Lecture XXIII: Field Theory of Superconductivity

Following our discussion of the field theory of BEC and superfluidity in the weakly interacting Bose gas, we turn now to condensation phenomena in Fermi systems

▷ Starting point is BCS Hamiltonian for local pairing interaction: $g \equiv \frac{V}{L^d}$

$$\hat{H} = \int d^d r \left[\sum_\sigma c^\dagger_\sigma(\mathbf{r}) \frac{\hat{\mathbf{p}}^2}{2m} c_\sigma(\mathbf{r}) - g\, c^\dagger_\uparrow(\mathbf{r}) c^\dagger_\downarrow(\mathbf{r}) c_\downarrow(\mathbf{r}) c_\uparrow(\mathbf{r}) \right]$$

▷ Quantum partition function: $\mathcal{Z} = \mathrm{tr}\, e^{-\beta(\hat{H} - \mu \hat{N})}$

$$\mathcal{Z} = \int_{\psi(\beta)=-\psi(0)} D(\bar{\psi}, \psi) \exp\Big\{ - \overbrace{\int_0^\beta d\tau d^d r}^{x\equiv(\tau,\mathbf{r}) \quad \int dx} \Big[\sum_\sigma \bar{\psi}_\sigma \overbrace{\left(\partial_\tau + \frac{\hat{\mathbf{p}}^2}{2m} - \mu\right)}^{[\hat{G}_0^{(\mathrm{p})}]^{-1}} \psi_\sigma - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \Big] \Big\}$$

where $\psi_\sigma(\mathbf{r}, \tau)$ denote Grassmann (anticommuting) fields

Analysis of interacting QFT?

- Perturbative expansion in g?

  Transition to condensate non-perturbative in g

- "Mean-field" analysis:

  condensation of pair wavefunction signalled by "anomalous average" $\langle c^\dagger_\uparrow c^\dagger_\downarrow \rangle$

▷ Hubbard-Stratonovich" decoupling: introducing complex commuting field $\Delta(\mathbf{r}, \tau)$

$$e^{g \int dx\, \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} = \int D(\bar{\Delta}, \Delta) \exp\Big\{ - \int dx\, \overbrace{\Big[\frac{1}{g} |\Delta(\mathbf{r}, \tau)|^2 + (\bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow) \Big]}^{\frac{1}{g}(\bar{\Delta} + g\bar{\psi}_\uparrow \bar{\psi}_\downarrow)(\Delta + g\psi_\downarrow \psi_\uparrow) - g\bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} \Big\}$$

$$\mathcal{Z} = \int D(\bar{\psi}, \psi) \int D(\bar{\Delta}, \Delta) e^{-\int dx \frac{|\Delta|^2}{g}} \exp\Big[- \int dx\, \overbrace{\begin{pmatrix} \bar{\psi}_\uparrow & \psi_\downarrow \end{pmatrix}}^{\text{Nambu spinor}} \overbrace{\begin{pmatrix} [\hat{G}_0^{(\mathrm{p})}]^{-1} & \Delta \\ \bar{\Delta} & [\hat{G}_0^{(\mathrm{h})}]^{-1} \end{pmatrix}}^{\text{Gor'kov Ham. } \hat{\mathcal{G}}^{-1}} \begin{pmatrix} \psi_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix} \Big]$$

'free particle/hole' Hamiltonian: $[\hat{G}_0^{(\mathrm{p/h})}]^{-1} = \partial_\tau {}^+/_- (\frac{\hat{\mathbf{p}}^2}{2m} - \mu)$

N.B. $\int dx\, \bar{\psi}_\downarrow \partial_\tau \psi_\downarrow = -\int dx\, (\partial_\tau \bar{\psi}_\downarrow) \psi_\downarrow = \int dx\, \psi_\downarrow \partial_\tau \bar{\psi}_\downarrow$

Using Gaussian field integral: $\int D(\bar{\psi}, \psi) \exp[-\int \bar{\psi} \hat{A} \psi] = \det \hat{A} = \exp[\ln \det \hat{A}]$

$$\mathcal{Z} = \int D(\bar{\Delta}, \Delta) \exp\left[- \int dx \frac{1}{g} |\Delta|^2 + \ln \det \hat{\mathcal{G}}^{-1}[\Delta] \right]$$

i.e. $\mathcal{Z}$ expressed as functional field integral over single complex scalar field $\Delta(x)$

▷ To proceed, we must invoke some approximation:

- Mean-field theory: far below transition ($T \ll T_c$, $\beta \gg \beta_c$) fluctuations small
  $\rightsquigarrow$ saddle-point approximation in Δ constant — Gap equation

- Ginzburg-Landau theory: since transition is continuous, close to T_c,
  we may develop perturbative expansion in (small) Δ

$$\text{Noting :} \qquad \hat{\mathcal{G}}^{-1} = \hat{\mathcal{G}}_0^{-1}\left[1 + \hat{\mathcal{G}}_0 \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix}\right], \qquad \hat{\mathcal{G}}_0 \equiv \hat{\mathcal{G}}(\Delta = 0)$$

$$\ln\det\hat{\mathcal{G}}^{-1} = \mathrm{tr}\ln\hat{\mathcal{G}}^{-1} = \mathrm{tr}\ln\hat{\mathcal{G}}_0^{-1} - \frac{1}{2}\mathrm{tr}\left[\hat{\mathcal{G}}_0 \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix}\right]^2 + \cdots$$

N.B. $\ln(1+z) = -\sum_{n=1}^{\infty}(-z)^n/n$

- Zeroth order term $\rightsquigarrow$ 'free particle' contribution, viz. $\mathcal{Z}_0 = \det\hat{\mathcal{G}}_0^{-1}$

$$\text{Using} \quad \text{id.} = \sum_{k\equiv\mathbf{k},\omega_n} |k\rangle\langle k|, \qquad \Delta_k = \frac{1}{\sqrt{\beta L^d}}\int dx \; \overbrace{e^{ik\cdot x}}^{e^{i\omega_n\tau - i\mathbf{k}\cdot\mathbf{r}}} \; \Delta(x)$$

- Second order term

$$\begin{aligned}\mathrm{tr}\,\hat{G}_0^{(\mathrm{p})}\Delta\hat{G}_0^{(\mathrm{h})}\bar{\Delta} \quad &= \quad \sum_{kk'} G_0^{(\mathrm{p})}(k) \; \overbrace{\langle k|\Delta|k'\rangle}^{\Delta_{k'-k}/\sqrt{\beta L^d}} \; G_0^{(\mathrm{h})}(k')\langle k'|\bar{\Delta}|k\rangle \\ &\stackrel{q=k'-k}{=} \quad \sum_q \Delta_q\bar{\Delta}_{-q} \overbrace{\frac{1}{\beta L^d}\sum_k G_0^{(\mathrm{p})}(k)G_0^{(\mathrm{h})}(k+q)}^{\text{pairing susceptibility }\Pi(q)}\end{aligned}$$

Combined with bare term, one obtains

$$\mathcal{Z} = \int D[\bar{\Delta},\Delta]e^{-S[\bar{\Delta},\Delta]}, \qquad S = \sum_{\omega_n,\mathbf{q}}\left[\frac{1}{g} + \Pi(\omega_n,\mathbf{q})\right]|\Delta_{\omega_n,\mathbf{q}}|^2 + O(\Delta^4)$$

In principle, one can evaluate $\Pi(\mathbf{q},\omega_n)$ explicitly;
however we can proceed more simply by considering...

$\triangleright$ 'Gradient expansion': $\Pi(\mathbf{q},\omega_n) = \Pi(0,0) + \dfrac{\mathbf{q}^2}{2}\partial^2_{|\mathbf{q}|}\Pi(0,0) + O(i\omega_n,\mathbf{q}^4)$
$\rightsquigarrow$ <u>GINZBURG-LANDAU THEORY</u>

$$S[\Delta] = \beta\int d^dr\left[\frac{t}{2}|\Delta|^2 + \frac{K}{2}|\partial\Delta|^2 + u|\Delta|^4 + \cdots\right]$$

$\frac{t}{2} = \frac{1}{g} + \Pi(0,0)$, $K = \partial^2_{|\mathbf{q}|}\Pi(0,0) > 0$ and $u > 0$

Note structural similarity to weakly interacting Bose gas

▷ LANDAU THEORY: If we assume that dominant contribution to $\mathcal{Z} = e^{-\beta F}$ arises from minumum action, i.e. spatially homogeneous Δ that minimises

$$\frac{S[\Delta]}{\beta L^d} = \frac{t}{2}|\Delta|^2 + u|\Delta|^4$$

$$\text{i.e.} \quad |\Delta|\left(t + 4u|\Delta|^2\right) = 0, \qquad |\Delta| = \begin{cases} 0 & t > 0 \\ \sqrt{t/4u} & t < 0 \end{cases}$$

i.e. for $t < 0$, spontaneous breaking of continuous U(1) symmetry associated with phase $\rightsquigarrow$ gapless fluctuations — Goldstone modes

▷ TRANSITION TEMPERATURE: Using identity $\frac{1}{L^d}\sum_{\mathbf{k}} = \int \frac{d^dk}{(2\pi)^d} = \int \nu(\epsilon) d\epsilon$

$$\Pi(0,0) = -\frac{1}{\beta L^d}\sum_{\omega_n,\mathbf{k}} \frac{1}{\omega_n^2 + (\mathbf{k}^2/2m - \mu)^2} \simeq -\frac{1}{\beta}\sum_{\omega_n}\int_{-\infty}^{\infty} d\zeta \frac{\nu(\zeta+\mu)}{\omega_n^2 + \zeta^2} \simeq -\frac{\pi\nu(\mu)}{\beta}\sum_{\omega_n}\frac{1}{|\omega_n|}$$

Introducing energy cut-off at Debye frequency $\omega_D = (2n_{\max}+1)\pi/\beta$

$$\Pi(0,0) \simeq -\nu(\mu)\sum_{n=-n_{\max}}^{n_{\max}} \frac{1}{2n+1} \simeq -2\nu(\mu)\int_0^{n_{\max}} \frac{dn}{2n+1} \simeq -\nu(\mu)\ln\left(\frac{\beta\omega_D}{\pi}\right)$$

Transition when $t/2 \equiv 1/g + \Pi(0,0) = 0$, i.e. $k_{\mathrm{B}}T < k_{\mathrm{B}}T_c = \pi\omega_D \exp\left[-\frac{1}{\nu(\mu)g}\right]$

$$\begin{aligned} \text{Near } T_c \qquad \frac{t}{2} &= \frac{g}{2} - \nu(\mu)\ln\left(\frac{\beta\omega_D}{\pi}\right) - \overbrace{\left(g - \nu(\mu)\ln\left(\frac{\beta_c\omega_D}{\pi}\right)\right)}^{=0} \\ &= \nu(\mu)\ln(T/T_c) = \nu(\mu)\ln(1 + (T-T_c)/T_c) \simeq \nu(\mu)\left(\frac{T-T_c}{T_c}\right) \end{aligned}$$

i.e. physically t is 'reduced temperature'