Lecture II: Landau's Fermi-Liquid Theory

...a triumph of phenomenology!

▷ References:

Original papers: L. D. Landau, *The Theory of the Fermi Liquid*, Sov. Phys. JETP **3**, 920 (1956); *Oscillations in a Fermi Liquid*, **5**, 101 (1957)

D. Pines and P. Nozières, *The Theory of Quantum Liquids, vol. 1*, Addison-Wesley 1989

▷ Motivation: Why does the free electron theory work so well as a description of 'real' metals and Fermi systems (viz. ^{3}He, ^{6}Li, etc.) at low temperatures?

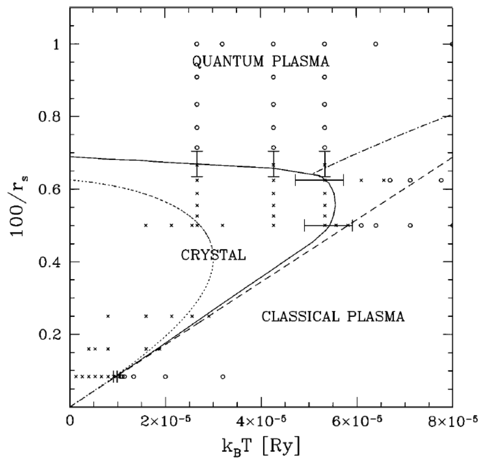
Is the interaction energy small? With d the average electron separation, $\frac{4}{3}\pi d^3 \rho_e = 1$,

$$\langle \text{k.e.} \rangle \sim \frac{\hbar^2}{2md^2}, \qquad \langle \text{p.e.} \rangle \sim \frac{e^2}{4\pi\varepsilon_0 d}$$

$$\text{i.e. } \frac{\langle \text{p.e.} \rangle}{\langle \text{k.e.} \rangle} \sim \frac{me^2}{2\pi\varepsilon_0\hbar^2} d = \frac{d}{a_B} \equiv r_s, \qquad a_B \text{ — electron Bohr radius}$$

Interactions should dominate if $r_s \gg 1$ (i.e. ρ_e small)

Indeed, for $r_s \sim O(10^2)$, electrons known (via QMC) to crystallise (Wigner)



Ceperly, ...

For metals, typical values:

$$\text{Be}: \ 1.87 \quad \leq r_s \leq \quad \text{Cs}: \ 5.62$$

not high enough to crystallize but Coulomb interaction is strong

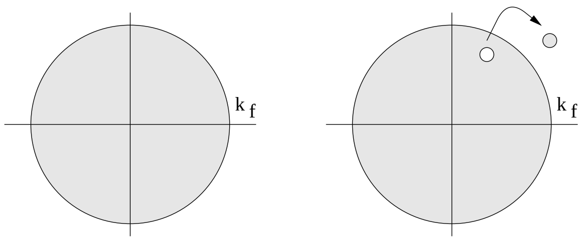
...so why does free-electron theory work so well?

▷ *(As discussed in first lecture)* Answer lies in principle of adiabatic continuity

Landau: while the fundamental symmetries remain unbroken, there will be a one-to-one correspondence between new and old states — new states can be labelled by the same quantum numbers

▷ Application to Fermi systems

Ground state of non-interacting Fermi gas is Fermi sphere with electrons filled up to the Fermi surface



Elementary excitations correspond to particle-hole excitations with allowed quantum numbers fixed by Fermi surface and Pauli principle

$\mathbf{k}\sigma$ — empty state below E_F — hole
$\mathbf{k}'\sigma'$ — newly occupied state above E_F — particle

Adiabatic continuity $\mapsto$ same true of interacting system: Elementary excitations are quasi-particles and quasi-holes about a filled quasi-particle Fermi surface

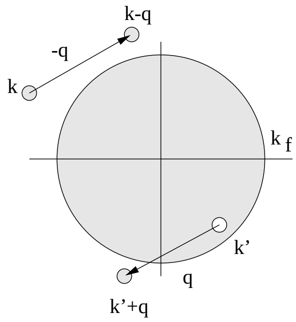
Corollary: consistency of quasi-particle picture relies on well-defined particle-like excitations — i.e. long lifetime...

▷ Consistency?

Lifetime of quasi-particle of energy $\epsilon \equiv E_k = |\epsilon_k - \epsilon_F|$ above Fermi surface?

Fermi's Golden rule:

$$\frac{1}{\tau_{\rm e}}(\epsilon) = \frac{2\pi}{\hbar} \sum_{\text{final states j}} |V_{ij}|^2 \delta(\epsilon_j - \epsilon)$$



Taking into account energy and momentum conservation,

$$\frac{1}{\tau_{\rm e}}(\epsilon) \sim \frac{2\pi}{\hbar} \int d\mathbf{k}' \int d\mathbf{q}\, |V(\mathbf{q})|^2 \delta(E_{k-q} + E_{k'+q} - E_{k'} - E_k)$$

Restriction that excitation energies $\{E_i > 0\}$ presents severe constraint on momentum sum (exercise)

$$\frac{1}{\tau_{\rm e}}(\epsilon) \sim \frac{\pi}{\hbar} \frac{|V|^2}{\epsilon_F^3} \epsilon^2 \qquad (\sim T^2 \quad \text{at finite T})$$

for slow decay, require $1/\tau_e\epsilon \ll 1$; consistent for $\epsilon/\epsilon_F \ll 1$

▷ Physical Consequences:

Entropy: as with non-interacting system, depends on allowed configurations
cf. $P_{\mathbf{k}} = G_{\mathbf{k}}!/N_{\mathbf{k}}!(G_{\mathbf{k}} - N_{\mathbf{k}})!$, *etc.*

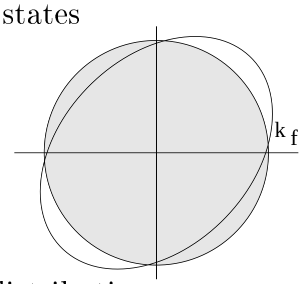
$$S = -k_B \sum_i p_i \ln p_i = -k_B \sum_{\mathbf{k}\sigma} [n_{\mathbf{k}\sigma} \ln n_{\mathbf{k}\sigma} + (1 - n_{\mathbf{k}\sigma}) \ln(1 - n_{\mathbf{k}\sigma})]$$

As usual, minimising S on $n_{k\sigma}$ subject to constraint $\sum_{\mathbf{k}\sigma} \delta n_{\mathbf{k}\sigma} = 0$ and $\sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} \delta n_{\mathbf{k}\sigma} = 0$, obtain Fermi-Dirac distribution

$$n_{\mathbf{k}\sigma} = \frac{1}{1 + e^{(\epsilon_{\mathbf{k}} - \mu)/k_B T}}$$

Energy: transfer to quasi-particle picture may

(i) result in modified single-particle spectrum, and
(ii) energy can depend on occupancy of other quasi-particle states



Setting $n_{\mathbf{k}\sigma} = \bar{n}_{\mathbf{k}\sigma} + \delta n_{\mathbf{k}\sigma}$, where $\bar{n}_{\mathbf{k}\sigma}$ is thermal equilibrium distribution

$$\epsilon^{\mathrm{qp}}_{\mathbf{k}\sigma} = \epsilon_{\mathbf{k}\sigma} - \epsilon_F + \frac{1}{V} \sum_{\mathbf{k}'\sigma'} f(\mathbf{k}, \sigma; \mathbf{k}'\sigma') \delta n_{\mathbf{k}'\sigma'}$$

Landau parameters $f(\mathbf{k}, \sigma; \mathbf{k}'\sigma')$ can depend both on angle $\theta_{\mathbf{k}\mathbf{k}'}$ between $\mathbf{k}$ and $\mathbf{k}'$ as well as relative spin orientation

Interactions effect both $f(\mathbf{k}, \sigma; \mathbf{k}', \sigma')$ and (linearising around ϵ_F)

$$\epsilon_{\mathbf{k}\sigma} - \epsilon_F \simeq \frac{\hbar^2 k_F}{m^*}(k - k_F)$$

(N.B. $m^* \neq m$ is interaction, and not lattice, effect)

For spherical Fermi surface, $f(\mathbf{k}, \sigma; \mathbf{k}', \sigma')$ depend only on $\theta_{\mathbf{k}\mathbf{k}'}$
i.e. separable into symmetric and antisymmetric spin components,

$$f(\mathbf{k}, \sigma; \mathbf{k}'\sigma') = \sum_\ell [f^s_\ell P_\ell(\cos\theta_{\mathbf{k}\mathbf{k}'}) + f^a_\ell P_\ell(\cos\theta_{\mathbf{k}\mathbf{k}'})\sigma\sigma']$$

Legendre polynomials: $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, etc.

Quasi-particle Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} (\epsilon_{\mathbf{k}\sigma} - \epsilon_F)\delta n_{\mathbf{k}\sigma} + \frac{1}{2V} \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} \delta n_{\mathbf{k}\sigma} f(\mathbf{k}, \sigma; \mathbf{k}'\sigma')\delta n_{\mathbf{k}'\sigma'}$$

▷ Experiment

(1) Specific heat: Only change reflected in modification of DoS through effective mass m^*: $\nu(0) = \dfrac{m^* k_F}{\pi^2 \hbar^2}$,

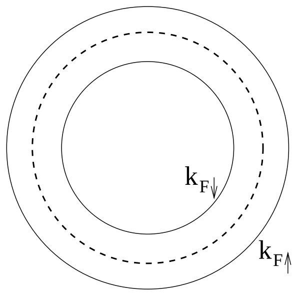
$$C_v = \frac{\pi^2}{3}\nu(\epsilon_F) k_\mathrm{B}^2 T = \frac{m^* k_F}{3\hbar^2} k_\mathrm{B}^2 T$$

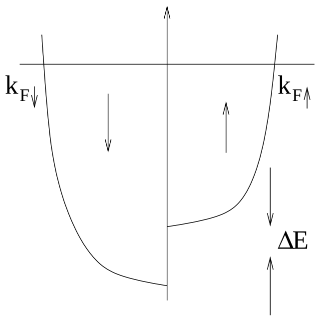
(2) Pauli spin susceptibility: response to external magnetic field

with $\sum_\mathbf{k} (\delta n_{\mathbf{k}\uparrow} + \delta n_{\mathbf{k}\downarrow}) = 0$, and

$$M = -\mu_B \sum_\mathbf{k} (\delta n_{\mathbf{k}\uparrow} - \delta n_{\mathbf{k}\downarrow})$$

can infer $\displaystyle\sum_\mathbf{k} \delta n_{\mathbf{k}\uparrow} = -\sum_\mathbf{k} \delta n_{\mathbf{k}\downarrow} = -\frac{M}{2\mu_B}$





Shift in density distribution isotropic; shift in energy

$$\Delta E = \mu_B B + \frac{1}{V} \sum_\mathbf{k} [(f_0^s + f_0^a)\delta n_{\mathbf{k}\uparrow} + (f_0^s - f_0^a)\delta n_{\mathbf{k}\downarrow}] = \mu_B B - \frac{M f_0^a}{\mu_B V}$$

Therefore, noting $M = \mu_B \overbrace{V\nu(0)\Delta E}^{\text{polarisation}}$

$$M = \frac{\mu_B^2 V \nu(0)}{1 + \nu(0) f_0^a} B, \qquad \chi = \frac{\mu_B^2 \nu(0)}{1 + F_0^a}$$

where $F_0^a = \nu(0) f_0^a$ is (dimensionless) "Landau parameter"

When $F_0^a \mapsto -1$, divergence (i.e. transition to ferromagnetism)

(3) Compressibility: $K = -\frac{1}{V}\frac{\partial V}{\partial P} = \frac{1}{n^2}\frac{\partial n}{\partial \mu}$ involves isotropic change in density (exercise)

$$K = \frac{V\nu(0)}{n^2}\frac{1}{1+F_0^s}, \qquad v_s^2 = \frac{1}{Kmn} = \frac{\hbar^2 k_F^2}{3m^* m}(1+F_0^s)$$

F_0^s positive — requires more energy to distort Fermi surface

▷ Landau parameters are not independent:

Galilean invariance — uniform translation of Fermi sphere
can be interpreted through F_1^s as quasi-particle/quasi-hole excitation

Mass transfer by quasi-particles — *no. quasi-particles equals no. particles*

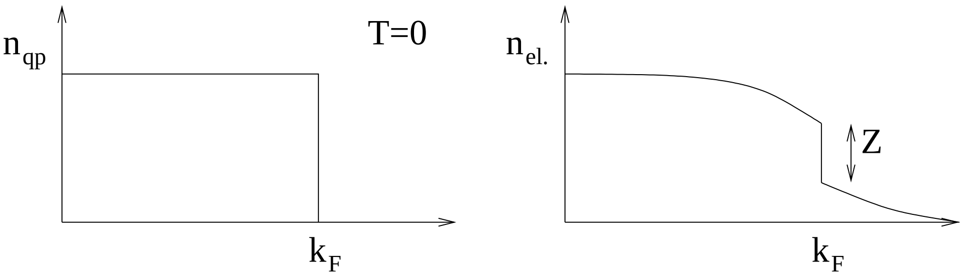
$$\sum_\sigma \int d\mathbf{p}\,\mathbf{p}\,n_{\mathbf{p}\sigma} \stackrel{!}{=} \sum_\sigma \int d\mathbf{p}\,m\frac{\partial \epsilon^{\rm qp}_{\mathbf{p}\sigma}}{\partial \mathbf{p}} n_{\mathbf{p}\sigma}$$

translates to condition (exercise)

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3}$$

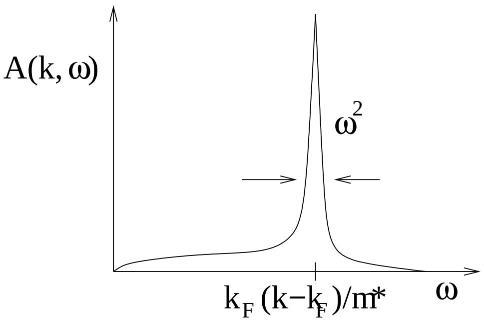
▷ What happens to bare electrons?

$$|\psi^{\rm qp}_{\mathbf{k}\sigma}\rangle = Z^{1/2}|\psi^{\rm el.}_{\mathbf{k}\sigma}\rangle + \text{p/h excitations}$$



i.e. Fermi-surface discontinuity

Spectral function $A(\mathbf{k},\omega)$— probability of finding electron $\mathbf{k}$ at energy ω — broadened



▷ Applications

^{3}He

	0 bar	27 bar
m^*/m	2.8	5.26
F_0^s	9.28	68.17
F_1^s	5.39	12.79
F_0^a	−0.696	−0.76

Metals: compelling evidence for Fermi-Liquid theory from heavy fermion materials

e.g. In UPt_3, $m^*/m_{\text{band}} \sim 17$ (measured by de Hass-van Alphen
and confirmed by specific heat $\gamma = C/T$)