Lecture III: Quantising the Classical Field

Having established that the low energy properties of the atomic chain are represented by a free scalar classical field theory, we now turn to the formulation of the quantum system.

▷ Canonical Quantisation procedure: recall point particle mechanics

1. Define canonical momentum $p = \partial_{\dot{x}} L$
2. Construct Hamiltonian $H = p\dot{x} - L(p, x)$
3. Promote position and momentum to operators with canonical commutation relations

$$x \mapsto \hat{x}, \qquad p \mapsto \hat{p}, \qquad [\hat{p}, \hat{x}] = -i\hbar, \qquad H \mapsto \hat{H}$$

1. Canonical momentum: natural generalisation to continuous field

$$\boxed{\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}}$$

applied to atomic chain, $\pi = \partial_{\dot{\phi}}(m\dot{\phi}^2/2) = m\dot{\phi}$

2. Classical Hamiltonian:

$$H[\phi, \pi] \equiv \int dx \ \overbrace{\left[\pi\dot{\phi} - \mathcal{L}(\partial_x \phi, \dot{\phi})\right]}^{\text{Hamiltonian density } \mathcal{H}(\phi,\pi)}$$

$$\text{i.e.} \qquad \mathcal{H}(\phi, \pi) = \frac{1}{2m}\pi^2 + \frac{k_s a^2}{2}(\partial_x \phi)^2$$

3. Canonical Quantisation:

(a) promote $\phi(x)$ and $\pi(x)$ to operators: $\phi \mapsto \hat{\phi}$, $\pi \mapsto \hat{\pi}$

(b) generalise the canonical commutation relations

$$[\hat{\pi}(x), \hat{\phi}(x')] = -i\hbar\delta(x - x')$$

N.B. $[\delta(x - x')] = [\text{Length}]^{-1}$ (exercise)

Operator-valued functions $\hat{\phi}$ and $\hat{\pi}$ referred to as quantum fields

Comments: $\hat{H}$ *represents a quantum field theoretical formulation of elastic chain, but not yet a solution. In fact, the development of a variety of methods for the analysis of quantum field theoretical models will represent major part of course. Here, objective is merely to exemplify how physical information can be extracted from this particular model.*

As with any fn, operator-valued fns. can be expressed as Fourier series expansion:

$$\left\{\begin{array}{l}\hat\phi(x)\\ \hat\pi(x)\end{array}\right. = \frac{1}{L^{1/2}}\sum_k e^{\pm ikx}\left\{\begin{array}{l}\hat\phi_k\\ \hat\pi_k\end{array}\right., \qquad \left\{\begin{array}{l}\hat\phi_k\\ \hat\pi_k\end{array}\right. \equiv \frac{1}{L^{1/2}}\int_0^{L=Na} dx\, e^{\mp ikx}\left\{\begin{array}{l}\hat\phi(x)\\ \hat\pi(x)\end{array}\right.$$

$\sum_k$ runs over all quantised wavevectors $k = 2\pi m/L$, $m \in \mathcal{Z}$

Exercise: confirm $[\hat\pi_k, \hat\phi_{k'}] = -i\hbar\delta_{kk'}$

Advice: *Maintain strict conventions(!) — we will pass freely between real and Fourier space (and we will not care to write a tilde in each case).*

Hermiticity: $\hat\phi^\dagger(x) = \hat\phi(x)$, implies $\hat\phi_k^\dagger = \hat\phi_{-k}$ (similarly $\hat\pi$). Using

$$\int_0^L dx\,(\partial\hat\phi)^2 = \sum_{k,k'}(ik\hat\phi_k)(ik'\hat\phi_{k'})\overbrace{\frac{1}{L}\int_0^L dx\, e^{i(k+k')x}}^{\delta_{k+k',0}} = \sum_k k^2\hat\phi_k\hat\phi_{-k}\ \left(=\sum_k k^2|\hat\phi_k|^2\right)$$

$$\hat H = \sum_k\left[\frac{1}{2m}\hat\pi_k\hat\pi_{-k} + \overbrace{\frac{k_s a^2}{2}}^{m\omega_k^2/2} k^2\ \hat\phi_k\hat\phi_{-k}\right]$$

$\omega_k = v|k|$, $v = a(k_s/m)^{1/2}$

In Fourier representation, 'modes k' decoupled

Comments:

- $\hat H$ provides explicit description of the low energy excitations of the system (waves) in terms of their microscopic constituents (atoms)
- However, it would be much more desirable to develop a picture where the relevant excitations appear as fundamental units...

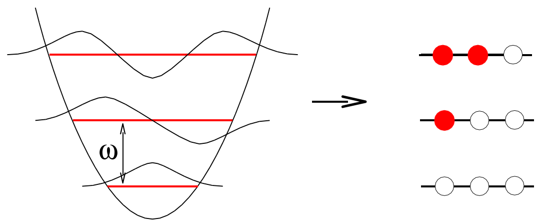
  to learn how, noting the structural similarity, let us digress and discuss/revise the...

▷ Quantum Harmonic Oscillator (Revisited)

$$\hat H = \frac{\hat p^2}{2m} + \frac{1}{2}m\omega^2\hat q^2$$

Although a single-particle problem, its property of equidistant energy level separation, $\epsilon_n = \hbar\omega\left(n+\frac{1}{2}\right)$ suggests alternative interpretation:

State with energy ϵ_n can be viewed as an "assembly" of n elementary, structureless (i.e. the only 'quantum number' is their energy $\hbar\omega$), bosonic particles (state can be multiply occupied) each having an energy $\hbar\omega$



▷ Formally, defining the ladder operators

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \qquad \hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$\rightsquigarrow$ canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ (characteristic of bosons)

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}$$

If we find state $|0\rangle$: $\hat{a}|0\rangle = 0 \rightsquigarrow \hat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle$, i.e. $|0\rangle$ provides ground state

Using commutation relations, one may show $|n\rangle \equiv \frac{1}{(n!)^{1/2}}\,\hat{a}^{\dagger n}|0\rangle$

is (normalised) eigenstate with eigenvalue $\hbar\omega(n + \frac{1}{2})$

<u>Comments</u>: a-representation affords a many-particle interpretation

- $|0\rangle$ represents 'vacuum', i.e. state with no particles
- $\hat{a}^\dagger|0\rangle$ represents state with a single particle of energy $\hbar\omega$
- $\hat{a}^{\dagger n}|0\rangle$ is many-body state with n particles

  i.e. $\hat{a}^\dagger$ is an operator that <u>creates</u> particles
- In 'diagonal' form $\hat{H} = \hbar\omega(\hat{a}^\dagger a + \frac{1}{2})$ simply counts number of particles,

  i.e. $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$, and assigns an energy $\hbar\omega$ to each

▷ Returning to quantum harmonic chain, let us then introduce new representation:

$$a_k \equiv \sqrt{\frac{m\omega_k}{2\hbar}}\left(\hat{\phi}_k + \frac{i}{m\omega_k}\hat{\pi}_{-k}\right), \qquad a_k^\dagger \equiv \sqrt{\frac{m\omega_k}{2\hbar}}\left(\hat{\phi}_{-k} - \frac{i}{m\omega_k}\hat{\pi}_k\right)$$

N.B. By convention, drop hat from operators a

with $[a_k, a_{k'}^\dagger] = \frac{i}{2\hbar}\Big(\overbrace{[\hat{\pi}_{-k}, \hat{\phi}_{-k'}]}^{-i\hbar\delta_{kk'}} - [\hat{\phi}_k, \hat{\pi}_{k'}]\Big) = \delta_{kk'}$

i.e. bosonic commutation relations

▷ And obtain (exercise — PS I)

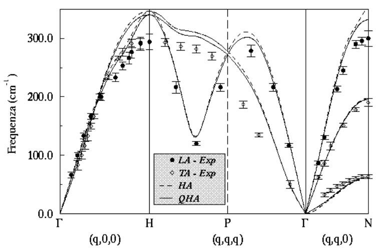
$$\boxed{\hat{H} = \sum_k \hbar\omega_k\left(a_k^\dagger a_k + \frac{1}{2}\right)}$$

Elementary collective excitations of quantum chain (phonons)
created/annihilated by bosonic operators $a_k^\dagger$ and a_k

Spectrum of excitations is linear $\omega_k = v|k|$ (cf. relativistic)

LESSONS:

- Low-energy excitations of discrete model involve slowly varying collective modes; i.e. each mode involves many atoms
- Low-energy ($k \to 0$) $\mapsto$ long-wavelength excitations — i.e. universal, insensitive to microscopic detail;
- This fact allows many different systems to be mapped onto a few (hopefully simple) classical field theories;
- Canonical quantisation procedure for point mechanics generalises to quantum field theory;
- Simplest model actions (such as the one considered here) are quadratic in the fields — known as free field theory;
- More generally, interactions $\rightsquigarrow$ non-linear eqs. of motion and interacting quantum field theories



▷ Other examples? †Quantum Electrodynamics

EM field — specified by 4-vector potential $A(x) = (\phi(x), \mathbf{A}(x))$ $\qquad (c = 1)$

$$\text{Classical action :} \quad \boxed{S[A] = \int d^4x \; \mathcal{L}(A), \qquad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ — EM field tensor

Classical equation of motion:

$$\overbrace{\partial_{A^\alpha} \mathcal{L} - \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = 0}^{\text{Euler} - \text{Lagrange eqns.}} \qquad \mapsto \qquad \overbrace{\partial_\alpha F^{\alpha\beta} = 0}^{\text{Maxwell's eqns.}}$$

Quantisation of classical field theory identifies elementary excitations: photons

for more details, see handout, or go to QFT!