Appendix A

*Quantum Friction

In section 6.4 we found that the statistical mechanics of a classical one-dimensional Ising model could be mapped onto the quantum mechanics of a point particle confined to a double well potential. The existence of unbound topological configurations, the instanton gas, implied a *delocalisation* of the particle and reflected the absence of long-range order in the one-dimensional Ising model at all finite, i.e. non-zero, temperatures. This contrasts with the behaviour of the two-dimensional XY-model studied in section 5.2. There, the existence of non-trivial topological instanton configurations of the field, the vortices in this case, displayed qualitatively different behaviour. At high temperatures, there is a proliferation of defects which form a plasma of unbound vortices. This disordered phase is analogous to the instanton gas in the one-dimensional Ising model. However, at low but finite temperatures, the topological defects bind to form vortex/antivortex pairs. Can an analogous phase be induced in the one-dimensional quantum Hamiltonian?

In this appendix we will study the *macroscopic* tunnelling of a quantum particle coupled to external degrees of freedom. Remarkably the external coupling can induce a quantum phase transition which separates a disordered high temperature phase from one in which kink/antikink pairs are bound. In the *confined* phase, there is a "quantum localisation" of the particle. Although first discussed by Caldiera and Leggett (Physica **121A**, 587 (1983); Ann. Phys. (N.Y.) **149**, 374 (1983)), we will follow more closely the work of Schmid (Phys. Rev. Lett. **51**, 1506 (1983)).

A.1 Macroscopic Quantum Tunnelling

Suppose the transverse fluctuations u of a string, with an action

$$S_{\text{string}} = \int d\tau \int dx \left[\frac{\rho}{2}\dot{u}^2 + \frac{\sigma}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right],$$

where ρ denotes the mass per unit length and σ the tension, are coupled at a point $x = 0$ to a particle

$$q(\tau) = u(\tau, x = 0),$$

which is itself confined to a potential $U(q)$ and described by the action

$$S_{\text{part.}} = \int d\tau \left[\frac{m}{2}\dot{q}^2 + U[q(\tau)]\right].$$

Later we will explain the physical motivation which stands behind the consideration of such models. Together, the total partition function is given by

$$\mathcal{Z} = \int Du(x,\tau) \int Dq(\tau)\ \delta(q(\tau) - u(\tau,0))\ e^{-S_{\text{string}} - S_{\text{part.}}}.$$

To determine the effect of the string on the dynamics of the particle it is necessary to integrate out all of the degrees of freedom associated with the fluctuation of the string (subject to the constraint $q(\tau) = u(\tau, 0)$), and write an effective action for q. Such an integral is conveniently performed in reciprocal space.[1] We begin by expressing the δ-function constraint as a functional integral over some set of auxiliary variables

$$\delta(q(\tau) - u(\tau,0)) = \int Df(\tau) \exp\left[i \int d\tau f(\tau)(q(\tau) - u(\tau,0))\right].$$

Since we will shortly perform in the inverse transform we will leave the subtleties associated with the measure unspecified. Switching to reciprocal space in x and τ, we can express the string dependent part of the partition function together with the constraint as

$$\begin{aligned}\mathcal{Z} = \int Dq(\omega) \int Df(\omega) \exp\left[-S_{\text{part.}} + i\int \frac{d\omega}{2\pi} f(\omega)q(-\omega)\right] \\ \times \int Du(\omega,k) \exp\left[-\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\left(\frac{\rho\omega^2 + \sigma k^2}{2}|u(\omega,k)|^2 + if(\omega)u(-\omega,-k)\right)\right].\end{aligned}$$

The Gaussian integral over $u(k,\omega)$ is performed straightforwardly and leads to the result

$$\mathcal{Z} = \int Dq(\omega) \int Df(\omega) \exp\left[-S_{\text{part.}} + i\int \frac{d\omega}{2\pi} f(\omega)q(-\omega) - \int \frac{dk}{2\pi}\frac{|f(\omega)|^2}{2(\rho\omega^2 + \sigma k^2)}\right].$$

where we have absorbed the constant term in S_{part}. Finally, taking the integral over k and performing the Gaussian integral over f we obtain

$$\mathcal{Z} = \int Dq(\omega) e^{-S_{\text{eff.}}}$$

where

$$\begin{aligned} S_{\text{eff.}}[q(\omega)] &= S_{\text{part.}} + \frac{\eta}{2}\int \frac{d\omega}{2\pi}|\omega||q(\omega)|^2, \\ S_{\text{eff.}}[q(\tau)] &= S_{\text{part.}} + \frac{\eta}{4\pi}\int\int d\tau d\tau' \frac{(q(\tau) - q(\tau'))^2}{(\tau - \tau')^2}, \end{aligned} \tag{A.1}$$

[1] Our convention will be to choose

$$u(\tau, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int e^{i(kx + \omega\tau)} u(\omega, k).$$

with $\eta = 2\sqrt{\rho\sigma}$.

The effect of the string is to induce a dissipative term into the classical action. The parameter η serves a "friction coefficient". To understand how this term leads to a collapse of particle tunneling in the double well (i.e. confinement) let us first recall the conclusion of our earlier study of $S_{\text{part.}}$ and examine the limit of *strong coupling*. To be concrete, we will study the transition in a periodic potential

$$U(q) = -g\cos\left(\frac{2\pi q}{q_0}\right),$$

as opposed to the double well cited previously.

Taking our notation from Fisher and Zwerger (Phys. Rev. B **32**, 6190 (1985)) we begin by reexpressing the action in terms of dimensionless parameters

$$\phi(\tau) = \frac{2\pi q(\tau)}{q_0}, \qquad \alpha = \frac{\eta q_0^2}{2\pi}, \qquad g_0 = \frac{g}{\Lambda},$$

where $\Lambda = (2\pi)^2/mq_0^2$ denotes the quantum mechanical energy necessary to confine a particle of mass m within a lattice spacing. In this notation, the total action takes the form

$$S = S_0 + S_1, \quad S_0 = \frac{1}{2}\int \frac{d\omega}{2\pi} K_\Lambda(\omega)|\phi(\omega)|^2, \quad S_1 = -g_0\Lambda \int d\tau \cos[\phi(\tau)], \tag{A.2}$$

where

$$K_\Lambda(\omega) = \frac{\alpha}{2\pi}|\omega| + \frac{\omega^2}{\Lambda}.$$

A.1.1 Strong-coupling

In the limit of strong coupling ($g_0 \gg 1$), the tunnelling of the quantum mechanical particle between adjacent minima of the potential can be well-described in the WKB approximation where the dominant contribution to the action comes from the classical path corresponding to the motion of the undamped particle in the "inverted" potential — the instanton:

$$\ddot{\bar{\phi}} - \Lambda^2 g_0 \sin(\bar{\phi}) = 0.$$

In this case, the one-instanton or kink solution can be obtained explicitly

$$\bar{\phi}(\tau) = \pm f(\tau) \equiv \pm 4\arctan\left(\exp[\omega_0\tau]\right),$$

and describes a trajectory where ϕ changes by $\pm 2\pi$ in a lapse time of $\omega_0 = (\Lambda^2 g_0)^{1/2}$. It gives a contribution to the action, S_{part} of $s = 8g_0^{1/2} \gg 1$.

More generally, one can construct a gas of instantons with a sequence of kinks ($e_j = +1$) and antikinks ($e_j = -1$) located at the positions $\tau_1 \cdots \tau_n$,

$$\bar{\phi} = \sum_i e_i f(\tau - \tau_i),$$

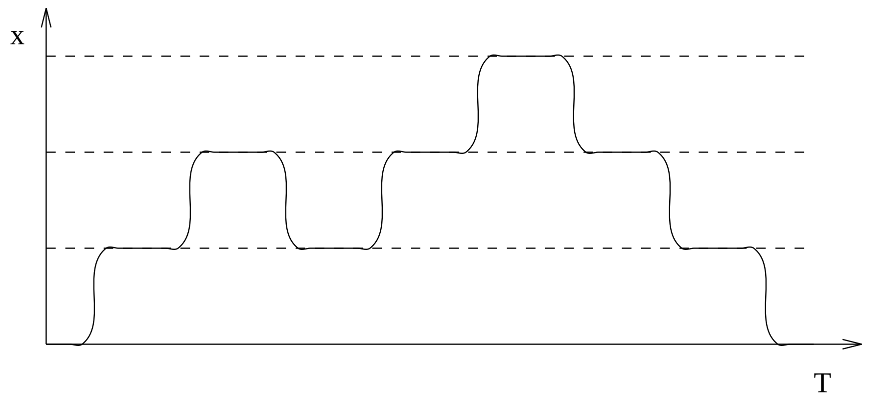

Figure A.1: An instanton configuration in a periodic potential.

which are approximate solutions provided that $|\tau_j - \tau_k| \gg \omega_0^{-1}$. Note that, in contrast to the double well potential, the sequence of kink/antikink pairs is arbitrary (see fig. A.1.1).

Substituting this solution into the effective action in Eq. (A.1), up to constant, we obtain[2]

$$S_{\text{eff.}}[\phi] = ns + \frac{1}{2}\sum_{jk} e_j e_k \Delta(\tau_j - \tau_k),$$

where the effective interaction between the kinks is given by

$$\Delta(\tau) = \frac{\alpha}{2\pi}\int \frac{d\omega}{2\pi}|\omega||f(\omega)|^2 e^{-i\omega\tau} = \begin{cases} -c\frac{\alpha}{2\pi}\omega_0^2\tau^2 & \omega_0|\tau| \ll 1, \\ -2\alpha\ln(\omega_0|\tau|) & \omega_0|\tau| \gg 1, \end{cases} \tag{A.3}$$

where $c \sim 0(1)$. This is analogous to the energy of a vortex configuration in the XY-model. The factor e^{-s} serves as a fugacity for the defects, while the second term is associated with the pairwise interaction of the topological defects. We remark that unless the charge of the gas is neutral $\sum_i e_i = 0$ the action is infinitely large. This means that $n = 2N$.

This interaction suggests the existence of phase transition at $\alpha^* = 1$. To see this, consider the average distance between a kink/antikink pair.

$$T = \int d\tau_{12}\tau_{12}e^{-2\alpha\ln t_{12}} = \begin{cases} \text{finite} & \alpha > 1, \text{ neutral gas,} \\ \infty & \alpha < 1, \text{ plasma.} \end{cases}$$

In the neutral phase the kink is always confined to an antikink. An excursion of the particle to a neighbouring well is inevitably succeeded by its return. The particle is *localised*, unable to escape. Our approach so far has been based on a strong-coupling expansion. Does this picture survive in the limit of weak-coupling? This question was answered by Schmid who found an ingenious *duality* transformation which maps the strong-coupling action onto its weak-coupling limit.

[2]A more careful treatment that includes the effect of Gaussian fluctuations around $\bar{\phi}$ (as discussed in chapter 6) yields the correction $e^{-s} \to \omega_0(2s/\pi)^{1/2}e^{-s}$.

A.1.2 Weak-coupling

Following Schmid let us examine the expansion

$$\exp\left[g_0\Lambda\int d\tau\cos(\phi)\right]=\sum_{n=0}^{\infty}\int d\tau_1\cdots d\tau_n\frac{(g_0\Lambda/2)^n}{n!}\sum_{\{e_j\}}\exp\left[ie_j\int d\tau\delta(\tau-\tau_j)\phi(\tau)\right],$$

where the quantities $e_j=\pm 1$ may be considered as charges of classical particles located at τ_j. Since $K_\Lambda|_{\omega=0}=0$, the functional integral with respect to ϕ can only be performed if the time average of $\sum_j e_j\delta(\tau-\tau_j)$ vanishes. This means that the total charge $\sum_j e_j$ must be zero. Hence $n=2N$ and the "plasma" is neutral.

There are $(2N)!/(N!)^2$ equivalent ways to distribute the charges on the coordinates. Choosing a charge density

$$\rho(\tau)=\delta(\tau-\tau_1)+\cdots+\delta(\tau-\tau_n)-\delta(\tau-\tau_{n+1})-\cdots-\delta(\tau-\tau_N),$$

and performing the Gaussian integration over ϕ, we obtain

$$\begin{aligned}\mathcal{Z}=\sqrt{\det K_\Lambda}\sum_N\int d\tau_1\cdots d\tau_{2N}\left[\frac{(g_0\Lambda/2)^N}{N!}\right]^2\\ \times\exp\left[-\frac{1}{2}\int d\tau d\tau'\rho(\tau)K_\Lambda^{-1}(\tau-\tau')\rho(\tau')\right],\end{aligned}$$

where

$$K_\Lambda^{-1}(\tau)=\int\frac{d\omega}{2\pi}K_\Lambda^{-1}(\omega)e^{-i\omega\tau}=\begin{cases}-\frac{\Lambda}{2}|\tau| & \alpha\Lambda|\tau|\ll 1,\\ -\frac{2}{\alpha}\ln\left(\frac{\alpha\Lambda|\tau|}{2\pi}\right) & \alpha\Lambda|\tau|\gg 1.\end{cases}\tag{A.4}$$

This expression can be interpreted as the grand canonical partition function of a neutral plasma. At short distances the interaction is Coulomb-like whereas at large separation it is dominated by a logarithmic dependence.

Remarkably, if we neglect the differences in the core regions of the interaction, the weak-coupling limit can be mapped onto the strong-coupling limit by the substitution

$$\alpha\to\frac{1}{\alpha}.$$

The fixed point of this mapping is $\alpha^*=1$.[3]

To further test the validity of the weak-coupling expansion at α^* it is possible to examine the phase transition for weak-coupling making use of a renormalisation group calculation.

[3]Taking into account the exponential prefactor (with Gaussian fluctuations) and the measure we require also the mapping $g_0\Lambda\times\alpha/\Lambda\to\omega_0 s^{1/2}e^{-s}\times(1/\omega_0)$. This shows that the fixed parameter $s^*=1.5$ corresponds to neither weak nor strong coupling. It means that the estimate of α^* can not be considered as completely reliable and motivates the RG analysis.

A.1.3 Renormalisation Group

To apply the renormalisation group we will follow the notation of Fisher and Zwerger. We start by recognising that the higher-order gradient term in the action due to the kinetic energy simply provides a natural high frequency cut-off Λ, inversely proportional to the mass. By simple power counting, it is expected to be irrelevant. Therefore, in what follows it will be dropped,

$$K_\Lambda(\omega) \to K(\omega) = \frac{\alpha}{2\pi}|\omega|,$$

and instead frequency integrals will be restricted to $|\omega| < \Lambda$.

To apply a renormalisation group analysis in the weak coupling limit we will work perturbatively in powers of g_0 and derive scaling equations for g_0 using the finite-momentum shell recursion method discussed in section 4.4. As a first step, we will separate the function $\phi(\tau)$ into components which describe slow $\phi_<(\tau)$ and fast $\phi_>(\tau)$ degrees of freedom

$$\phi(\omega) \equiv \begin{cases} \phi_<(\omega) & |\omega| < \Lambda/b, \\ \phi_<(\omega) & |\omega| > \Lambda/b, \end{cases}$$

where $b \gg 1$.

The second step is to integrate out the fast modes by expanding in powers of g_0

$$\mathcal{Z} = \int D\phi e^{-S_0 - S_1} = \int D\phi_< e^{-\widetilde{S}},$$

where

$$\widetilde{S} = \frac{\alpha}{4\pi}\int \frac{d\omega}{2\pi}|\omega||\phi_<(\omega)|^2 + \langle S_1 \rangle - \frac{1}{2}\left\langle (S_1 - \langle S_1 \rangle)^2 \right\rangle + O(g_0^3).$$

Dealing with the term first order in g_0, and using the expressions

$$\left\langle e^{i(\phi_>(\tau)+\phi_<(\tau))} \right\rangle_{\phi_>} = e^{i\phi_<(\tau)} \exp\left[-\frac{\langle \phi_>(\tau)^2 \rangle_{\phi_>}}{2} \right],$$

with

$$\left\langle \phi_>(\tau)^2 \right\rangle_{\phi_>} = \int_{\Lambda/b}^{\Lambda} \frac{d\omega}{2\pi} K^{-1}(\omega) = \frac{2}{\alpha}\ln b,$$

we obtain

$$\langle S_1 \rangle = -g_0 \Lambda b^{-1/\alpha} \int d\tau \cos[\phi_<(\tau)].$$

To complete the renormalisation group transformation, we must rescale

$$\tau \to \tau' = b\tau,$$

to bring $\widetilde{S}$ back into its original form. No rescaling of ϕ (i.e. wavefunction renormalisation) is necessary since the theory has an underlying $\phi \to \phi + 2\pi$ symmetry.

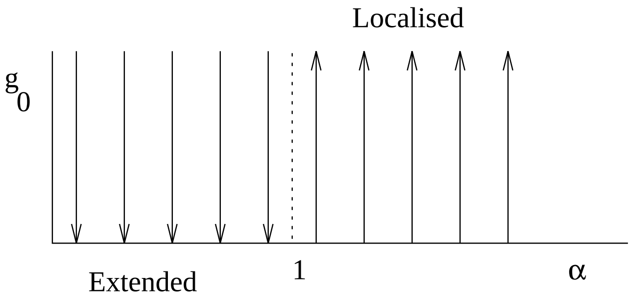

Figure A.2: RG flow for macroscopic quantum tunnelling.

After rescaling, we have to leading order

$$g_0(\Lambda/b) = g_0(\Lambda) b^{1-1/\alpha}.$$

To obtain the differential flow equations, we differentiate $g_0(\Lambda/b)$ with respect to Λ/b keeping the bare parameters Λ and $g_0(\Lambda)$ fixed. Defining $b = e^{\ell}$, we obtain

$$\frac{\partial g_0}{\partial \ell} = -\left(\frac{1}{\alpha} - 1\right) g_0(\ell) + O(g_0^2).$$

which shows that to lowest order g_0 scales to zero if $\alpha < 1$ and grows if $\alpha > 1$.

A.2 Applications

Models which involve quantum phase transitions of this kind have appeared in many applications in the recent literature:

A.2.1 Quantum Transport in a One-Channel Luttinger Liquid

Of these, the most recent concerned the study of the quantum mechanical tunnelling of strongly interacting fermions through a barrier in one-dimension. (See, for example, C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. **68**, 1220 (1992); Phys. Rev. B **46**, 15233 (1992).) Such systems can be described by an effective action describing the hydrodynamics of the correlated electronic system known as a *Luttinger* liquid.

A particularly simple way of describing the low-energy physics of a Luttinger liquid is via the method of "Bosonization". For spinless fermions, the transformation from fermionic to (hard-core) bosonic operators is achieved by the *Jordan-Wigner* transformation, $c_i^\dagger \to \exp[i\pi \sum_{j<i} n_i] b_i^\dagger$. The Fermi statistics of the original particles is accounted for by a phase factor that "counts" the electrons "to the left". The interacting bosons are then represented in the number-phase representation $b_i^\dagger = \sqrt{n_i}\exp[i\chi_i^\dagger]$. In the continuum limit, bosonic fields θ and ϕ are defined such that $n_i \to \bar{\rho} + \nabla\theta(x)/\sqrt{\pi}$ and $\chi_i \to \sqrt{\pi}\phi(x)$ where $\bar{\rho} = k_F/\pi$ denotes the mean density. It then follows from the canonical commutation relation between number and phase that θ and ϕ satisfy

$$[\phi(x), \theta(x')] = -i\Theta(x - x'),$$

where $\Theta(x)$ denotes the Θ-function. Thus $\nabla\theta$ may be regarded as the momentum conjugate to ϕ. Alternatively $\nabla\phi$ is the momenta conjugate to θ. Expressed in terms of these bosonic fields, the fermionic field operator may be written

$$\psi^\dagger(x) \approx \sum_{nodd} \exp[in(\sqrt{\pi}\theta(x) + k_F x)] \exp[i\sqrt{\pi}\phi(x)].$$

The sum on n enforces the constraint that the particle density be discrete. By having n odd, the Jordan-Wigner string is included and Fermi statistics is obeyed. Terms which depend on $\nabla\theta$ arising from the expansion of $\sqrt{n}$ can be neglected at long-wavelengths.

The interactions between the bosons arise from a combination of the hard-core constraint and the interaction in the original fermion problem. Provided these interactions are short-ranged, the effective Hamiltonian may be written in the generic form

$$H = \int dx v \left[\frac{\alpha}{2}(\nabla\phi)^2 + \frac{1}{2\alpha}(\nabla\theta)^2\right],$$

where $v = 1/\sqrt{\kappa m a}$ denotes the sound-wave velocity. Although the parameter $\alpha = \pi\hbar\kappa v$ depends in a complicated way on the microscopic parameters, qualitatively it decreases with increasing repulsion. It is directly related to the compressibility κ of the gas, and $\alpha = 1$ describes the non-interacting system.

By passing from the Hamiltonian to the Lagrangian in either ϕ or θ we obtain

$$S = \int dx d\tau \frac{v}{2\alpha}\left[(\nabla\theta)^2 + \frac{1}{v^2}\dot{\theta}^2\right]. \tag{A.5}$$

A small potential perturbation described by the Hamiltonian

$$\delta H = \int dx V(x)\psi^\dagger(x)\psi(x),$$

can be expanded in terms of the bosonised operators and leads to the action

$$\delta S \approx \sum_{n=-\infty}^{\infty} \frac{V(2nk_F)}{2} \int d\tau e^{i2n\sqrt{\pi}\theta(x=0,\tau)}, \tag{A.6}$$

where $V(2nk_F)$ denote the Fourier coefficients. When the potential barrier has inversion symmetry $V(x) = V(-x)$ the Fourier coefficients are real. Physically the interaction corresponds to a process where n fermions are backscattered, each by a momenta $2k_F$, from one Fermi-point to another. Alternatively, Eq. (A.6) can be viewed as an effective potential which is invariant under the transformation $\theta \to \theta + \sqrt{\pi}$. Since $\theta/\sqrt{\pi}$ is the number of particles to the left of $x = 0$, the interaction may be regarded as a weak pinning potential of the liquid. Keeping the most dominant $2k_F$ term we obtain the effective action described in section A.1.

The scaling flow diagram described in section A suggests the remarkable result that the transmission coefficient of a gas of particles with repulsive interaction through an infinitesimal barrier scales to zero, while an attractive interaction scales to unity. Non-interacting particles describe the critical case.

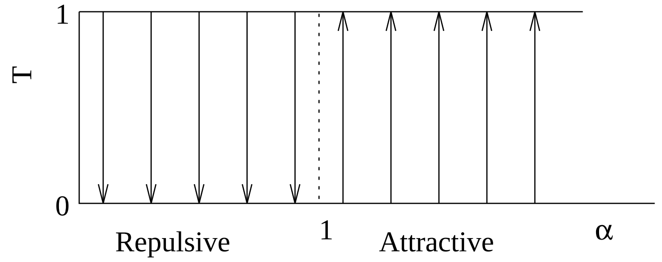

Figure A.3: RG flow diagram of the transmission coefficient of a one-dimensional strongly interacting gas of particles through a weak barrier.

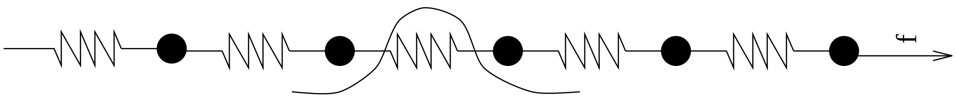

Figure A.4: Model of quantum friction described by a chain of springs coupled to a pinning potential.

Models of this kind are believed to describe, amongst other systems, chiral edge states in the fractional quantum Hall regime. After the original discussion by Kane and Fisher, the promise of experimental detection of Non-Fermi Liquid like behaviour spawned a flurry of activity. Generalisations to include Coulomb Blockade and Aharonov-Bohm effects, resonant tunnelling, and superconductor tunnel junctions were quickly proposed. At present, the experimental situation in this area seems less certain but active.

A.2.2 Quantum Friction

The action studied in section A.1 is easily generalised to study the leading contribution to the tunnelling exponent when a charge density wave is driven through a pinning centre (see L. S. Levitov, A. V. Shytov, and A. Yu. Yakovets, "Quantum Breaking of an Elastic String", Preprint 1995 for an approach different to the one studied above). To the action of Eq. (A.1) we need add a driving force to the action

$$S_{\text{force}} = -\int d\tau F u(x=0,\tau), \tag{A.7}$$

where for simplicity we suppose that the force acts on the tunnelling particle.

Previously we found that a kink-antikink pair experiences a binding potential which varied logarithmically with their separation. Below the delocatisation threshold the string is pinned by the potential. With a force, the action of kink-antikink pair is easily obtained

$$S_{\text{eff}}(\tau) = 2\alpha \ln(\omega_0|\tau|) - 2\pi F|\tau|, \tag{A.8}$$

The action is minimised for $\tau^* = \alpha/\pi F$ and implies a total tunneling or slip probability

$$W \sim e^{-S_{\text{eff}}(\tau^*)/\hbar} = \left(\frac{\alpha\omega_0}{e\pi F}\right)^{-2\alpha}. \tag{A.9}$$

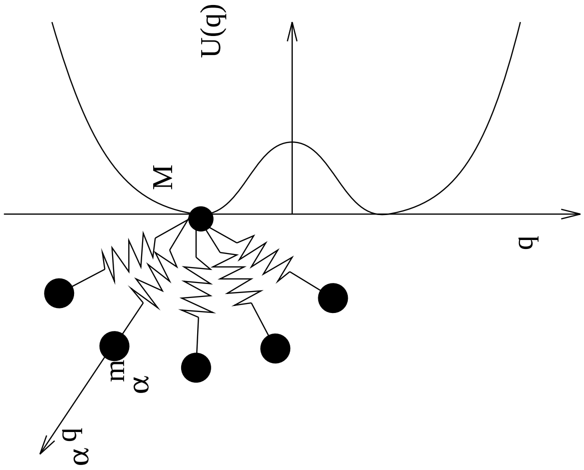

Figure A.5: Diagram showing the macroscopic quantum tunnelling of a particle coupled to an external bath.

The sliding velocity $v \sim W^{1/2} \sim F^{\alpha}$ follows a power law. The hydrodynamic description of the electron liquid implies an Ohm's law for free fermions ($\alpha = 1$, $v \sim I$, and $F \sim V$). When $\tau^* > 1/\omega_0$, there is no time for the lattice to relax before the tunnelling repeats and the pinning fails catastrophically.

A.2.3 Quantum Dissipation

The influence of an external ensemble of oscillators on the tunnelling of a quantum particle can be studied using the Hamiltonian introduced by Caldeira and Leggett (Phys. Rev. Lett. **46**, 211 (1981)),

$$H = \frac{M}{2}\dot{q}^2 + U(q) + \sum_{\alpha}\left[\frac{m_\alpha}{2}\dot{q}_\alpha^2 + \frac{m_\alpha}{2}\omega_\alpha^2 q_\alpha^2 - f_\alpha(q)q_\alpha\right], \tag{A.10}$$

where $f_\alpha(q)$ defines the coupling of the particle to the bath. The effective action for the particle can be obtained by integrating out the degrees of freedom corresponding to the bath. In fact, in this form, the action is a generalisation of one considered earlier.

$$\begin{aligned} S_{\text{eff}} &= S_{\text{part}} + S_{\text{bath}}, \\ S_{\text{bath}} &= \sum_{\alpha}\int \frac{d\omega}{2\pi}\frac{f_\alpha[q(\omega)]f_\alpha[q(-\omega)]}{2m_\alpha(\omega_\alpha^2 + \omega^2)}. \end{aligned}$$

If we assume that the coupling is linear $f_\alpha[q(\tau)] = c_\alpha q(\tau)$, we obtain

$$S_{\text{bath}} = \int d\tau d\tau' q(\tau)q(\tau')K(\tau - \tau'), \qquad K(\tau) = \sum_{\alpha}\frac{c_\alpha^2 e^{-\omega_\alpha|\tau|}}{4m_\alpha\omega_\alpha},$$

which is known as the *Caldiera-Leggett action.*

Typically the bath can be considered as a continuum in which the number of oscillators diverges while their weight $c_\alpha^2/m_\alpha\omega_\alpha^2$ vanishes. In this limit it is more convenient to express the effective interaction in terms of a spectral density of the thermal bath

$$K(\tau) = \int \frac{d\omega}{2\pi} J(\omega) e^{-\omega|\tau|}, \qquad J(\omega) = \frac{\pi}{2} \sum_\alpha \frac{c_\alpha^2}{m_\alpha\omega_\alpha} \delta(\omega - \omega_\alpha).$$

For a spectral density that scales as a power law $J(\omega) = A|\omega|^\beta$, we obtain

$$S_{\text{bath}} = A' \int dt dt' \frac{q(\tau)q(\tau')}{|\tau - \tau'|^{1+\beta}} = A'' \int \frac{d\omega}{2\pi} |\omega|^\beta |q(\omega)|^2. \tag{A.11}$$

This implies that for $\beta > 1$ the tunnelling amplitude is reduced, while for $\beta > 1$ the tunnelling is suppressed completely. The "Ohmic" case of $\beta = 1$ or $J = A|\omega|$ is critical and coincides with the case studied in the text.

Caldeira and Leggett originally introduced the Hamiltonian above to study the macroscopic quantum tunnelling of the phase difference q across a Josephson element in a rf SQUID ring.