Lecture XV: Many-body (Coherent State) Path Integral

Could formulate many-body propagator (Green function), but here, convenient to focus on partition function.

▷ Quantum partition function

$$\mathcal{Z} = \sum_{\{n\}\in\text{Fock Space}} \langle n|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle, \qquad \beta = 1/k_BT, \qquad \mu \text{ chemical potential}$$

Coherent state representation of $\mathcal{Z}$ — insert resolution of id. (fermions/bosons)

$$\int d[\bar{\psi},\psi]e^{-\sum_i \bar{\psi}_i\psi_i}|\psi\rangle\langle\psi| = \mathbf{1}_{\mathcal{F}}, \qquad d[\bar{\psi},\psi] \equiv \prod_i \frac{d\bar{\psi}_i d\psi_i}{\pi^{(1-\zeta)/2}}$$

each element ψ_i associated with one basis state, viz. $a_i^\dagger$
— e.g. i may include position, momentum, spin, lattice site, etc.

$$\mathcal{Z} = \int d[\bar{\psi},\psi]e^{-\sum_i \bar{\psi}_i\psi_i}\sum_n \langle n|\psi\rangle\langle\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle$$

Elimination of $|n\rangle$ requires identity: $\langle n|\psi\rangle\langle\psi|n\rangle = \langle -\zeta\psi|n\rangle\langle n|\psi\rangle$

Proof: E.g. $|n\rangle = a_1^\dagger a_2^\dagger \cdots a_n^\dagger|0\rangle$

$$\begin{aligned}
\langle n|\psi\rangle &= \langle 0|a_n\cdots a_2a_1|\psi\rangle = \psi_n\cdots\psi_2\psi_1\langle 0|\psi\rangle = \psi_n\cdots\psi_2\psi_1 \\
\langle\psi|n\rangle &= \bar{\psi}_1\bar{\psi}_2\cdots\bar{\psi}_n \\
\langle n|\psi\rangle\langle\psi|n\rangle &= \psi_n\cdots\psi_2\psi_1\bar{\psi}_1\bar{\psi}_2\cdots\bar{\psi}_n = \psi_1\bar{\psi}_1\psi_2\bar{\psi}_2\cdots\psi_n\bar{\psi}_n \\
&= (-\zeta\bar{\psi}_1\psi_1)(-\zeta\bar{\psi}_2\psi_2)\cdots(-\zeta\bar{\psi}_n\psi_n) = \langle -\zeta\psi|n\rangle\langle n|\psi\rangle
\end{aligned}$$

commute through and erase $\sum_n |n\rangle\langle n|$

$$\boxed{\mathcal{Z} = \int d[\bar{\psi},\psi]e^{-\sum_i \bar{\psi}_i\psi_i}\langle -\zeta\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|\psi\rangle}$$

▷ Coherent State Path Integral

Applied to general Hamiltonian

$$\hat{H} - \mu\hat{N} = \sum_{ij}(h_{ij} - \mu\delta_{ij})a_i^\dagger a_j + \sum_{ijkl} V_{ijkl}a_i^\dagger a_j^\dagger a_k a_l$$

N.B. operators are normal ordered

Follow general strategy of Feynman:

(i) Divide 'time interval' β into N segments $\Delta\beta = \beta/N$

$$\langle -\zeta\psi| e^{-\beta(\hat{H}-\mu\hat{N})}|\psi\rangle = \langle -\zeta\psi| e^{-\Delta\beta(\hat{H}-\mu\hat{N})}{}_{\wedge} e^{-\Delta\beta(\hat{H}-\mu\hat{N})}{}_{\wedge}\cdots|\psi\rangle$$

(ii) At each $\wedge$ insert resolution of id.

$$\mathbf{1}_{\mathcal{F}} = \int d[\bar{\psi}_n,\psi_n] e^{-\bar{\psi}_n\cdot\psi_n}|\psi_n\rangle\langle\psi_n|$$

i.e. N-independent sets N.B. each ψ_n is a vector with elements $\{\psi_i\}_n$

(iii) Expand exponent in $\Delta\beta$

$$\begin{aligned}
\langle\psi'|e^{-\Delta\beta(\hat{H}-\mu\hat{N})}|\psi\rangle &= \langle\psi'|\left[1-\Delta\beta(\hat{H}-\mu\hat{N})\right]|\psi\rangle + O(\Delta\beta)^2 \\
&= \langle\psi'|\psi\rangle - \Delta\beta\langle\psi'|(\hat{H}-\mu\hat{N})|\psi\rangle + O(\Delta\beta)^2 \\
&= \langle\psi'|\psi\rangle\left[1-\Delta\beta\left(H(\psi',\psi)-\mu N(\psi',\psi)\right)\right] + O(\Delta\beta)^2 \\
&\simeq e^{\psi'\cdot\psi}e^{-\Delta\beta(H(\psi',\psi)-\mu N(\psi',\psi))}
\end{aligned}$$

$$\text{with} \qquad H(\psi',\psi) = \frac{\langle\psi'|\hat{H}|\psi\rangle}{\langle\psi'|\psi\rangle} = \sum_{ij} h_{ij}\bar{\psi}'_i\psi_j + \sum_{ijkl} V_{ijkl}\bar{\psi}'_i\bar{\psi}'_j\psi_k\psi_l$$

similarly $N(\psi',\psi)$ *N.B.* $\langle\psi'|\psi\rangle$ *bilinear in* ψ*, i.e. commutes with everything*

$$\mathcal{Z} = \int_{\bar{\psi}_N=-\zeta\bar{\psi}_0,\psi_N=-\zeta\psi_0} \prod_{n=0}^{N} d[\bar{\psi}_n,\psi_n] e^{-\sum_{n=1}^{N}\left[\bar{\psi}_n\cdot(\psi_n-\psi_{n-1})+\Delta\beta(H(\bar{\psi}_n,\psi_{n-1})-\mu N(\bar{\psi}_n,\psi_{n-1}))\right]}$$

Continuum limit $N\to\infty$

$$\Delta\beta\sum_{n=0}^{N}\to\int_0^\beta d\tau, \qquad \frac{\psi_n-\psi_{n-1}}{\Delta\beta}\to\partial_\tau\psi\Big|_{\tau=n\Delta\beta}, \qquad \prod_{n=0}^{N} d[\bar{\psi}_n,\psi_n]\to D(\bar{\psi},\psi)$$

comment on "small" Grassmann nos.

$$\boxed{\mathcal{Z} = \int_{\substack{\bar{\psi}(\beta)=-\zeta\bar{\psi}(0)\\ \psi(\beta)=-\zeta\psi(0)}} D(\bar{\psi},\psi)e^{-S[\bar{\psi},\psi]}, \qquad S[\bar{\psi},\psi] = \int_0^\beta d\tau\left(\bar{\psi}\cdot\partial_\tau\psi + H(\bar{\psi},\psi) - \mu N(\bar{\psi},\psi)\right)}$$

With particular example:

$$S[\bar{\psi},\psi] = \int_0^\beta d\tau\left[\sum_{ij}\bar{\psi}_i(\tau)\left[(\partial_\tau-\mu)\delta_{ij}+h_{ij}\right]\psi_j(\tau) + \sum_{ijkl}V_{ijkl}\,\bar{\psi}_i(\tau)\bar{\psi}_j(\tau)\psi_k(\tau)\psi_l(\tau)\right]$$

i.e. Quantum partition function expressed as path integral over fields $\psi_j(\tau)$

▷ Matsubara frequency representation

Often convenient to express path integral in frequency domain

$$\psi(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} \psi_n e^{i\omega_n \tau}, \qquad \psi_{\omega_n} = \frac{1}{\sqrt{\beta}} \int_0^\beta d\tau \, \psi(\tau) e^{-i\omega_n \tau}$$

where, since $\psi(\tau) = -\zeta\psi(\tau+\beta)$

$$\omega_n = \left\{ \begin{array}{ll} 2n\pi/\beta, & \text{bosons,} \\ (2n+1)\pi/\beta, & \text{fermions} \end{array} \right. , \qquad n \in \mathcal{Z}$$

ω_n are known as Matsubara frequencies

Using $\displaystyle\frac{1}{\beta}\int_0^\beta d\tau \; e^{i(\omega_n-\omega_m)\tau} = \delta_{\omega_n\omega_m}$

$$S[\bar{\psi},\psi] = \sum_{ij\omega_n} \bar{\psi}_{i\omega_n} \left[(i\omega_n - \mu)\, \delta_{ij} + h_{ij} \right] \psi_{j\omega_n} + \\ + \frac{1}{\beta} \sum_{ijkl} \sum_{\omega_{n_1}\omega_{n_2}\omega_{n_3}\omega_{n_4}} V_{ijkl} \bar{\psi}_{i\omega_{n_1}} \bar{\psi}_{j\omega_{n_2}} \psi_{k\omega_{n_3}} \psi_{l\omega_{n_4}} \delta_{\omega_{n_1}+\omega_{n_2},\omega_{n_3}+\omega_{n_4}}$$