = 0$) is given by $\rho_0(T) = \rho[1 - (\frac{T}{T_c})^{3/2}]$.

▷ Info. Since its prediction in the early 20s, the phenomenon of Bose–Einstein condensation has been a standard component of undergraduate texts. However, it took some seven decades

[9] The chemical potential μ can always be adjusted so as to meet this condition.

before the condensation of bosonic particles was directly[10] observed in experiment. The reason for this delay is that the critical condensation temperature of particles that are comfortably accessible to experiment — atoms — is absurdly low.

In 1995 the groups of Cornell and Wieman at Colorado University and, soon after, Ketterle at MIT succeeded in cooling a system of rubidium atoms down to temperatures of 20 billionths(!) of a Kelvin.[11] To reach these temperatures, a gas of rubidium atoms was caught in a magnetic trap, i.e. a configuration of magnetic field gradients that couple to the magnetic moments of the atoms so as to keep the system spatially localized (see the schematic).

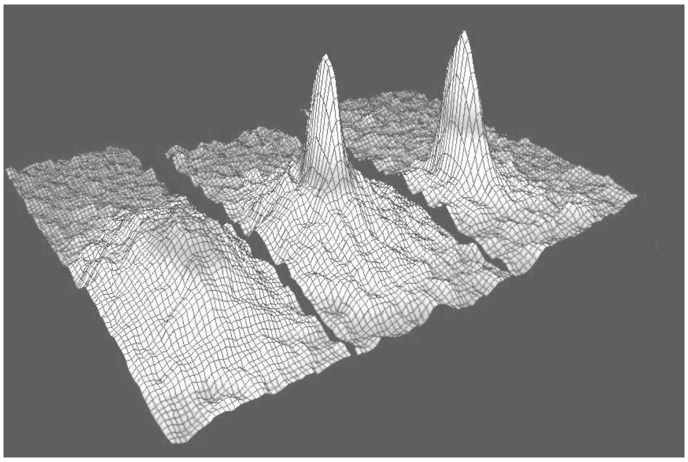


Figure 5.6: Spectroscopic images of a gas of atoms at 400nK (left), 200nK (middle), and 50nK (right). The peak in the density distribution signals the onset of condensation. Courtesy of JILA institute, University of Colorado

The gas of atoms was then brought to a temperature of $\mathcal{O}(10^{-5})$K — still much too hot to condense — by 'laser cooling'; crudely speaking, a technique where atoms, subjected to a suitably adjusted ray of monochromatic light, may transmit more of their kinetic energy to the photons than they get back. To lower the temperature still further, the principle of 'evaporative cooling' was applied: By lowering the potential well of the trap, a fraction of the atoms, namely those with large kinetic energy, is allowed to escape. The remaining atoms have a low kinetic energy and, therefore, a low temperature. What sounds like a simple recipe actually represents a most delicate experimental procedure. (For example, if the trap potential is lowered too strongly, all atoms escape and there is nothing left to condense. If, on the other hand, trapping is too strong, the atoms remain too hot, etc.) However, after more than a decade of intensive experimental preparation, the required temperatures have been reached.

Spectroscopic images of the Bose–Einstein condensation process are shown in the figure (courtesy of the JILA institute, University of Colorado) for three values of temperature (400 nK, 200 nK, and 50 nK from left to right). The peak in the density distribution signals the onset of condensation. On lowering the temperature, one may observe the transition to a condensed phase by monitoring the formation of a peak in the density distribution. The preparation of

[10]Here, by 'direct' we refer to the controlled preparation of a state of condensed massive bosonic particles. There are numerous 'indirect' manifestations of condensed states, e.g. the anomalous properties of the Helium liquids at low temperatures, or of Cooper–pair condensates in superconductors.

[11]M. H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E. A. Cornell, *Observation of Bose–Einstein Condensation in a Dilute Atomic Vapor*, Science **269**, 198 (1995).

a Bose condensed state of matter was recognized with the award of the 2001 Nobel prize in physics. Since 1995, research on atomic condensates has blossomed into a broad arena of research. Already, it is possible to prepare complex states of Bose condensed matter such as atomic vortices in rotating Bose–Einstein condensates, condensates in different dimensionalities, or even an artificial crystalline state of matter. A detailed discussion of these interesting developments is beyond the scope of the present text.

With this background, let us now try to understand how the phenomenon of Bose–Einstein condensation can be implemented into the functional integral representation. Evidently, the characteristics of the condensate will be described by the zero field component $\psi_0(\tau)$. The problem with this zero mode is that, below the condensation transition, its action apears to be unbound: both the chemical potential and the eigenvalue are zero. This means that the action of the zero Matsubara component $\psi_{0,0}$ vanishes. We will deal with this difficulty in a pragmatic way. That is, we will not treat $\psi_0(\tau)$ as an integration variable but rather as a time–independent Lagrange multiplier to be used to fix the number of particles below the transition. More precisely, we introduce a reduced action of the form

$$S_0[\bar{\psi}_0, \psi_0] = -\bar{\psi}_0 \beta\mu\psi_0 + \sum_{a\neq 0,n} \bar{\psi}_{an}\left(i\omega_n + \epsilon_a - \mu\right)\psi_{an}\,,$$

where we did not yet set $\mu = 0$ (since we still need μ as a differentiation variable). To understand the rational behind this simplification one may note that

$$N = -\partial_\mu F_0|_{\mu=0^-} = T\partial_\mu \ln \mathcal{Z}_0|_{\mu=0^-} = \bar{\psi}_0\psi_0 + T\sum_{a\neq 0,n}\frac{1}{i\omega_n - \epsilon_a} = \bar{\psi}_0\psi_0 + N_1 \qquad (5.10)$$

determines the number of particles. According to this expression, $\bar{\psi}_0\psi_0 = N_0$ sets the number of particles in the condensate. Now, what enables us to regard ψ_0 as a time–independent field? Remembering the construction of the path integral, we note that the introduction of time–dependent fields, or 'time slicing' was necessitated by the fact that the operators appearing in the Hamiltonian of a quantum theory do not, in general, commute. (Otherwise we could have decoupled the expression $\mathrm{tr}\,(e^{-\beta(\hat{H}-\mu\hat{N})(a^\dagger,a)}) \simeq \int d(\bar{\psi},\psi)e^{-\beta(H-\mu N)(\bar{\psi},\psi)}$ in terms of a single coherent state resolution, i.e. a 'static' configuration). Reading this observation in reverse, we conclude that the dynamic content of the field integral represents the quantum character of a theory. (Alluding to this fact, the temporal fluctuations of field variables are often referred to as **quantum fluctuations**.) Conversely, *a static approximation in a field integral* $\psi(\tau) = \psi_0 =$ const. *amounts to replacing a quantum degree of freedom by its classical approximation.*
(In order to distinguish them from quantum, fluctuations in the 'classical' static sector of the theory are called **thermal fluctuations**.) To justify the approximation of $a_0 \leftrightarrow \psi_0$ by a classical object, notice that, upon condensation, $N_0 = \langle a_0^\dagger a_0\rangle$ will assume 'macroscopically large' values. On the other hand, the commutator $[a_0, a_0^\dagger] = 1$, continues to be of $\mathcal{O}(1)$. It thus seems to be legitimate to neglect all commutators of the zero operator a_0 in comparison with its expectation value — a classical approximation.[12]

[12]Notice the similarity of that reasoning to the arguments employed in connection with the semi–

Now, we are still left with the problem that the ψ_0–integration appears to be un–defined. The way out is to remember that the partition function should extend over those states that contain an (average) number of N particles. That is, Eq. (5.10) has to be interpreted as a relation that fixes the modulus $\bar{\psi}_0\psi_0$ so as to adjust the appropriate value of N. (For a more rigorous discussion of the choice of the thermodynamic variables in the present context, we again refer to Ref.[1].)

5.3.2 The Weakly Interacting Bose Gas

Now, with this background, let us restore the interaction focussing on a small but finite coupling constant g. To keep the discussion concrete, we specialize to the case of a free single-particle system, $\hat{H}_0 = \hat{\mathbf{p}}^2/2m$. (Notice that the ground state wavefunction of this system describes a spatially constant zero momentum state.) By adiabatic continuity we expect that much of the picture developed above will survive generalization to non–zero interaction strengths. In particular, the ground state, which in the case under consideration corresponds to a temporally and spatially constant mode ψ_0, will continue to be macroscopically occupied. Under these circumstances, the dominant contribution to the action will again come from the classical ψ_0 sector:

$$TS[\bar{\psi}_0, \psi_0] = -\mu\bar{\psi}_0\psi_0 + \frac{g}{2L^d}(\bar{\psi}_0\psi_0)^2 \,. \tag{5.11}$$

Crucially, the stability of the action is now guaranteed by the interaction vertex, no matter how small is $g > 0$ (see the schematic plot of the action in the figure). Accordingly, we will no longer treat ψ_0 as a fixed parameter but rather as an ordinary intergration variable. Integration over all field components will produce a partition function $\mathcal{Z}(\mu)$ that depends parametrically on the chemical potential. As usual in statistial physics, the latter can then be employed to fix the particle number. (Notice that, vis–a–vis aspects of thermodynamics, the interacting system appears to behave more 'naturally' than its ideal, non–interacting approximation. This reflects a general feature of bosonic systems; interactions 'regularize' a number of pathological features of the ideal gas.)

Returning to the ψ_0–integration, we observe that, for low enough temperatures, the problem is an ideal candidate for saddle–point analysis. Variation of the action with respect to ψ_0 obtains

$$\psi_0\left(-\mu + \frac{g}{L^d}\bar{\psi}_0\psi_0\right) = 0.$$

This equation is solved by any constant complex field configuration ψ_0 with modulus $|\psi_0| = \sqrt{\mu L^d/g} \equiv \gamma$. Inspite of its innocent appearance, this equation reveals much about the nature of the system:

classical treatment of spin systems in the limit of large S (section 2.2.4). Unfortunately, the actual state of affairs with the classical treatment of the condensate is somewhat more complex than the simple argument above suggests. (For a good discussion, see Ref. [1].) However, the net result of a more thorough analysis, i.e. an integration over all dynamically fluctuating components $\psi_{0,n\neq 0}$, shows that the treatment of ψ_0 as classical represents a legitimate approximation.

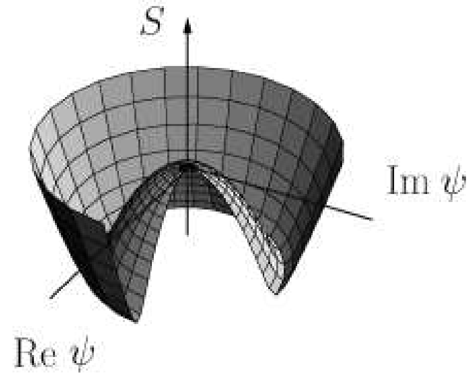


Figure 5.7: The action $S[\bar\psi_0,\psi_0]$ shown as a function of real and imaginary part of the condensate field (part removed for clarity). The most important features of the action are (a) the existence of a degenerate minimum determined by the set of solutions of the equation $\partial_{|\psi_0|}S = 0$, and (b) the large amplitude asymptotics $\sim \beta g|\psi_0|^4$ stabilizing the ψ_0-integration.

- ▷ For $\mu < 0$ (i.e. above the condensation threshold of the non–interacting system), the equation exhibits only the trivial solution $\psi_0 = 0$. This means that no stable condensate amplitude exists.
- ▷ Below the condensation threshold (i.e. for $\mu \geq 0$),[13] the equation is solved by any configuration with $|\psi_0| = \gamma \equiv \sqrt{\mu L^d/g}$. (Notice that $\bar\psi_0\psi_0 \propto L^d$, reflecting the macroscopic population of the ground state.)
- ▷ The equation couples only to the modulus of ψ_0. I.e. the solution of the stationary phase equation is continuously degenerate: Each configuration $\psi_0 = \gamma\exp(i\phi), \phi \in [0, 2\pi]$ is a solution.

For our present discussion, the last of the three aspects mentioned above is the most important. It raises the question as to which of the configurations $\psi_0 = \gamma\exp(i\phi)$ is the 'right' one?

Without loss of generality, we may chose $\psi_0 = \gamma \in \mathbb{R}$ as a reference configuration for our theory. This choice amounts to selecting a particular minimum lying in the 'mexican hat' profile of the action shown above. However, it is clear that an expansion of the action around that minimum will be singular: Fluctuations $\psi_0 \to \psi_0 + \delta\psi$ that do not leave the azimuthally symmetric well of degenerate minima do not change the action and, therefore, have vanishing expansion coefficients. As a result, in the present situation, we will not be able to implement a simple scheme viz. 'saddle–point plus quadratic fluctuations'. (There is nothing that constrains the deviations $\delta\psi$ to be small.) The integral over fluctuations around the mean–field configuration has to be undertaken in a more careful way.

▷ Info. The mechanism encoutered here is one of **spontaneous symmetry breaking**. To understand the general principle, consider an action $S[\psi]$ with a global continuous symmetry

[13] Due to the stabilization of the zero mode integration by the interaction constant, $\mu \leq 0$ is no longer a strict condition.

under some transformation g (not to be confused with the afore mentioned coupling constant of the Bose gas): Specifically, the action remains invariant under a global transformation of the fields such that, $\forall i \in M: \psi_i \to g\psi_i$, where M is known as the "base manifold", i.e. $S[\psi] = S[g\psi]$. The transformation is 'continuous' in the sense that g takes values in some manifold, typically a group G.

Examples: The action of a Heisenberg ferromagnet is invariant under **rotation** of all spins simultaneously by the same amount, $\mathbf{S}_i \to g\mathbf{S}_i$. In this case, $g \in G = \mathrm{O}(3)$, the three–dimensional group of rotations. The action of the displacement fields $\mathbf{u}$ describing elastic deformations of a solid (phonons) is invariant under simultaneous **translation** of all displacements $\mathbf{u}_i \to \mathbf{u}_i + \mathbf{a}$, i.e. the symmetry manifold is the the d–dimensional translation group $G \simeq \mathbb{R}^d$. In the example above, we encountered a U(1) symmetry under phase multiplication $\psi_0 \to e^{i\phi}\psi_0$. This phase freedom expresses the **global gauge symmetry** of quantum mechanics under transformation by a phase, a point we will discuss in more detail below.

Now, given a theory with globally G invariant action, two scenarios are conceivable: Either the ground states share the invariance properties of the action or they do not. The two alternatives are illustrated in the figure for the example of the Bose system. For $\mu > 0$, the action $S[\bar{\psi}_0, \psi_0]$ has a single ground state at $\psi_0 = 0$. This state is trivially symmetric under the action of $G = \mathrm{U}(1)$. However, for negative μ, i.e. in the situation discussed above, there is an entire manifold of degenerate ground states, defined through the relation $|\psi_0| = \gamma$. These ground states transform into each other under the action of the gauge group. However, none of them is individually invariant.

With the other examples mentioned above, the situation is similar. For symmetry groups more complex than the one–dimensional manifold U(1), the ground states will, in general, be invariant under transformation by the elements of a certain subgroup $H \subseteq G$ (that includes the two extremes $H = \{\mathbf{1}\}$ and $H = G$). For example, below the transition temperature, the ground state of the Heisenberg magnet will be given by (domainwise) aligned configurations of spins. Assuming that the spins are oriented along the z–direction, the ground state is invariant under the abelian subgroup $H \subset \mathrm{O}(3)$ containing all rotations around the z–axis. However, invariance under the full rotation group is manifestly broken. Solids represent states where the translation symmetry is fully broken, i.e. all atoms collectively occupy a fixed pattern of spatial positions in space, $H = \{\mathbf{1}\}$, etc.

Inspite of the undeniable existence of solids, magnets, and Bose condensates of definite phase, the notion of a ground state that does not share the full symmetry of the theory may appear paradoxical, or at least 'unnatural'. For example, even if any particular ground state of the 'Mexican hat' potential shown in the figure above 'breaks' the rotational symmetry, shouldn't all these states enter the partition sum with equal statistical weight, such that the net outcome of the theory is again fully symmetric?

To understand why symmetry breaking is a 'natural' and observable phenomenon, it is instructive to perform a gedanken experiment: To this end, consider the partition function of a classical[14] ferromagnet,

$$\mathcal{Z} = \mathrm{tr}\left(e^{-\beta(H - \mathbf{h}\cdot\sum_i \mathbf{S}_i)}\right),$$

where H is the rotationally invariant part of the energy functional and $\mathbf{h}$ represents a weak external field. (Alternatively, we can think of $\mathbf{h}$ as an internal field, caused by a slight structural imperfection of the system.) In the limit of vanishing field strength, the theory becomes

[14]The same argument can be formulated for the quantum magnet.

manifestly symmetric. Symbolically,

$$\lim_{N\to\infty}\lim_{h\to 0}\mathcal{Z}\longrightarrow \text{rot. sym.}\,,$$

where the limit $N\to\infty$ serves as a mnemonic indicating that we consider systems of macroscopic size. However, keeping in mind the fact that the model ought to describe a physical magnetic system, the order of limits taken above appears questionable. Since the external perturbation couples to a macroscopic number of spins, a more natural description of an 'almost' symmetric situation would be

$$\lim_{h\to 0}\lim_{N\to\infty}\mathcal{Z}\longrightarrow\ ?$$

The point is that the two orders of limits lead to different results. In the latter case, for any $\mathbf{h}$, the $N\to\infty$ system is described by an explicitly symmetry broken action. No matter how small the magnetic field, the energetic cost to rotate $N\to\infty$ spins against the field is too high, i.e. the ground state $|\mathbf{S}\rangle$ below the transition temperature will be uniquely aligned, $\mathbf{S}_i \parallel \mathbf{h}$. When we then send $\mathbf{h}\to 0$ in a subsequent step, that particular state will remain the observable reference state of the system. Although, formally, a spontaneous thermal fluctuation rotating all spins by the same amount $|\mathbf{S}\rangle\to|g\mathbf{S}\rangle$ would not cost energy, that flucutation can be excluded by entropic reasoning.[15] (By analogy, one rarely observes kettles crashing into the kichen wall as a consequence of a concerted thermal fluctuation of the water molecules!)

However, the appearance of non–trivial ground states is just one manifestation of spontaneous symmetry breaking. Equally important, residual fluctuations around the ground state lead to the formation of **soft modes** (sometimes known as **massless modes**), i.e. field configurations $\phi_{\mathbf{q}}$ whose action $S[\phi]$ vanishes in the limit of long wavelengths $\mathbf{q}\to 0$. Specifically, the soft modes formed on top of a symmetry broken ground state are called **Goldstone modes**. As a rule, the presence of soft modes in a continuum theory has important phenomenological consequences. To understand this point, notice that the general structure of a soft mode action is given by

$$S[\phi]=\sum_{\mathbf{q},i}\phi_{\mathbf{q}}\left[c_1^i|q_i|+c_2^iq_i^2\right]\phi_{-\mathbf{q}}+\mathcal{O}(\phi^4,q^3)\,, \tag{5.12}$$

where $c^i_{1,2}$ are coefficients. The absence of a constant contribution to the action (i.e. a contribution that does not vanish in the limit $q\to 0$) signals the existence of long–ranged power–law correlations in the system. As we will see shortly, the vanishing of the action in the long wavelength limit $q\to 0$ further implies that the contribution of the soft modes dominates practically all observable properties of the system.

5.3.3 Superfluidity

As we have seen, the theory of the weakly interacting superfluid to be discussed below was originally conceived by Bogoliubov, then in the language of second quantisation.[16] In the following, we will reformulate the theory in the language of the field integral starting

[15]Note that this (overly) simple picture in fact breaks down in dimensions $d\le 2$, cf. our discussion of the thermal fluctuations of the ferromagnet in chapter 2.

[16]Bogoliubov, N. N., *On the Theory of Superfluidity*, J. Phys. (USSR) **11**, 23 (1947), (reprinted in D. Pines, *The Many–body Problem*, W. A. Benjamin, New York, 1961).

with the action of the weakly interacting Bose gas (5.9). Focussing on temperatures below T_c, ($\mu > 0$), let us expand the theory around the particular mean–field ground state $\bar{\psi}_0 = \psi_0 = (\mu L^d/g)^{1/2} = \gamma$. (Of course, any other state lying in the 'Mexican hat' minimum of the action would be just as good.) Notice that the quantum ground state corresponding to the configuration ψ_0 is unconventional in the sense that it cannot have a definite particle number. The reason is that, according to the correspondence $\psi \leftrightarrow a$ between coherent states and operators, respectively, a non–vanishing functional expectation value of ψ_0 is equivalent to a non–vanishing quantum expectation value $\langle a_0 \rangle$. Assuming that, at low temperatures, the thermal average $\langle \ldots \rangle$ will project onto the ground state $|\Omega\rangle$, we conclude that $\langle \Omega | a_0 | \Omega \rangle \neq 0$, i.e. $|\Omega\rangle$ cannot be a state with a definite number of particles.[17]

The symmetry group U(1) acts on this state by multiplication, $\psi_0 \to e^{i\phi}\psi_0$ and $\bar{\psi}_0 \to e^{-i\phi}\bar{\psi}_0$. Knowing that the action of a weakly modulated field $\phi(\mathbf{r}, \tau)$ will be massless, let us introduce coordinates

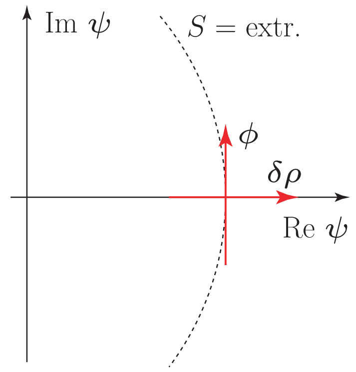


Figure 5.8: Schematic diagram showing the coordinates of the massive ($\delta\rho$) and massless ($\delta\phi$) fluctuations.

$$\begin{aligned}\psi(\mathbf{r}, \tau) &= [\rho_0 + \delta\rho(\mathbf{r}, \tau)]^{1/2} e^{i\phi(\mathbf{r}, \tau)}, \\ \bar{\psi}(\mathbf{r}, \tau) &= [\rho_0 + \delta\rho(\mathbf{r}, \tau)]^{1/2} e^{-i\phi(\mathbf{r}, \tau)},\end{aligned}$$

where $\rho_0 \equiv \gamma^2 = \bar{\psi}_0 \psi_0$ is the condensate density. Evidently, the variable $\delta\rho$ parameterizes deviations of the field $\psi(\mathbf{r}, \tau)$ from the extremum. These excursions are energetically costly, i.e. $\delta\rho$ will turn out to be a massive mode. Also notice that the transformation of coordinates $(\bar{\psi}, \psi) \to (\delta\rho, \phi)$, viewed as a change of integration variables, has a Jacobian of unity.

▷ INFO. As we are dealing with a (functional) integral, there is a lot of freedom as to the choice of integration parameters. (I.e. in contrast to the operator formulation, there is no *a priori* constraint for a transform to be 'canonical'.) However, physically meaningful changes of representation will usually be **canonical transformations**, in the sense that the corresponding

[17] However, as usual with grand canonical descriptions, in the thermodynamic limit, the relative uncertainty in the number of particles, $(\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2)/\langle \hat{N} \rangle^2$ will become vanishingly small.

transformations of operators would conserve the commutation relations. Indeed, one may confirm that the operator transformation $a(\mathbf{r}) \equiv \hat{\rho}(\mathbf{r})^{1/2}\, e^{i\hat{\phi}(\mathbf{r})}$, $a^\dagger(\mathbf{r}) \equiv e^{-i\hat{\phi}(\mathbf{r})}\, \hat{\rho}(\mathbf{r})^{1/2}$, fulfills this criterion (exercise).

We next substitute the density–phase relation into the action and expand to second order around the reference mean–field. Ignoring gradients acting on the density field (in comparison with the 'potential' cost of these fluctuations), we obtain

$$S[\delta\rho,\phi] \approx \int d^d r \int d\tau \left[i\rho\partial_\tau\phi + \frac{\rho_0}{2m}(\partial\phi)^2 + \frac{g\delta\rho^2}{2}\right]. \tag{5.13}$$

The first term of the action has the canonical structure 'momentum $\times\,\partial_\tau$(coordinate)' indicative of a canonically conjugate pair. The second term measures the energy cost of spatially varying phase flucutations. Notice that fluctuations with $\phi(\mathbf{r},\tau) =$ const. do not incur an energy cost — ϕ is a Goldstone mode. Finally, the third term records the energy cost of massive fluctuations from the potential minimum. Eq. (5.13) represents the Hamiltonian version of the action, i.e. an action comprising coordinates ϕ and momenta $\delta\rho$. Gaussian integration over the field $\delta\rho$ leads us to the Lagrangian form of the action (exercise):

$$S[\phi] \approx \frac{1}{2}\int d^d r \int d\tau \left[\frac{1}{g}(\partial_\tau\phi)^2 + \frac{\rho_0}{m}(\partial\phi)^2\right]. \tag{5.14}$$

Comparison with Eq. (1.2) identifies this action as the familiar the d–dimensional oscillator. Drawing on the results of chapter 1 (see, e.g., Eq. (1.15)), we find that the energy $\omega_{\mathbf{k}}$ carried by elementary excitations of the system scales linearly with momentum, $\omega_{\mathbf{k}} = |\mathbf{k}|\rho_0/mg$.

Let us now discuss the physical ramifications of these results. The actions (5.13) and (5.14) describe the phenomenon of superfluidity. To make the connection between the fundamental degree of freedom of a superfluid system, the **supercurrent**, and the phase field explicit, let us consider the quantum mechanical current operator

$$\begin{aligned}\hat{\mathbf{j}}(\mathbf{r},\tau) &= \frac{i}{2m}\left[(\nabla a^\dagger(\mathbf{r},\tau))a(\mathbf{r},\tau) - a^\dagger(\mathbf{r},\tau)\nabla a(\mathbf{r},\tau)\right] \xrightarrow{\text{fun. int}} \\ &\to \frac{i}{2m}\left[(\nabla\bar{\psi}(\mathbf{r},\tau))\psi(\mathbf{r},\tau) - \bar{\psi}(\mathbf{r},\tau)\nabla\psi(\mathbf{r},\tau)\right] \approx \frac{\rho_0}{m}\nabla\phi(\mathbf{r},\tau),\end{aligned} \tag{5.15}$$

where the arrow indicates the functional integral correspondence of the operator description and we have neglected all contributions arising from spatial fluctuations of the density profile. (Indeed, these — massive — fluctuations describe the 'normal' contribution to the current flow.)

▷ INFO. **Superfluidity** is one of the most counterintuitive and fascinating phenomena displayed by condensed matter systems. Experimentally, the most straightforward access to superfluid states of matter is provided by the Helium liquids. Representative of many other effects displayed by superfluid states of Helium, we mention the capability of thin films to flow up the walls of a vessel (if the reward is that on the outer side of the container a low lying basin can be reached — the fountain experiment) or to effortlessly propagate through porous media that no normal fluid may penetrate.

The gradient of the phase variable is therefore a measure of the (super)current flow in the system. The behaviour of that degree of freedom can be understood by inspection of the stationary phase equations — alias, the Hamilton or Lagrange equations of motion — associated with the actions (5.13) or (5.14). Turning to the Hamiltonian formulation, one obtains (exercise)

$$i\partial_\tau \phi = -g\delta\rho, \qquad i\partial_\tau \delta\rho = \frac{\rho_0}{m}\partial^2\phi = \nabla\cdot\mathbf{j}.$$

The second of these equations represents (the Euclidean time version) of a continuity equation. A current flow with non–vanishing divergence is accompanied by dynamical distortions in the density profile. The first equation tells us that the system adjusts to spatial fluctuations of the density by a dynamical phase fluctuation. The most remarkable feature of these equations is that they possess steady state solutions with non–vanishing current flow. Setting $\partial_\tau\phi = \partial_\tau\delta\rho = 0$, we obtain the conditions $\delta\rho = 0$ and $\nabla\cdot\mathbf{j} = 0$, i.e. below the condensation temperature, a configuration with a uniform density profile can support a steady state divergenceless (super)current. Notice that a 'mass term' in the ϕ action would spoil this property, i.e. within our present approach, the phenomenon of supercurrent flow is intimately linked to the Goldstone mode character of the ϕ field.

▷ EXERCISE. Add a ficticious mass term to the ϕ–action (viz. $\delta\mathcal{L} = m\phi^2$) and explore its consequences. How do the features discussed above present themselves in the Lagrange picture?

It is very instructive to interpret the phenomenology of supercurrent flow from a different, more microscopic perspective. Steady state current flow in normal environments is prevented by the mechanism of **energy dissipation**, i.e. particles constituting the current flow scatter off imperfections inside the system thereby converting part of their energy into the creation of elementary excitations. (Macroscopically, the conversion of kinetic energy into the creation of excitations manifests itself as heat production.) Apparently, this mechanism is inactivated in superfluid states of matter, i.e. the current flow is dissipationless.

How can the dissipative loss of energy be avoided. Trivially, no energy can be exchanged if there are no elementary excitations to create. In reality, this means that the excitations of the system are energetically high–lying such that the kinetic energy stored in the current–carrying particles is insufficient to create them. But this is not the situation that we encounter in the superfluid! As we saw above, there is no energy gap separating the quasi–particle excitations of the system from the ground state. Rather, the dispersion $\omega(\mathbf{k})$ vanishes linearly as $\mathbf{k}\to 0$. However, there is an ingenuous argument due to Landau showing that a linear excitation spectrum indeed suffices to stabilize dissipationless transport:

▷ INFO. Consider the flow of some fluid through a pipe (cf. Fig. 5.9 top left). To be concrete, let us assume that the flow occurs at a uniform velocity $\mathbf{V}$. Taking the mass (of a certain portion of the fluid) to be M, the current carries a total kinetic energy $E_1 = M\mathbf{V}^2/2$. Now, suppose we view the situation from the point of view of the fluid, i.e. we perform a Galileian transformation into its own rest frame (see Fig. 5.9, top right). From the perspective of the fluid, the walls of the pipe appear as though they were moving with velocity $-\mathbf{V}$. Now, suppose that frictional forces between fluid and the wall lead to the creation of an elementary excitation of momentum $\mathbf{p}$ and energy $\epsilon(\mathbf{p})$, i.e. the fluid is no longer at rest but carries kinetic energy. After a Galileian transformation back to the laboratory frame, one finds that the energy of the fluid after the creation of the excitation is given by (excercise)

$$E_2 = \frac{M\mathbf{V}^2}{2} + \mathbf{p}\cdot\mathbf{V} + \epsilon(\mathbf{p})\,.$$

Now, since all of the energy needed to manufacture the excitation must have been provided by

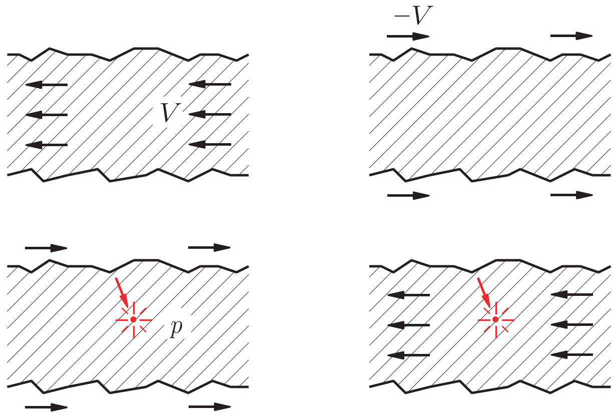


Figure 5.9: Top left: Flow of a fluid through a rough pipe. Top right: The same viewed from the rest frame of the fluid. Bottom left: Dissipative creation of a (quasi–particle) excitation. Bottom right: The same viewed from the laboratory frame.

the liquid itself, energy conservation requires that $E_1 = E_2$, or $-\mathbf{p}\cdot\mathbf{V} = \epsilon(\mathbf{p})$. Since $\mathbf{p}\cdot\mathbf{V} > -|\mathbf{p}||\mathbf{V}|$, this condition can only be met if $|\mathbf{p}||\mathbf{V}| > \epsilon(\mathbf{p})$. While systems with a 'normal' gapless dispersion, $\epsilon(\mathbf{p}) \sim \mathbf{p}^2$ are compatible with this energy–balance relation (i.e. no matter how small $|\mathbf{V}|$, quasi–particles of low momentum can always be excited), both gapped dispersions $\epsilon(\mathbf{p}) \stackrel{\mathbf{p}\to 0}{\longrightarrow}$ const. and linear dispersions are incompatible if $\mathbf{V}$ becomes smaller than a certain **critical velocity** V_*. Specifically for a linear dispersion $\epsilon(\mathbf{p}) = v|\mathbf{p}|$, the critical velocity is given by $V_* = v$. For currents slower than that, the flow is necessarily dissipationless.

Let us conclude our preliminary discussion of the weakly interacting Bose gas with a very important remark. Superficially, Eqs. (5.13) and (5.14) suggest that we have managed to describe the long–range behaviour of the condensed matter system in terms of a free Gaussian theory. However, one must recall that ϕ is a phase field, defined only modulo 2π. (In Eqs. (5.13) and (5.14) this condition is understood implicitly. At this point, it is perhaps worth reiterating that when dealing with Goldstone modes it is important to keep the underlying geometry in mind and not too tightly focus on a specific coordinate representation.) The fact that ϕ is defined only up to integer multiples of 2π manifests itself in the formation of the most interesting excitations of the superfluid; **vortices**, i.e. phase configurations $\phi(\mathbf{r},\tau)$ that change by a multiple of 2π as one moves around a certain reference coordinate, the vortex centre. Existing in parallel with harmonic phonon–like excitations discussed above, these excitations lead to a wealth of observable phenomena. However, leaving such effects aside, let us turn to the discussion of another prominent superfluid, the condensate of Cooper pairs, more generally known as the superconductor.

5.4 Superconductivity

The electrical resistivity of many metals and alloys drops suddenly to zero when the specimen is cooled to a sufficiently low temperature. This phenomenon, which goes by the name **superconductivity**, was first observed by Kammerlingh Onnes in Leiden in 1911, three years after he first liquefied Helium.[18] Perhaps more striking, a superconductor cooled below its transition temperature in a magnetic field expels all magnetic flux from its interior. This phenomenon of *perfect diamagnetism* is known as the **Meissner effect** and is characteristic of superconductivity.

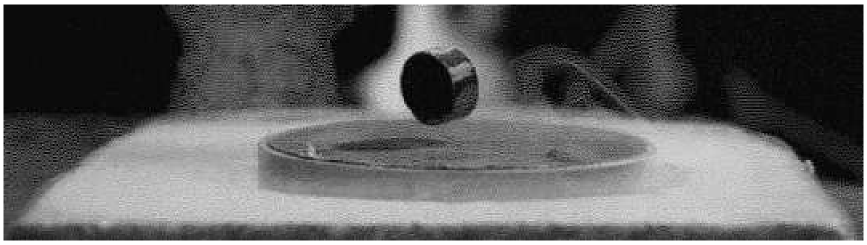
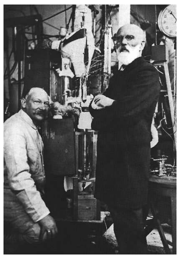


Figure 5.10: The levitation of a magnet above a (high temperature) superconductor due to the expulsion of magnetic flux.

The superconducting state is an ordered state of the conduction electrons of the metal. The nature and origin of the ordering was explained by Bardeen, Cooper and Schrieffer.[19] At low temperatures, the presence of an attractive pairwise interaction can induce an instability of the electron gas towards the formation of bound pairs of time-reversed states

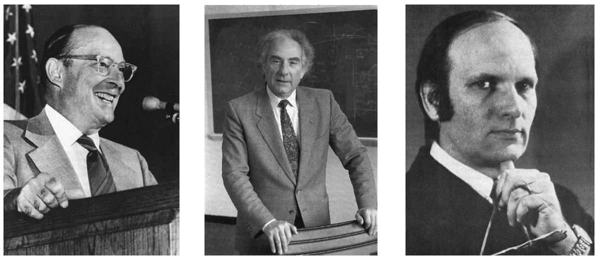
[18]

Kammerlingh Onnes 1853-1926 (left, photographed with van der Waals): 1913 Nobel Laureate in Physics for his investigations on the properties of matter at low temperatures which led, *inter alia* to the production of liquid helium.



[19] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957).

John Bardeen 1908-1991 (left), Leon N. Cooper 1930- (centre), and J. Robert Schrieffer 1931- (right): 1972 Nobel Laureate in Physics for their jointly developed theory of superconductivity. (Bardeen was also recipient of the 1956 Nobel Laureate in Physics for his research on semiconductors and discovery of the transistor effect.



$\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$ in the vicinity of the Fermi surface. We have already seen (in section 2.2.6) how the exchange of lattice vibrations or phonons can induce an attractive interaction of electrons within the Debye frequency ω_D of the Fermi surface. Being made up of two electrons, these *composite objects*, known as **Cooper pairs** behave as bosons. At low temperatures, these bosonic degrees of freedom form a condensate which is responsible for the remarkable properties of superconductors such as perfect diamagnetism.

To explore the phenomenology of the Cooper instability of the electron gas, we will adopt a simplified model known as the pairing or reduced Hamiltonian

$$\boxed{\hat{H} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} n_{\mathbf{k}\sigma} - g \sum_{kk'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}}$$

Although, strictly speaking, a realistic model of attraction would involve a more complicated momentum-dependent interaction such as the one obtained from the consideration of the electron-phonon interaction in section 2.2.6, the simple pairing interaction captures the essential physics. More importantly, to simplify our discussion, we will take the electrons to be otherwise non-interacting. In fact, the presence of a repulsive Coulomb interaction of the electrons plays a crucial role in the controlling properties of the superconductor, a point to which we will return later.

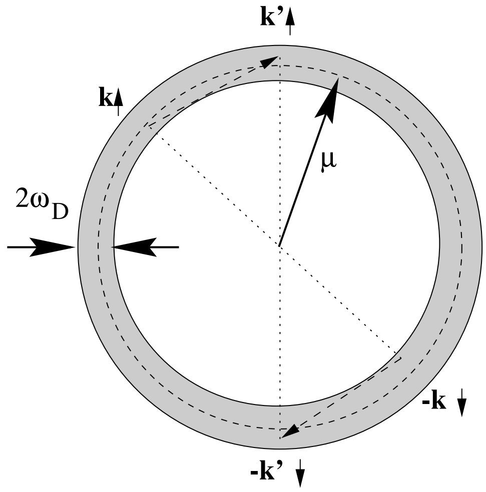


Figure 5.11: Schematic diagram showing the Fermi surface of the electron gas. The attractive interaction mediated by the exchange of phonons allows electrons within the Debye frequency ω_D of the Fermi surface to pair.

Before turning to the field theoretic formulation, we will begin by investigating a mean-field theory of the BCS transition from the stand point of the second quantisation.

5.4.1 Mean-Field Theory of Superconductivity

In its present form, the Hamiltonian explicitly involves a two-body interaction of the electrons. As such, it seems infeasible to develop an exact many-body treatment of the

Hamiltonian. Instead we will seek an approximation which renders the Hamiltonian bilinear in the electron operators and, therefore, tractable. As usual in physics, our method relies on the expected structure of the ground state wavefunction. In particular, anticipating that electrons in time-reversed states pair, let us suppose that

$$\Delta = g\sum_{\mathbf{k}}\langle \text{g.s.}|c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}|\text{g.s.}\rangle, \qquad \Delta^* = g\sum_{\mathbf{k}}\langle \text{g.s.}|c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}|\text{g.s.}\rangle$$

acquires a non-zero expectation value in the ground state. Here Δ represents an **order parameter** becoming non-zero in the condensed phase and therefore signalling the transition to the superconducting state. At first sight, a non-zero expectation value Δ looks strange: such a result would imply that the ground state wavefunction of the superconducting condensate is not an eigenstate of particle number (while one can see that the Hamiltonian commutes with $\hat{N}$). However, later, we will see that in the grand canonical ensemble the ground state wavefunction is a superposition of states involving many particles but strongly peaked around the thermodynamic density N/L^d.

To develop the mean-field approximation, let us set

$$g\sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} = \Delta + \overbrace{\left(g\sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} - \Delta\right)}^{\text{small}}$$

and keep only terms which depend up to quadratic order in the electron operators. Adding the chemical potential, the 'mean-field' Hamiltonian takes the form

$$\hat{H} - \mu\hat{N} \simeq \sum_{\mathbf{k}}\Big[\sum_{\sigma}\overbrace{(\epsilon_{\mathbf{k}}-\mu)}^{\xi_{\mathbf{k}}}\,c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} - \left(\Delta^* c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} + \Delta c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}\right)\Big] + \frac{|\Delta|^2}{g}$$

known as the **Bogoluibov** or **Gor'kov** Hamiltonian. In this simplified form, it is interesting to note that the Hamiltonian does not now conserve particle number. Instead, pairs of particles are born and annihilated out of the vacuum.

To bring the mean-field Hamiltonian to a diagonal form, it is convenient to recast it in a **Nambu spinor** representation defining

$$\Psi^\dagger_{\mathbf{k}} = \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow}\end{pmatrix}, \qquad \Psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow}\end{pmatrix}$$

after which the Hamiltonian takes the form (exercise: recall the fermionic anticommutation relations of the electron operators)

$$\boxed{\hat{H} - \mu\hat{N} = \sum_{\mathbf{k}}\left[\Psi^\dagger_{\mathbf{k}}\begin{pmatrix}\xi_{\mathbf{k}} & -\Delta \\ -\Delta^* & -\xi_{\mathbf{k}}\end{pmatrix}\Psi_{\mathbf{k}} + \xi_{\mathbf{k}}\right] + \frac{|\Delta|^2}{g}}$$

Now, being bilinear in the electron operators, the mean-field Hamiltonian can be brought to a diagonal form by employing the unitary transformation

$$\chi^\dagger_{\mathbf{k}} \equiv \begin{pmatrix}\alpha^\dagger_{\mathbf{k}\uparrow} \\ \alpha_{-\mathbf{k}\downarrow}\end{pmatrix} = \begin{pmatrix}\cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}}\end{pmatrix}\begin{pmatrix}c^\dagger_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}\end{pmatrix} \equiv U\psi^\dagger_{\mathbf{k}},$$

(under which the anticommutation relations of the new electron operators $\alpha_{\mathbf{k}\sigma}$ are maintained: exercise). Note that the notation is purely symbolic: $\alpha^\dagger_{\mathbf{k}\uparrow}$ involves a superposition of $c^\dagger_{\mathbf{k}\uparrow}$ and $c_{-\mathbf{k}\downarrow}$. Choosing Δ to be real,[20] and setting $\tan(2\theta_{\mathbf{k}}) = -\Delta/\xi_{\mathbf{k}}$, i.e. $\cos(2\theta_{\mathbf{k}}) = \xi_{\mathbf{k}}/\lambda_{\mathbf{k}}$, $\sin(2\theta_{\mathbf{k}}) = -\Delta/\lambda_{\mathbf{k}}$, where $\lambda_{\mathbf{k}} = (\Delta^2 + \xi^2_{\mathbf{k}})^{1/2}$, the transformed Hamiltonian takes the form (exercise)

$$\begin{aligned} \hat{H} - \mu\hat{N} &= \sum_{\mathbf{k}} \lambda_{\mathbf{k}} \left(\alpha^\dagger_{\mathbf{k}\uparrow}\alpha_{\mathbf{k}\uparrow} - \alpha_{-\mathbf{k}\downarrow}\alpha^\dagger_{-\mathbf{k}\downarrow}\right) + \sum_{\mathbf{k}} \xi_{\mathbf{k}} + \frac{\Delta^2}{g} \\ &= \sum_{\mathbf{k}\sigma} \lambda_{\mathbf{k}}\alpha^\dagger_{\mathbf{k}\sigma}\alpha_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - \lambda_{\mathbf{k}}) + \frac{\Delta^2}{g} \end{aligned}$$

This result shows that the elementary excitations or quasi-particle states, known as "**Bogoluibons**", created by $\alpha^\dagger_{\mathbf{k}\sigma}$, have a minimum energy Δ, the energy gap.

To determine the ground state wavefunction one simply has to identify the state which is annihilated by all the quasi-particle annihilation operators $\alpha_{\mathbf{k}\sigma}$. This condition is met uniquely by the state

$$\boxed{|\text{g.s.}\rangle = \prod_{\mathbf{k}} \alpha_{\mathbf{k}\uparrow}\alpha_{-\mathbf{k}\downarrow}|\Omega\rangle \propto \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}\right)|\Omega\rangle}$$

where $|\Omega\rangle$ represents the vacuum state, and

$$2\sin^2\theta_{\mathbf{k}} = 1 - \frac{\xi_{\mathbf{k}}}{\lambda_{\mathbf{k}}}$$

Physically, in the limit $\Delta \to 0$, $\sin^2\theta_{\mathbf{k}} \to \theta(\mu - \epsilon_{\mathbf{k}})$, and the ground state collapses to the filled Fermi sea with chemical potential μ. As Δ becomes non-zero, states in the vicinity of the Fermi surface rearrange themselves into a bound state condensate and lower their energy.

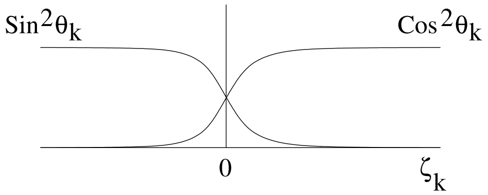


Figure 5.12: Schematic diagram showing the variation of the occupancy of the momemtum basis states in the ground state; $\sin^2\theta_k$ represents the occupancy of the $\mathbf{k}$ states. Note that the wavefunction of the ground state condensate involves the occupation of basis states with momentum in excess of $p_F = \sqrt{2m\mu}$.

An estimate of the corresponding ground state energy obtains

$$E_{\text{g.s.}} \equiv \langle\text{g.s.}|\hat{H} - \mu\hat{N}|\text{g.s.}\rangle = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - \lambda_{\mathbf{k}}) + \frac{\Delta^2}{g},$$

[20] One may show that the phase of Δ may be chosen arbitrarily, a fact to which we will return later.

a result which always lowers the energy below the non-interacting $g = 0$ theory (exercise).

Finally, to determine the scale of the order parameter we have to determine self-consistently the order parameter, viz.,

$$\begin{aligned}
\Delta &= g\sum_{\mathbf{k}}\langle \text{g.s.}|c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}|\text{g.s.}\rangle = -g\sum_{\mathbf{k}}\sin\theta_{\mathbf{k}}\cos\theta_{\mathbf{k}}\\
&= \frac{g}{2}\sum_{\mathbf{k}}\frac{\Delta}{(\Delta^2+\xi_{\mathbf{k}}^2)^{1/2}} \simeq \frac{gL^3\Delta}{2}\nu(\mu)\int_{-\omega_D}^{\omega_D}\frac{d\xi}{(\Delta^2+\xi^2)^{1/2}} = gL^3\Delta\nu(\mu)\sinh^{-1}(\omega_D/\Delta)
\end{aligned}$$

where $\sum_{\mathbf{k}} \to \int d\xi\ \nu(\xi)$ and $\nu(\xi)$ denotes the density of states. (It is left as an exercise to show that a minimisation of the g.s. energy obtains the same self-consistent equation for the order parameter.) Here we have assumed that the pairing interaction g extends over an energy scale set by ω_D. Physically, for pairing mechanisms which derive from the exchange of phonons, this energy scale is set by the corresponding Debye frequency, the maximum energy phonon. Finally rearranging this equation, one obtains

$$\boxed{\Delta = \frac{\omega_D}{\sinh(1/gL^3\nu(\mu))} \simeq 2\omega_D\exp\left[-\frac{1}{gL^3\nu(\mu)}\right]}$$

This completes our formal investigation of the BCS transition from the mean-field Hamiltonian. How does the same transition emerge from the corresponding field theory? In the following, we will develop a theory of superconductivity from the coherent state path integral for the quantum partition function.

5.4.2 Superconductivity from the Path Integral

To investigate the BCS transition within the framework of the coherent state path integral, it is convenient to abandon the long-ranged pairing Hamiltonian considered above and introduce a space-local attractive interaction contained within the **BCS Hamiltonian**,

$$\hat{H}_{\text{BCS}} = \int d^dr\left[\sum_\sigma c_\sigma^\dagger(\mathbf{r})\frac{\hat{\mathbf{p}}^2}{2m}c_\sigma(\mathbf{r}) - gc_\uparrow^\dagger(\mathbf{r})c_\downarrow^\dagger(\mathbf{r})c_\downarrow(\mathbf{r})c_\uparrow(\mathbf{r})\right].$$

Expressed in the form of the coherent state path integral, the corresponding quantum partition function takes the form

$$\mathcal{Z} = \int_{\text{a.p.b.c.}} D(\bar\psi,\psi)\exp\left\{-\int_0^\beta d\tau\int d^dr\left[\sum_\sigma\bar\psi_\sigma\left(\partial_\tau+\frac{\hat{\mathbf{p}}^2}{2m}-\mu\right)\psi_\sigma - g\bar\psi_\uparrow\bar\psi_\downarrow\psi_\downarrow\psi_\uparrow\right]\right\},$$

where $\psi(\mathbf{r},\tau)$ represent anticommuting or Grassmann fields and the nemonic a.p.b.c. denotes anti-periodic boundary conditions. As usual the quartic interaction of the fields prevents the partition function from being evaluated explicitly. Moreover, anticipating the existence of a transition of the electron gas to a condensed phase in which electrons in the vicinity of the Fermi surface are paired, we can expect that a perturbative expansion in the coupling constant g will be inadequate. Motivated by the mean-field theory discussed above, we will instead introduce a bosonic field Δ to decouple the interaction and which will have the physical significance of the complex order parameter.

The decoupling is arranged using a **Hubbard-Stratonovich** transformation

$$e^{g\int d\tau\int d^dr\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow} = \int D(\Delta^*,\Delta)\exp\left\{-\int d\tau\int d^dr\left[\frac{1}{g}|\Delta|^2 - \left(\Delta^*\psi_\downarrow\psi_\uparrow + \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow\right)\right]\right\}$$

where $\Delta(\mathbf{r},\tau)$ represents a dynamically fluctuating **bosonic** or complex field with a symmetry that reflects that of the bilinear $\psi_\downarrow\psi_\uparrow$. Taking Δ to be homogeneous in space and time, the quantum Hamiltonian corresponding to the action coincides with that of the mean-field Hamiltonian considered in the previous section. Motivated by that analysis, we turn to the **Nambu spinor** representation

$$\bar{\Psi} = \begin{pmatrix}\bar{\psi}_\uparrow & \psi_\downarrow\end{pmatrix}, \qquad \Psi = \begin{pmatrix}\psi_\uparrow \\ \bar{\psi}_\downarrow\end{pmatrix}.$$

wherein the quantum partition function assumes the form

$$\begin{aligned}\mathcal{Z} &= \int D(\bar{\psi},\psi)\int D(\Delta^*,\Delta)\exp\left\{-\int d\tau\int d^dr\left[\frac{1}{g}|\Delta|^2 + \bar{\Psi}\hat{\mathcal{G}}^{-1}\Psi\right]\right\},\\ \hat{\mathcal{G}}^{-1} &= \begin{pmatrix}[\hat{G}_0^{(\mathrm{p})}]^{-1} & -\Delta \\ -\Delta^* & [\hat{G}_0^{(\mathrm{h})}]^{-1}\end{pmatrix},\end{aligned}$$

where $[\hat{G}_0^{(\mathrm{p})}]^{-1} = \partial_\tau + (\epsilon_{\hat{\mathbf{p}}} - \mu)$ and $[\hat{G}_0^{(\mathrm{h})}]^{-1} = \partial_\tau - (\epsilon_{\hat{\mathbf{p}}} - \mu)$ represents the non-interacting Green function of the particle and hole respectively, and $\hat{\mathcal{G}}$ is known as the **Gor'kov Green function**.

Being Gaussian in the fermionic fields, the functional integral over the Grassmann fields can be performed straightforwardly, and yields the formal expression

$$\mathcal{Z} = \int D(\Delta^*,\Delta)\exp\left[-\int d\tau\int d^dr\frac{1}{g}|\Delta|^2 + \ln\det\hat{\mathcal{G}}^{-1}\right],$$

where we have written $\det\mathcal{G}^{-1} = \exp[\ln\det\mathcal{G}^{-1}]$. By introducing a Hubbard-Stratonovich decoupling of the local interaction, we have succeeded in expressing the quantum partition function as a path integral over an auxiliary bosonic field Δ. Further progress is possible only within some approximation. Empirically, we know that the superconducting transition is *second order*, i.e. the order parameter Δ develops a non-zero expectation value below a critical temperature T_c growing *continuously from zero.* At temperatures $T \ll T_c$, spatial and temporal fluctuations around the expectation value $\bar{\Delta}$ can be treated as small. In this limit, the action can be treated within a **mean-field** approximation where the partition function is dominated by the **saddle-point configuration** of $\bar{\Delta}$. The saddle-point analysis, which leads to a self-consistent equation for $\bar{\Delta}$ known as the **Gap Equation**, is left as an exercise in Problem Set 4. Instead we will focus on temperatures $T \sim T_c$ in which an effective action for Δ can be obtained.

In the vicinity of the transition temperature T_c, the order parameter Δ is expected to be small. We are therefore at liberty to look for a perturbative expansion in powers of Δ. Setting

$$\hat{\mathcal{G}}^{-1} = \hat{\mathcal{G}}_0^{-1}\left[1 - \hat{\mathcal{G}}_0\begin{pmatrix}0 & \Delta \\ \Delta^* & 0\end{pmatrix}\right],$$

where, by definition, $\hat{\mathcal{G}}_0 \equiv \hat{\mathcal{G}}(\Delta = 0)$, and expanding the action to second order in Δ,[21] the quantum partition function takes the form (exercise)

$$\mathcal{Z} = \mathcal{Z}_0 \int D(\Delta^*, \Delta) e^{-S}, \qquad S = \sum_{\omega_n, \mathbf{q}} \left[\frac{1}{g} + \Pi(\omega_n, \mathbf{q})\right] |\Delta_{\omega_n, \mathbf{q}}|^2 + O(\Delta^4), \tag{5.16}$$

with (see Fig. 5.13a)

$$\Pi(\omega_m, \mathbf{q}) = \frac{1}{\beta L^d} \sum_{\omega_n, \mathbf{k}} G_0^{(\mathrm{p})}(\omega_n, \mathbf{k}) G_0^{(\mathrm{h})}(\omega_n + \omega_m, \mathbf{k} + \mathbf{q}).$$

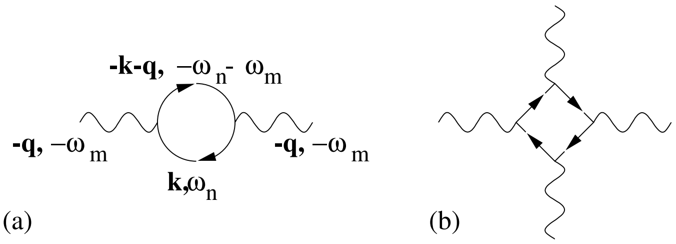


Figure 5.13: Diagrammatic representation of (a) the response function $\Pi(\omega_m, \mathbf{q})$, and (b) the quartic vertex.

An instability of the electron gas towards the formation of a paired condensate is signalled by the appearance of a non-zero expectation value of the anomalous average $\langle \psi_\uparrow \psi_\downarrow \rangle$ (i.e. $\bar{\Delta} \neq 0$). Moreover, intuitively, we would expect the action to be minimised by a spatially and temporally uniform field configuration of Δ. Applying this *Ansatz*, we are led to consider a gradient expansion of the action in powers of Δ.

Neglecting temporal fluctuations altogether, a gradient expansion in powers of $\mathbf{q}$ obtains

$$\Pi(\omega_m, \mathbf{q}) = \Pi(0,0) + \frac{1}{2}\mathbf{q}^2 \overbrace{\partial_{|\mathbf{q}|}\Pi(0,0)}^{K>0} + O(i\omega_m, \mathbf{q}^4),$$

where, by symmetry, we have made use of the fact that $\Pi = \Pi(\mathbf{q}^2)$. Substituting this expansion into Eq. (5.16), and returning to the real space representation, we obtain the static effective action

$$\boxed{S[\Delta] = \beta \int d^d r \left[\frac{t}{2}|\Delta|^2 + \frac{K}{2}|\partial\Delta|^2 + u|\Delta|^4 + \cdots\right]} \tag{5.17}$$

where $t/2 = g + \Pi(0,0)$ represents an effective 'chemical potential' for Δ, and $u > 0$ represents the constant coefficient associated with the quartic vertex (see Fig. 5.13b). (The calculation of the coefficients K and u is left as an exercise in Matsubara frequency summations! However, in the following, we will not need to know their form explicitly.)

[21] Here we have made use of the identity $\ln\det\hat{\mathcal{G}}^{-1} = \mathrm{tr}\ln\hat{\mathcal{G}}^{-1}$ to form the expansion

$$\mathrm{tr}\ln\hat{\mathcal{G}}^{-1} = \mathrm{tr}\ln\hat{\mathcal{G}}_0^{-1} + \sum_{n=1}^{\infty} \frac{1}{n}\mathrm{tr}\left[\hat{\mathcal{G}}_0 \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}\right]^n$$

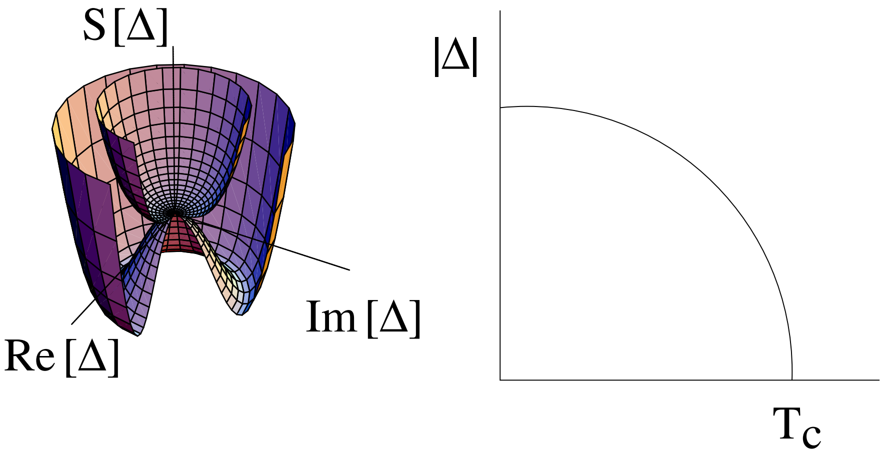


Figure 5.14: Dependence of the effective action S_0 on the complex mean-field order parameter Δ. At $t > 0$, fluctuations of Δ are energetically unfavourable whilst for $t < 0$, the action becomes unstable towards the formation of a non-zero expectation value of Δ. Also shown is the expectation value $\bar{\Delta}$ of the order parameter as a function of temperature.

5.4.3 Gap Equation

Applying a **mean-field approximation**, i.e. focusing on the spatially homogeneous component Δ_0, the effective action takes the form

$$\boxed{\frac{S[\Delta_0]}{\beta L^d} = \frac{t}{2}|\Delta_0|^2 + u|\Delta_0|^4.} \tag{5.18}$$

In particular, when the effective chemical potential t becomes negative, the expectation value of $\langle \Delta_0 \rangle$ becomes finite (see Fig. 5.14). Summing over momenta, and making use of the identity $\frac{1}{L^d}\sum_{\mathbf{k}} = \int_{-\infty}^{\infty} d\xi \nu(\xi)$ where, as usual, $\nu(\xi)$ denotes the density of states at energy ξ, one obtains

$$\Pi(0,0) = -\frac{1}{\beta L^d}\sum_{\omega_n,\mathbf{k}} \frac{1}{\omega_n^2 + (\epsilon_{\mathbf{k}} - \mu)^2} \simeq -\frac{1}{\beta}\sum_{\omega_n}\int_{-\infty}^{\infty} d\xi \frac{\nu(\xi+\mu)}{\omega_n^2 + \xi^2} \simeq -\sum_{\omega_n}\frac{\pi\nu(\mu)}{\beta\omega_n}.$$

Recalling that the attractive interaction of the electrons was mediated by the exchange of phonons, we note that the Matsubara summation should be cut-off at the scale of the Debye frequency. Setting $\omega_D = (2n_{\max}+1)\pi/\beta$, we obtain

$$\Pi(0,0) \simeq -\nu(\mu)\sum_{n=-n_{\max}}^{n_{\max}} \frac{1}{2n+1} \simeq -2\nu(\mu)\int_0^{n_{\max}} \frac{dn}{2n+1} \simeq -\nu(\mu)\ln(\beta\omega_D).$$

Equating this result with $1/g$, we deduce that the electron gas becomes unstable towards the formation of a pair condensate when

$$\boxed{T < T_c \equiv \omega_D \exp\left[-\frac{1}{\nu(\mu)g}\right]}$$

Substituting this result into the expression for t we find that, in the vicinity of T_c,

$$t = 2\nu(\mu)\ln\left(\frac{T}{T_c}\right) \simeq 2\nu(\mu)\left(\frac{T-T_c}{T_c}\right),$$

i.e. one may think of the parameter t as a '**reduced temperature**'.

Finally, taking the partition function to be dominated by the minimum of the mean-field action (5.18) $\mathcal{Z} \sim \exp[-S_0(|\bar{\Delta}|)]$ (i.e. applying a saddle-point approximation), we find a spontaneous breaking of the $U(1)$ symmetry of the complex order parameter — i.e. $\bar{\Delta}$ has a magnitude (see Fig. 5.14b),

$$|\bar{\Delta}| = \begin{cases} 0 & t > 0, \\ \sqrt{t/4u} & t < 0. \end{cases}$$

with arbitrary but constant phase. This situation is reminiscent of the Heisenberg ferromagnet: the complex order parameter is isomorphic to a 'two-component' spin where the phase degree of freedom is mirrored in the orientation of the moment. According to the Mermin-Wagner theorem, the breaking of the continuous $U(1)$ symmetry should be accompanied by the appearance of massless **Goldstone modes**. The stability of the mean-field solution towards fluctuations is governed by the effective action $S[\Delta]$ (5.17).

To summarise, the quantum partition function of an electron gas subject to a local attractive pairing interaction has been cast in the form of a quantum field theory involving a complex scalar field $\Delta(\mathbf{r},\tau)$ whose expectation value is connected to the anomalous average $\langle\psi_\uparrow\psi_\downarrow\rangle$. A gradient expansion of the effective action in powers Δ reveals an instability of the electron gas towards the formation of a spatially and temporally uniform pair condensate. Fluctuations of the order parameter Δ around its mean-field expectation value are described by a low energy effective action (5.17).

In fact, interpreted as an effective Free energy, this result might have been guessed on purely phenomenological grounds: indeed, identifying the anomalous average as an appropriate **order parameter**, Δ Eq. (5.17) is consistent with a gradient expansion of the Free energy (in powers of Δ) compatible with symmetry and the observed temperature dependences. The phenomenology of superconductivity (expressed by the free energy of (5.17) and known as the **Ginzburg-Landau Theory**) anticipated the microscopic theory of BCS.

5.4.4 †Superconductivity: Anderson-Higgs Mechanism

▷ Info. So far, our analysis of the quantum partition function associated with the BCS Hamiltonian is incomplete. Indeed, no reference has yet been made to the characteristic or defining properties: superconductivity, and perfect diamagnetism. To discover such phenomena within our theory we have to generalise our approach to accommodate an electromagnetic field. At the microscopic level, we can do so by incorporating into the BCS Hamiltonian the canonical momentum $\mathbf{p} \to \mathbf{p} - e\mathbf{A}/c$, and introducing the classical action associated with the electromagnetic field: $\mathcal{L} = -F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ represents the electromagnetic field tensor. Leaving the formal derivation as an exercise, we can, in the spirit of Ginzburg-Landau theory, guess the appropriate form of the resulting (time-independent) action:

$$\boxed{S = \beta \int d^d r \left[\frac{t}{2}|\Delta|^2 + \frac{K}{2}|(\partial + i2e\mathbf{A}/c)\Delta|^2 + u|\Delta|^4 + \frac{1}{2}(\partial \times \mathbf{A})^2 \right]} \tag{5.19}$$

This result can be inferred from gauge invariance of the order parameter under the local $U(1)$ transformation $\psi \to e^{ie\phi(\mathbf{r})/c}\psi$, $\Delta(\mathbf{r}) \to e^{2ie\phi(\mathbf{r})/c}\Delta(\mathbf{r})$ — see below — (or indeed obtained from first principles by generalising the derivation presented above). The apparent 'doubling of the electric charge' can be interpreted as reflecting the effective charge of a Cooper pair.

Gauge Invariance: In the presence of the electromagnetic field, the quantum partition function $\mathcal{Z} = \int D\mathbf{A} \int D[\Delta, \Delta^*] e^{-S}$ exhibits a gauge invariance under the transformation (exercise)

$$\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} - \partial\phi, \qquad \Delta \mapsto \Delta' = e^{2ie\phi/c}\Delta.$$

That is, under such a transformation, the action remains invariant. Therefore, by gauge fixing $\partial\phi = \mathbf{A}$, the phase of the order parameter can be eliminated from the action and the effective action takes the form

$$S = \beta \int d^d r \left[\frac{t}{2}|\Delta|^2 + \frac{K}{2}(\partial|\Delta|)^2 - \frac{m_\nu^2}{2}\mathbf{A}^2 + u|\Delta|^4 + \frac{1}{2}(\partial \times \mathbf{A})^2 \right],$$

where $m_\nu^2 = 4e^2 K|\Delta|^2/c^2$. As a result, we find that the massless phase degrees of freedom ϕ have disappeared from the action! They have been subsumed into the longitudinal mode of the vector field $\mathbf{A}$, which has itself become massive. This is an example of the celebrated **Anderson-Higgs mechanism**: below the transition temperature, the Goldstone bosons (in this case ϕ) and the gauge field (in this case the electromagnetic field — the photon) conspire to create massive excitations, and the massless excitations are unobservable.

It is instructive to interpret this result from the saddle-point or mean-field equations of the motion of the order parameter and vector potential. Minimising the effective action (5.19) with respect to spatial variations of $\bar{\Delta}$ and $\mathbf{A}$, one obtains the **Gross-Pitaevskii equations** (exercise)

$$\left[-K\left(\partial + 2ie\mathbf{A}/c\right)^2 + t + 4u|\Delta|^2 \right]\Delta = 0$$
$$\partial \times (\partial \times \mathbf{A}) \equiv \mathbf{j} - m_\nu^2 \mathbf{A}, \qquad \mathbf{j} = 2i\frac{e}{c}K(\Delta\partial\bar{\Delta} - \bar{\Delta}\partial\Delta).$$

Substituting the order parameter by its homogeneous mean field value and differentiating the second equation, one finds that $\partial \times (\partial \times \mathbf{B}) = -m_\nu^2 \mathbf{B}$, from which it follows that

$$\left(\partial^2 - m_\nu^2\right)\mathbf{B} = 0,$$

where $\mathbf{B} = \partial \times \mathbf{A}$. This result, known as the first **London Equation**, admits $\mathbf{B} = 0$ as the only constant spatially uniform solution. In a uniform superconductor the magnetic field is zero — the **Meissner effect**. At the edge of the superconductor, this equation can be integrated to give $\mathbf{B} \sim e^{-m_\nu x}$ showing the field to penetrate a distance m_ν^{-1} — the **Penetration depth** — into the sample.

That these equations imply superconductivity can be inferred from the time derivative of the current $\partial_t \mathbf{j} = m_\nu^2 \mathbf{E}$, where $\mathbf{E}$ denotes an external electric field (the second London equation). If a uniform field is applied for a time t_0, a current $m_\nu^2 \mathbf{E} t_0$ builds up. This current remains even if the electric field is subsequently switched off. This contrasts with conventional conductors where there is a relaxation of the current.

Superconductivity is destroyed by a sufficiently strong magnetic field. An estimate of the critical field can be made by studying the Gross-Pitaevskii equation

$$-K\left(\partial + 2ie\mathbf{A}/c\right)^2 \Delta(\mathbf{r}) = t\Delta(\mathbf{r})$$

This equation is formally equivalent to a Schrödinger equation describing a particle of charge $2e$ and mass $m = 1/2K$ in a uniform magnetic field. The lowest Landau level is defined by the condition $t/K = 2eB_c/c$. This defines the highest field where superconductivity can occur.

Finally, we remark that, to expel a magnetic field from a sample we require an energy of $B^2/2$ per unit volume to resist the magnetic pressure. This must be compensated by the condensation energy $S/\beta L^d$. If the threshold field is smaller than the critical field B_c (Type II) magnetic field penetrates the sample in the form of flux tubes. At low temperatures the latter arrange themselves in a hexagonal configuration known as an **Abrikosov vortex lattice**. Superconductors where the situation is opposite are known as Type I.

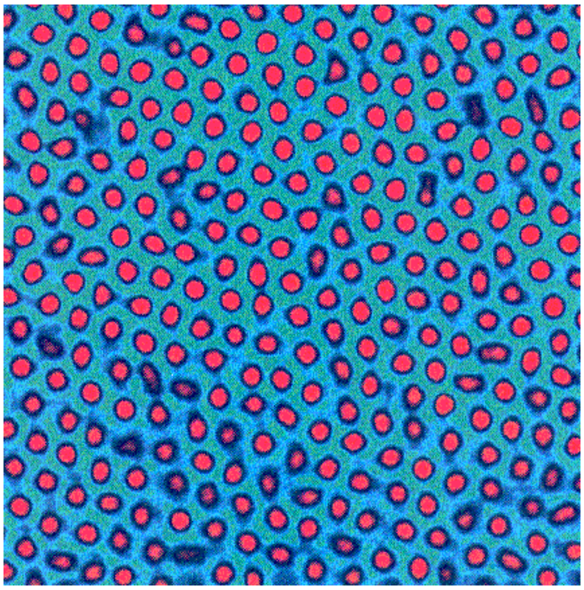


Figure 5.15: Bitter patter of an Abrikosov vortex lattice.

This concludes our formal discussion of the instability of the electron gas. For the sake of clarity we avoided a detailed discussion of the physical manifestations of the Cooper instability of the electron gas — it was used here merely as a vehicle for illustrating the general approach of the coherent state path integral. However, before leaving this section, a few remarks concerning the question of universality are in order.

5.4.5 Statistical Field Theory: Ferromagnetism Revisited

The motivation that stands behind the phenomenology of Ginzburg-Landau theory has profound implications that go beyond its application to superconductivity pervading all areas of physics. To illustrate the generality of the concept, let us temporarily leave behind superconductivity and consider the *classical* equilibrium statistical mechanics of a 'one-component' or **Ising ferromagnet** (i.e. spin degrees of freedom can take only two values: $S_i = \pm 1$). Our previous considerations in chapter 2 have emphasised that, when viewed microscopically, the development of magnetic moments on the atomic lattice sites of a crystal and the subsequent ordering of the moments is a complex process involving the cooperative behaviour of many interacting electrons. However, at first sight, this picture seems to be at odds with the empirical observation that thermodynamic properties of different macroscopic ferromagnetic systems seem to be the same — e.g. temperature dependence of the specific heat, susceptibility, etc. Moreover, the thermodynamic critical properties of completely different physical systems, such as an Ising ferromagnet and a liquid at its boiling point, show the same dependence on, say, temperature. What is the physical origin of this **Universality**?

Suppose we take a ferromagnetic material and measure some of its material properties such as its magnetisation. Dividing the sample into two roughly equal halves, keeping the internal variables like temperature and magnetic field the same, the macroscopic properties of each piece will then be the same as the whole. The same holds true if the process is repeated. But eventually, after many iterations, something different must happen because we know that the magnet is made up of electrons and ions. The characteristic length scale at which the overall properties of the pieces begins to differ markedly from those of the original defines a **correlation length**. It is the typical length scale over which the fluctuations of the microscopic degrees of freedom are correlated.

Now experience tells us that a ferromagnet may abruptly change its macroscopic behaviour when the external conditions such as the temperature or magnetic field are varied. The points at which this happens are called **critical points**, and they mark a **phase transition** from one state to another. In the ferromagnet, there are essentially two ways in which the transition can occur (see Fig. 5.16). In the first case, the two states on either side of the critical point (spin up) and (spin down) coexist at the critical point. Such transitions, involve **discontinuous** behaviour of thermodynamic properties and are termed **first-order**. (c.f. melting of a three-dimensional solid) The correlation length at such a first-order transition is generally finite.

In the second case, the transition is **continuous**, and the correlation length becomes effectively infinite. Fluctuations become correlated over all distances, which forces the whole system to be in a unique, critical, phase. The two phases on either side of the transition (paramagnetic and ferromagnetic) must become identical as the critical point is approached. Therefore, as the correlation length diverges, the magnetisation goes smoothly to zero. The transition is said to be **second-order**.

The divergence of the correlation length in the vicinity of a second order phase transition suggests that properties near the critical point can be accurately described within an effective theory involving only long-range collective fluctuations of the system. This invites the construction of a phenomenological Hamiltonian or Free energy which is constrained only by the fundamental symmetries of the system — **Ginzburg-Landau theory**. Although the detailed manner in which the material properties and microscopic couplings of the ferromagnet influence the parameters of the effective theory might be unknown, qualitative properties such as the **scaling** behaviour are completely defined.

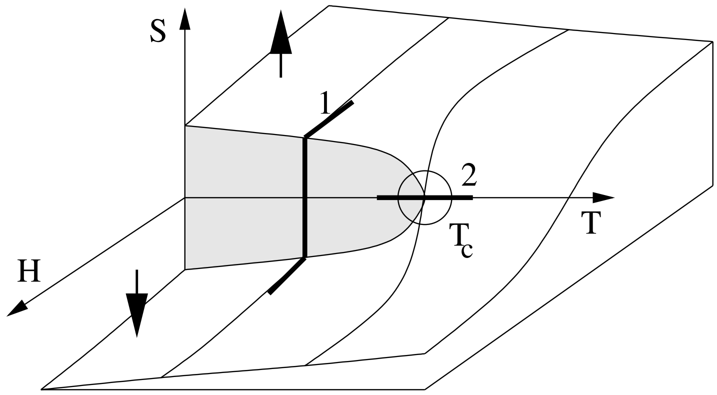


Figure 5.16: Phase diagram of the Ising ferromagnet showing the average magnetisation S as a function of magnetic field H and Temperature T. Following trajectory 1 by changing the magnetic field at constant temperature $T < T_c$, the sample undergoes a first order phase transition from an average 'spin-up' phase to an average 'spin-down'. By changing the temperature at fixed zero magnetic field, the system undergoes a second order phase transition at $T = T_c$ where the average magnetisation grows continuously from zero. This second order transition is accompanied by a *spontaneous symmetry breaking* in which the system chooses to be in either an up or down-spin phase. (Contrast this phase diagram with that of the liquid-gas transition — magnetisation $S \to$ density ρ, and magnetic field $H \to$ pressure.) The circle marks the region in the vicinity of the critical point where the correlation length is large as compared to the microscopic scales of the system, and Ginzburg-Landau theory applies.

Following this philosophy, the Ginzburg-Landau theory of the Ising ferromagnet is defined by gradient expansion of the effective Free energy in powers of the order parameter, the local magnetisation $S(\mathbf{r})$. Respecting the symmetry properties of the microscopic Hamiltonian (translational and rotational invariance in the spatial degrees of freedom, up-down or Z_2 invariance in the internal spin degrees of freedom, etc), the partition function of the Ising ferromagnet can be expressed in the form of a functional field integral over different spin configurations $S(\mathbf{r})$

$$\mathcal{Z} = \int DS(\mathbf{r}) e^{-\beta H[S(\mathbf{r})]},$$

with the effective Ginzburg-Landau Hamiltonian or Free energy functional

$$\beta H[S(\mathbf{r})] = \int d^d r \left[\frac{t}{2} S^2 + \frac{K}{2} (\partial S)^2 + u S^4 + \cdots + HS \right].$$

At the mean-field level (i.e. neglecting fluctuations of the magnetisation field) a minimisation of the effective Free energy (in the absence of an external magnetic field)

$$\frac{\beta H(S)}{V} = \frac{t}{2} S^2 + u S^4,$$

known as the **Landau Free energy**, leads to the average magnetisation

$$\bar{S} = \begin{cases} 0 & t > 0, \\ \sqrt{t/4u} & t < 0. \end{cases}$$

Thus a non-zero magnetisation develops when $t < 0$ identifying this parameter as the reduced temperature $t = (T - T_c)/T_c$.

The Ginzburg-Landau Free energy can be compared with that obtained from the microscopic Hamiltonian for the superconductor above (5.17). Apart from the complex nature of the order parameter Δ, the action coincides. In fact, we might very well have circumvented the analysis of the microscopic Hamiltonian, and written the Ginzburg-Landau Free energy on purely phenomenological grounds. The dependence of the parameters K, u, etc., on the microscopic or material properties of the system would have been unavailable, but the nature of the critical point and the physical properties associated with the transition would have been accessible. Indeed, as mentioned above much of the phenomenology of conventional superconductivity was developed and understood in this framework even prior to the discovery of the BCS theory.

From this result, we can draw important conclusions: Critical properties in the vicinity of a both classical and quantum second order phase transitions fall into a limited number of **universality classes** defined, not by detailed material parameters, but by the fundamental symmetries of the system. When we study the critical properties of the Ising transition in a one-component ferromagnet, we learn about the nature of the liquid-gas transition! Similarly, in the jargon of statistical field theory, the superconductor, with its complex order parameter Δ is in the same universality class as the two-component or XY Heisenberg ferromagnet. The analyses of critical properties associated with different universality classes is the subject of **Statistical field theory**.