Lecture XVIII: Random Phase Approximation

▷ Previously, we have seen that the quantum partition function of the weakly interacting electron gas can be written as field integral

$$\mathcal{Z} = \mathcal{Z}_0 \int D\phi\, e^{-S[\phi]}, \qquad S[\phi] = \frac{1}{2}\sum_{q=(\omega_m,\mathbf{q})} \overbrace{\left(\frac{\mathbf{q}^2}{4\pi} - e^2\Pi(q)\right)}^{D^{-1}(q)} |\phi_q|^2 + O(e^4)$$

where dielectric properties found to be controlled by density-density response function

$$\boxed{\Pi(q) = \frac{2}{\beta L^3}\sum_k \frac{1}{i\omega_n - \epsilon_{\mathbf{k}} + \mu}\,\frac{1}{i\omega_n + i\omega_m - \epsilon_{\mathbf{k}+\mathbf{q}} + \mu}}$$

To understand form of $\chi(q)$, we have to digress and discuss

▷ <u>Matsubara Summations</u>

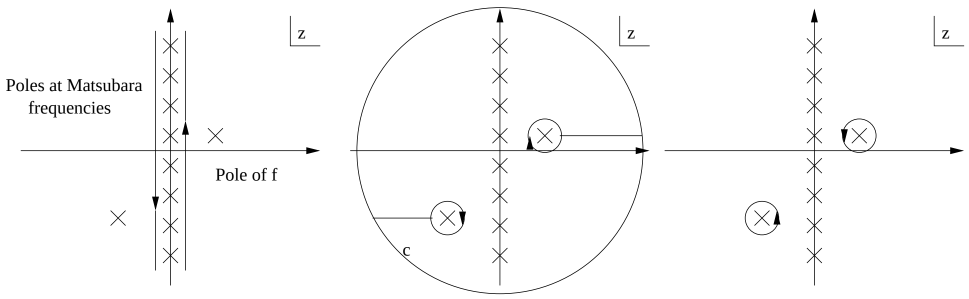
Basic idea: by introducing auxiliary function $g(z)$ that has simple poles of strength unity at $z = i\omega_n$, Cauchy's theorem implies

$$\sum_{\omega_n} f(i\omega_n) = \frac{1}{2\pi i}\oint_C dz\, g(z) f(z)$$

where contour C encloses only poles of $g(z)$

$$\text{e.g.} \quad g(z) = \begin{cases} \dfrac{\beta}{\exp(\beta z) - 1}, & \text{bosons} \\ -\dfrac{\beta}{\exp(\beta z) + 1}, & \text{fermions} \end{cases}$$

Then, moving contour to infinity



$$\sum_{\omega_n} f(i\omega_n) = \overbrace{\lim_{R\to\infty}\frac{R}{2\pi i}\int_0^{2\pi} d\theta g(Re^{i\theta}) f(Re^{i\theta})}^{\to 0} - \overbrace{\frac{1}{2\pi i}\sum_{P:f(z_P)=0}\oint dz g(z) f(z)}^{\text{if simple } \sum_P g(z_P) f(z_P)}$$

Applied to $\chi(q)$,

$$\chi(q) = -\frac{2}{\beta L^3}\sum_{\mathbf{k}}\left[\frac{g(\epsilon_{\mathbf{k}}-\mu)}{i\omega_m+\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}+\mathbf{q}}}+\frac{g(\epsilon_{\mathbf{k}+\mathbf{q}}-\mu-i\frac{2\pi m}{\beta})}{-i\omega_m-\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{k}+\mathbf{q}}}\right] = \frac{2}{L^3}\sum_{\mathbf{k}}\frac{n_F(\epsilon_{\mathbf{k}})-n_F(\epsilon_{\mathbf{k}+\mathbf{q}})}{i\omega_m+\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}+\mathbf{q}}},$$

where $n_{\mathrm{F}}(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}+1}$ is the Fermi distribution function

Finally, for $|\mathbf{q}| \ll k_F \equiv (2m\mu)^{1/2}$ and $k_{\mathrm{B}}T \ll \mu$, $\mathbf{k}$ summation $\rightsquigarrow$ Lindhard function

$$\chi(q) \simeq -2\nu(\mu)\left(1-\frac{\omega_m}{v_F|\mathbf{q}|}\tan^{-1}\left[\frac{v_F|\mathbf{q}|}{\omega_m}\right]\right)$$

where $\nu(\mu)$ is density of states at Fermi level

- Static Limit: For $|\omega_m| \ll k_F|\mathbf{q}|/m$, $\chi(0,\mathbf{q}) \simeq -2\nu(\mu)$, i.e.

$$D(0,\mathbf{q}) \simeq \frac{4\pi e^2}{\mathbf{q}^2}\frac{1}{1+2\frac{4\pi e^2}{\mathbf{q}^2}\nu(\mu)}$$

Fourier transformed, $\rightsquigarrow$ static screened Coulomb interaction $\dfrac{e^2}{|\mathbf{r}|}e^{-|\mathbf{r}|/\lambda_{\mathrm{TF}}}$

where $\lambda_{\mathrm{TF}} = 2\times 4\pi e^2\nu(\mu)$ — Thomas-Fermi screening length

i.e. At long time scales (low frequencies), bare Coulomb interaction is renormalised (screened) by collective charge fluctuations

Physically, focusing a single electron, because it is negatively charged, other electrons will be repelled. As a result, a positively charged cloud of radius λ_{TF} will form balancing the negative charge of the electron. When viewed from a distance larger than λ_{TF}, the electron+cloud behaves as a neutral particle.

- High Frequency Limit: For $|\omega_n| \gg k_F|\mathbf{q}|/m$, $\chi(\omega_m,\mathbf{q}) \simeq -\dfrac{\mathbf{q}^2}{m\omega_m^2}n$,

where $n = N/L^3$ is the total number density (including spin)

$$D(q) = \frac{4\pi e^2}{\mathbf{q}^2}\frac{1}{1+\frac{4\pi e^2 n}{m\omega_m^2}}$$

i.e. real time response ($i\omega_m \to \omega + i0$) singular when

$\omega_p = 4\pi e^2 n/m$ — Plasma frequency

In this case, there is a resonance which couples to the excitation mode where the positively charged background and the negatively charged electrons are moving uniformly against each other.

- Ground State Energy

$$\lim_{\beta\to\infty} \mathcal{Z} \sim e^{-\beta E_{\text{g.s.}}}.$$

In the RPA approximation, $\mathcal{Z} = \mathcal{Z}_0 \times \dfrac{\text{const.}}{\det D^{-1/2}} = \mathcal{Z}_0 \times \text{const.} \prod_{\mathrm{q}} \mathrm{D}(\mathrm{q})^{1/2}$

$$\text{i.e. } E_{\text{g.s.}} = E_{\text{g.s.}}(e=0) - \frac{1}{2\beta} \sum_q \ln D(q)$$