Lecture XIV: Anderson Localisation[5]

▷ References:

Original papers:

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- E. Abrahams, P. W. Anderson, D. C. Licciardello and T. V. Ramakrishnan, Physical Review Letters **42**, 673 (1979).[7] [AALR]
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Reviews:

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- B. L. Altshuler, A. G. Aronov, D. E. Khmelnitskii and A. I. Larkin in *Quantum Theory of Solids*, edited by I. M. Lifshits (MIR Publishers, Moscow, 1983).[9] [AAKL]
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- D. J. Thouless, Physics Reports **13**, 93 (1974).

▷ Why is disorder important in metals?

Disorder, even when weak, makes a qualitative difference to the transport properties of metals. The Boltzmann description of charges scattering off impurities predicts a residual resistivity that increases with disorder. This simple picture is insufficient to explain experimentally observed behaviour even for weak disorder. When disorder is weak, deviations are seen from Ohm's law in the sample-size dependence of resistance. These deviations are especially remarkable in one and two dimensions.

Physically, repeated scattering of an electron from impurities causes the electron wavefunction to quantum-interfere with itself. Such interference may even cause it to become localised which leads to an exponential increase of resistance with system size. This is completely unexpected in the semi-classical treatment of electrical conduction.

A phenomenon analogous to localisation in metals is observed commonly: a stack of transparencies is practically opaque while individual sheets are transparent.

[5] Lecture notes prepared by Vikram Tripathi
[6] Difficult reading.
[7] Also known as the Gang of Four.
[8] Perturbation theory, diagrammatic analysis.
[9] Hard to find. Can read LK above instead.

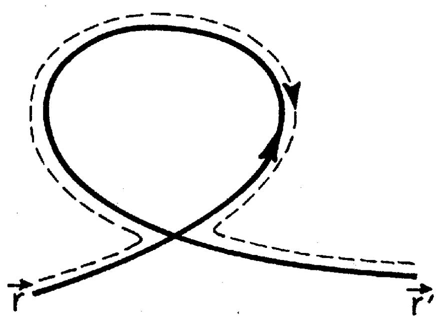

Figure 9: Simple physical picture of localisation due to AAKL (1983) and Bergmann [PRB **25**, 2397 (1982)]. The Green function $G(r, r'; E)$ can be constructed as a sum of amplitudes, $\sum_p A_p \, e^{i\theta_p(r,r')}$, of paths between r, r'. In the presence of impurities, most paths arrive with a random phase, with the exception of self-intersecting paths.

▷ How does interference encourage localisation?

Consider the probability to go from r to r', i.e., $|G(r, r'; E)|^2$. If the amplitude for the k^{th} path is $A_k e^{i\theta_k}$, the probability is

$$\left\langle \left| \sum_k A_k e^{i\theta_k} \right|^2 \right\rangle = \sum_k A_k^2 + 2 \sum_{j \neq k} A_j A_k \langle \cos(\theta_j - \theta_k) \rangle = \sum_k A_k^2,$$

since the average over the cosine vanishes. However if $r = r'$, then there are two paths k_1 and k_2 that are time-reversed (clockwise-anticlockwise) analogues of each other (loop part in Fig.9), so $A_{k_1} = A_{k_2} = A_k$, $\theta_{k_1} = \theta_{k_2} = \theta_k$. This contributes $4A_k^2$ to the return probability, as opposed to $2A_k^2$ in the absence of interference. Even though the enhancement is large, the number of such time-reversed pairs is a small fraction of the total number of possible paths. Nevertheless, total return probability is enhanced by interference!

- *Enhancement of probability to return is tantamount to decrease of conductivity*

The electron is a wave packet of transverse dimension λ_F and longitudinal dimension $v_F \tau_\varphi$, where τ_φ is the phase-breaking time,[10] assumed to be much larger than the mean free time for elastic collisions $\tau = l/v_F$. Since the electron motion is diffusive for times $t \gg \tau$, the electron is present in a volume $(Dt)^{d/2}$ where D is the diffusion constant ($\sigma = e^2 N_0 D$). In time t the wave packet charts out a volume $v_F t \lambda_F^{d-1}$, and the probability that it will intersect its course in a further time dt is $v_F \lambda_F^{d-1} dt/(Dt)^{d/2}$. Thus the relative change in conductivity is

$$\frac{\Delta\sigma}{\sigma} = -v_F \lambda_F^{d-1} \int_\tau^{\tau_\varphi} \frac{dt}{(Dt)^{d/2}}.$$

In three dimensions, the correction is

$$\frac{\Delta\sigma_{3\text{D}}}{\sigma_{3\text{D}}} = -\frac{2 v_F \lambda_F^2}{D^{3/2}} \left(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\tau_\varphi}} \right) = -\frac{2 v_F \lambda_F^2}{D} \left(\frac{1}{L_D} - \frac{1}{L_\varphi} \right),$$

[10] The characteristic time for the electron wavefunction to lose its phase memory due to inelastic collisions. One inelastic collision might not be sufficient and τ_φ can exceed inelastic scattering time.

where $L_D = \sqrt{D\tau}$ is the diffusion length and $L_\varphi = \sqrt{D\tau_\varphi}$ is the phase-breaking length. Note that the correction becomes a constant at large $L_\varphi = \sqrt{D\tau_\varphi}$.

In two dimensions, the correction is logarithmic:

$$\frac{\Delta\sigma_{2\mathrm{D}}}{\sigma_{2\mathrm{D}}} = -\frac{v_F\lambda_F}{D}\ln(\tau_\varphi/\tau) = -\frac{2v_F\lambda_F}{D}\ln(L_\varphi/L_D).$$

In one dimension, interference effects are most significant as L_φ increases:

$$\frac{\Delta\sigma_{1\mathrm{D}}}{\sigma_{1\mathrm{D}}} = -\frac{2v_F}{\sqrt{D}}\left(\sqrt{\tau_\varphi} - \sqrt{\tau}\right) = -\frac{2v_F}{D}\left(L_\varphi - L_D\right).$$

Such corrections to metallic conductivity are termed "weak localisation". If the phase breaking time τ_φ is temperature-dependent,[11] then the weak localisation corrections also acquire a temperature dependence (see Fig.10).

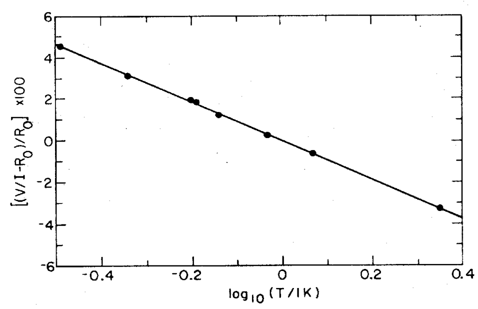

Figure 10: Resistivity change as a function of $\log_{10} T$ for PdAu film [Dolan and Osheroff, 1979].

- *A magnetic field suppresses interference effects*

We discussed how return probability is enhanced because of interference of time-reversed paths. Any looping path (with its time-reversed counterpart) contributes to enhancement of the return probability. In presence of a magnetic field, a loop and its time-reversed counterpart acquire excess Aharonov-Bohm phases,

$$\Delta\varphi = \pm(e/\hbar c)\oint \mathbf{A}\cdot d\mathbf{l} = \pm(e/\hbar c)HS,$$

where H is the component of the magnetic field normal to the loop and S is area of the loop. In time $t \gg \tau$, the area charted may be estimated as $S \sim Dt$. The Aharonov-Bohm phase differs from loop to loop, and once it exceeds 2π, we may disregard larger

[11] This is possible if phase breaking is due to inelastic processes that are temperature-dependent.

loops which will not contribute constructively to the return probability. That means a (magnetic) length scale $L_H = (eH/\hbar c)^{-1/2}$. Independently, loops that are larger than the phase-breaking length $L_\varphi = (D\tau_\varphi)^{1/2}$ will be dephased. The effect of a magnetic field becomes significant when L_H becomes the shorter phase-breaking length, i.e. when $L_H < L_\varphi$. [AKLL, 1980; Hikami, Larkin, Nagaoka, 1980] Since L_φ can be quite large in clean metals, the characteristic magnetic field required to suppress interference effects can be quite small, of the order of tens of gauss.

Suppression of interference by a magnetic field reduces localisation and therefore increases conductivity. Weak localisation corrections to magnetoresistance are thus positive.

The estimate for the conductivity is

$$\frac{\sigma(H,T) - \sigma(0,T)}{\sigma(0,T)} = v_F \lambda_F^{d-1} \int_0^{\tau_\varphi} \frac{dt}{(Dt)^{d/2}} \left[1 - \cos\left(\frac{eHDt}{\hbar c}\right)\right].$$

This expression can be simplified. In two dimensions,

$$\frac{\sigma_{2\mathrm{D}}(H,T) - \sigma_{2\mathrm{D}}(0,T)}{\sigma_{2\mathrm{D}}(0,T)} = \frac{v_F \lambda_F}{D} \int_0^{\frac{eH}{\hbar c} D\tau_\varphi} \frac{dz}{z} (1 - \cos z).$$

The result is

$$\frac{\sigma_{2\mathrm{D}}(H,T) - \sigma_{2\mathrm{D}}(0,T)}{\sigma_{2\mathrm{D}}(0,T)} \approx \begin{cases} \frac{v_F \lambda_F}{4D} \left(\frac{eH}{\hbar c} D\tau_\varphi\right)^2, & \frac{eH}{\hbar c} D\tau_\varphi \ll 1 \\ \frac{v_F \lambda_F}{D} \ln\left(\frac{eH}{\hbar c} D\tau_\varphi\right), & \frac{eH}{\hbar c} D\tau_\varphi \gg 1. \end{cases}$$

▷ Basic concepts of localisation

• Anderson (1958) argued that if disorder is *strong enough*, the electron wave function may get localised, $|\psi(\mathbf{r})| \sim \exp(-|\mathbf{r} - \mathbf{r}_0|/\xi)$. In one dimension, an arbitrarily *weak disorder* localises all states [Mott and Twose, 1961]. The traditional view had been that scattering by impurities causes the wavefunction to lose phase coherence over the mean free path l, nevertheless the wavefunction extends throughout the sample.

• Instead of varying disorder, let us vary the energy of the electron. States deep in the band tail are localised as they are formed from localised orbitals bound in the rarely occuring deep potential fluctuations. States near the centre of the band are likely to be extended for weak disorder. Below a certain energy E_c (mobility edge), all states will be localised. [Mott, 1967].

Q: As one crosses the mobility edge, does the metal-insulator transition happen continuously, or not?

• Mott (1973) argued that a *discontinuous* transition to the insulating state occurs once the mean free path $l \sim k_F^{-1}$. In the metallic state, $k_F l \gg 1$. Thus the minimum metallic conductivity can be estimated as

$$\begin{aligned}\sigma^{3D}_{\min} &= \frac{e^2}{\hbar}\frac{n}{k_F^2}(k_F l)\Big|_{k_F l\sim 1} = \frac{1}{3\pi^2}\frac{e^2}{\hbar}\frac{1}{a}\,\mathrm{in 3D, and a universal}\\ \sigma^{2D}_{\min} &\approx 0.1\frac{e^2}{\hbar}\,\mathrm{in 2D.}\end{aligned}$$

• The scaling theory of localisation which is the main subject of this lecture contests the existence of $\sigma_{\min}$ in both 3D and 2D. Instead it predicts that the metal-insulator transition is a *continuous* one in 3D and all states are localised in 2D.

▷ Scaling theory of localisation

Scaling theory describes localisation by considering the behaviour of the dimensionless *conductance* $g = G/(e^2/\hbar)$ as a function of system size $L \gg l$, or of other scale variables.[12]

Delocalised phase: The conductance of a $d-$dimensional hypercube of size $L > l$ and conductivity σ is given by Ohm's law,

$$g(L) = \sigma L^{d-2}.$$

Localised phase: DC conduction occurs by electron hopping from an occupied to an unoccupied state with nearly the same energy. However localised states close in energy are very far apart in space, so the tunnelling amplitude is exponentially small. Quite reasonably,

$$g(L) \sim e^{-L/\xi}.$$

(ξ is the localisation length, $l \ll \xi \ll L$). Note that this is a very non-Ohmic size-dependence.

• Single parameter scaling

AALR (1979) argued that

$$\beta(g) = \frac{d\ln g}{d\ln L} = \frac{L}{g}\frac{dg}{dL}$$

is a function of g alone. Physically, this means that the change in effective disorder as the size of the sample is increased a bit is determined by its value at the previous length scale, and the only measure of this effective disorder is the conductance.

The physical motivation behind such a picture is as follows. Consider the energy levels of a $d-$dimensional hypercube of size $2L$ in terms of the energy levels of the 2^d constituent blocks of size L. If $\delta = (N_0 L^d)^{-1}$ is the mean level spacing in the constituent blocks, and t is the overlap element between the blocks, then one expects the ratio t/δ to be a good measure of localisation. If the wavefunction is localised, t will be insensitive to the boundary conditions and is therefore exponentially small (so is t/δ). If however

[12] Note that the conductivity expressed in units of $e^2/\hbar$ is not a dimensionless parameter.

t/δ is large, the eigenstate of the $2L$ cube will be spread over all the 2^d cubes of size L, and is therefore extended. Therefore sensitivity to boundary conditions (or the ratio t/δ) appears to be the single parameter that controls the nature of the eigenstate as the system doubles its size. [Thouless, 1974]. It appears *plausible* that the sensitivity to boundary conditions of the larger block depends on the sensitivity to boundary conditions for the constituent blocks, i.e., $(t/\delta)_{2L}$ is a function of $(t/\delta)_L$.

- Scaling function $\beta(g)$

(a) Large conductance $g \gg g_c$:

$$\beta(g) = (d-2).$$

(b) Small Conductance $g \ll g_c$:

$$\beta(g) = \ln(g/g_c).$$

(c) Perturbation theory gives deviation from Ohm's law:

$$\beta(g) = (d-2) - \frac{g_c}{g}.$$

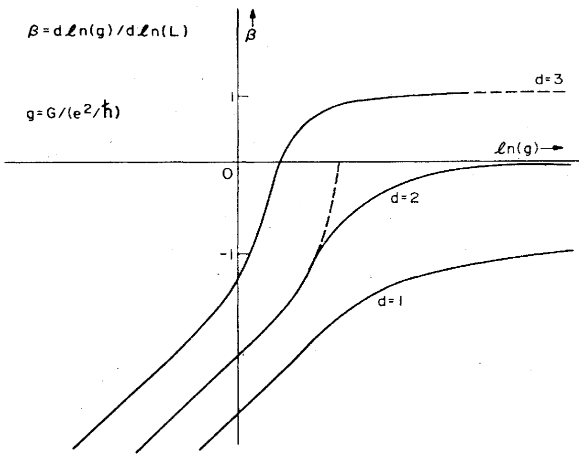

Figure 11: Scaling function $\beta(g) = d\ln g/d\ln L$ in various dimensions.

For electrons, $g_c = 1/\pi^2$. In two dimensions, integrating the scaling function $\beta(g)$ from the lower length scale l to the system size L gives the deviation of scale dependence of the conductance from Ohm's law:

$$g(L) = g(l) - \frac{1}{\pi^2}\ln\left[\frac{L}{l}\right],\ g(l) = \frac{k_F l}{2\pi}.$$

The logarithmic correction to conductance becomes comparable with the Boltzmann conductance $g(l)$ at the localisation length $L = \xi_{\rm loc}^{\rm 2D}$:

$$\xi_{\rm loc}^{\rm 2D} \sim l e^{\pi^2 g(l)} = l e^{\pi k_F l/2}.$$

A two-dimensional disordered metal is *non-Ohmic* at all length scales and for arbitrarily weak disorder!

In three dimensions, if $g > g_c$, then $\beta(g)$ is positive, i.e., conductance *increases* as the size of the conductor is increased. Thus we have a metal. If $g < g_c$, $\beta(g)$ is negative, which means the conductance decreases as the size increases. This is an insulator. At the critical value of conductance g_c, there is no change in the conductance as the sample size is increased!

I summarise below the scale-dependence in various dimensions when the conductivity is large.

D=3: $\sigma_{\rm 3D}(L) = \sigma(l) - \frac{e^2}{\hbar\pi^3}\left[\frac{1}{l} - \frac{1}{L}\right]$

D=2: $\sigma_{\rm 2D}(L) = \sigma(l) - \frac{e^2}{\hbar\pi^2}\ln\left[\frac{L}{l}\right]$

D=1: $\sigma_{\rm 1D}(L) = \sigma(l) - \frac{e^2}{\hbar\pi}(L - l).$

Observe that the above scaling is the same as we obtained from considerations of interference-enhanced return probability!

▷ Next lecture

We found that in the localised phase, the dc conductivity at zero temperature vanishes. The electrons are localised on the scale of the localisation length $\xi_{\rm loc}$. At a finite temperature, the conductivity is nonzero, and the mechanism of transport is electron hopping between different localised sites. What is the temperature dependence of such hopping conductivity?