Lecture notes from Sam Edwards' 1985 course on Analytical Dynamics delivered to second year ungraduates in Cambridge. Uploaded by Ben Simons, who took the course!

Analytical Dynamics

Books Landau and Lifshitz Brief but many useful examples. Will use as basic text for these lectures. Several brief books in Rayleigh Library

Leech Classical Mechanics

Ter Haar Hamiltonian Mechanics

Treatises are by E.T. Whittaker (C.U.P.) (which is good value as paperback) and by L. Pars which has many good examples but is rather long.

Several good intermediate sized books e.g. Goldstein.

Lectures to cover

1) Review
2) Invariants from Lagrange's Equations, virial theorem
3) Small oscillations; damping, resonance.
4) Friction, Rayleighan function
5) Angular motion
6) Gyroscopes, tops etc., Coriolis' forces
7) Constraints, holonomic and non holonomic systems
8) Least constraint; Gibbs-Appell equations
9) Hamiltonians, Liouville's equation
10) Hamilton-Jacobi theory; canonical transformations
11) Continuous systems

Analytical Dynamics Concerns itself with the expression of the laws of physics. Although historically the dynamics of particles and rigid bodies came first, the subject embraces the equations of wave motion and of quantum mechanical phenomena. One can regard physics as the investigation of nature which leads to powerful and succinct laws in which huge amounts of information are reduced to brief principles and equations.

Classical mechanics has reached this point in formulation (the last great work of formulation came in 1900) but there are still surprises appearing in the solution of the equations of motion.

To illustrate the fact that there are difference approaches we write down a brief preview of the formulations:

Lagrange's Equations : are the heir to Newton's equations and are differential equations for the coordinates or other descriptive variables.

Usually Lagranges equations are second order differential equations for say k dynamical variables: $\ddot{q}_r = f_r(\ldots q \ldots)$

Another method is to use Hamiltons equations where the equations appear in pairs for coordinate q and momentum p

$$\frac{\partial q_r}{\partial t} = \frac{\partial H}{\partial p_r}, \quad \frac{\partial p_r}{\partial t} = -\frac{\partial H}{\partial q_r}$$

where H(qp) is the energy written in terms of p, q when it is called the Hamiltonian.

The Hamiltonian form emphasizes an essential aspect of physical laws: they are Causal i.e. the future is determined by the past. (Note that Causality is not the same concept as determinism. Causal equations say that if we know a set of variables say p, q of Hamiltonian at time t, we can calculate them at a later time. Or given a wave function $\psi(r)$ at time t we can calculate it later. Determinism says that experimental measurement at time t permits the prediction of the results of experimental measurements at a later time. Classical physics is causal and deterministic, quantum physics is causal but not deterministic).

Both Lagrange's and Hamilton's equations give time dependent functions as their solutions which directly describe the system e.g. a particle has a coordinate X(t). An alternative is to ask for the probability of finding the particle at x, P(x,t) say. If a particle moves on a definite trajectory P is just $\delta(x - X(t))$, and if $\dot{x} = F(x)$, P satisfies the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} F(x)\right) P(x,t) = 0$$

or more generally for say the Hamiltonian variables

$$\left(\frac{\partial}{\partial t} + \Sigma \frac{\partial H}{\partial p_r}\frac{\partial}{\partial q_r} - \frac{\partial H}{\partial q_r}\frac{\partial}{\partial p_r}\right) P(\ldots q \ldots p \ldots, t) = 0$$

This is Liouville's equation and is the foundation of the statistical mechanics of any physical system. We start by studying Lagranges formulation of mechanics (1788). The usual cartesian variables labelling a particle, or the part of a rigid body is called x, (x, y, z) [3 Dim], (x_r, y_r, z_r) [several of them] but these are often not the best variables, for one likes to have a variable q such that all values of q correspond to all states of the system and unless it is totally unconstrained, the simple cartesians won't do this.

Lagrange showed that Newton's (and subsequently all other conservative) equations of motion could be derived from a Lagrangian

$$L(x, \dot{x}, \ddot{x} \,....) \text{ (usually only up to } \dot{x})$$

or in terms of our dynamical variables q

$$L(...q_a \;...\; \dot{q}_a \;...)$$

by the calculus of variations i.e. if we consider

$$\int L dt = S \qquad \delta S = 0 \text{ for the actual motion}$$

S is called Hamiltonians principal function, and its numerical value is called the Action. (Sometimes S is called the action, but I prefer to think of action as the numerical value of S in erg-seconds or Joule-hours or whatever, just as the Hamiltonian is a function, but its value is the energy measured in units of energy.) Rather confusingly the basic equation is refered to as Hamilton's principle:

$$\delta \int L \, dt = 0 \quad \text{or} \quad \delta S = 0$$

If $L = L(q,\dot{q})$ let us vary q to $q(t) + \delta q(t)$

$$\delta L + L = L(q + \delta q, \dot{q} + \delta\dot{q})$$

$$= L + \delta q \frac{\partial L}{\partial q} + \delta\dot{q} \frac{\partial L}{\partial \dot{q}} + O(\delta q)^2$$

but $\delta\dot{q}(t) = \frac{d}{dt} \delta q(t)$, so if $\delta S = 0$

$$0 = \int dt \left(\delta q(t) \frac{\partial L}{\partial q} + \delta\dot{q} \frac{\partial L}{\partial \dot{q}} \right)$$

$$= \int dt \left(\delta q(t) \frac{\partial L}{\partial q} + \frac{d}{dt}(\delta q) \frac{\partial L}{\partial \dot{q}} \right) \quad \text{Integrate by parts}$$

$$= \int dt \delta q(t) \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) + \left(\begin{array}{l} \text{end effects which} \\ = 0 \text{ for paths starting} \\ \text{and ending at same points} \end{array} \right)$$

Hence

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

or for several q's $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = 0.$

One integration is possible, for multiply by $\dot{q}_r$ and sum

$$\sum_r \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} \right) \dot{q}_r = 0 \quad \text{but we can write} \quad \Sigma \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r}$$

i.e. $$\frac{d}{dt} \left(\Sigma \, \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} \right) - \Sigma \left(\frac{\partial L}{\partial q_r} \dot{q}_r + \frac{\partial L}{\partial \dot{q}_r} \ddot{q}_r \right)$$

$$= \frac{d}{dt} \left(\Sigma \, \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L \right)$$

So that

$$\Sigma \, \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h \text{ a constant,}$$ called Jacobi's integration, the energy.

If the energy can be split into a kinetic energy T and a potential energy V

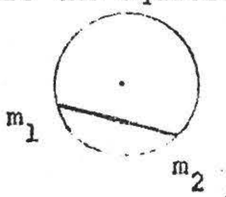
$$T + V = h \text{ (or often written as E)}$$

then $$T - V = L$$

e.g. for particle in a potential q is just x, $L = \frac{1}{2} m\dot{x}^2 - V(x)$.

In elementary mechanics a great advantage of Lagrange's approach compared to working directly from Newton's Laws is that the various reaction forces which come into N's equations and have then to be eliminated, just don't appear in Lagrange's equations so that one goes straight from T - V to the equation of motion.

Examples A light rod has two heavy rings at its ends which enclose a rigid wire. What are the equations of motion when 1) the wire is a circle in a vertical plane.

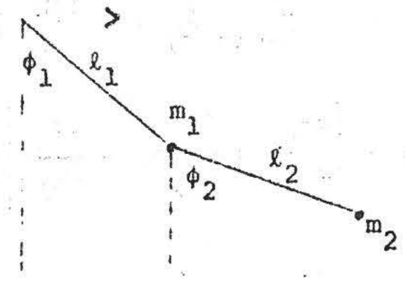


2) a vertical right helix $z = b\theta$

$x = a\sin\theta$

$y = a\cos\theta$

3) A double pendulum is



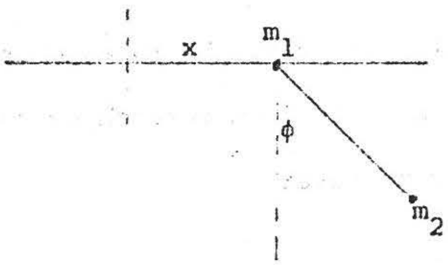
Show that

$$L = \tfrac{1}{2}(m_1+m_2)\ell_1^2\dot{\phi}_1^2$$

$$+ \tfrac{1}{2}m_2\ell_2^2\dot{\phi}_2^2 + m_2\ell_1\ell_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)$$

$$+(m_1+m_2)g\ell_1\cos\phi_1 + m_2g\ell_2\cos\phi_2.$$

4) Sliding pendulum



$$L = \tfrac{1}{2}(m_1+m_2)\dot{x}^2$$

$$+ \tfrac{1}{2}m_2(\ell^2\dot{\phi}^2 + 2\ell\dot{x}\dot{\phi}\cos\phi)$$

$$+ m_2\, g\ell\cos\phi.$$

Derive the equations of motion in these problems 1 - 4 (and try to do it using Newton's laws for comparison).

5) If one changes variables $x_a = f_a(q_1 \ldots q_s)$

$$a = 1 \ldots s$$

then $L = \Sigma \frac{1}{2} m_a(\dot{\underset{\sim}{r}}^2) - U$

becomes $L = \frac{1}{2} \Sigma a_{ih}(q)\dot{q}_i\dot{q}_h - U(q)$.

If one studies such an L with a general a_{ih} show that

$$\Sigma_{a\,ij}\ \ddot{q}_j + \Sigma \begin{bmatrix} jk \\ i \end{bmatrix} \dot{a}_j\dot{q}_k = -\frac{\partial U}{\partial q}$$

where $\begin{bmatrix} jk \\ i \end{bmatrix} = \frac{1}{2}\left[\frac{\partial a_{ki}}{\partial q_j} + \frac{\partial a_{ij}}{\partial q_k} - \frac{\partial a_{jk}}{\partial q_i}\right]$

(This is called a Christoffel symbol.)

A note on functional differentiation.

We got Lagranges Equation by putting $q(t) \to q(t) + \delta q$ but this is a bit pedestrian, for if we had $\delta(x)$, then by putting $f(x + dx) = f(x) + g(x)dx$ explicitly we can find $f'(x) = g(x)$, but normally one uses the rules of the calculus, and does not prove theorems like $\frac{d}{dx}x^2 = 2x$ from scratch everytime. So there should be an extension of the calculus to cover

$\frac{\delta}{\delta q(t)} F([q])$ directly. It is this:

I think of a set of variables x_1, x_2 .... then $\frac{\partial x_1}{\partial x_1} = 1$, $\frac{\partial x_2}{\partial x_1} = 0$ or briefly $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$.

In particular if $A = \Sigma a_j x_j$ $\frac{\delta A}{\delta x_i} = \frac{\delta}{\delta x_i}\Sigma a_j x_j = \Sigma a_j \delta_{ij} = a_i$

If we consider $A \to \int a(j)x(j)a_j$, $x \to x + \delta x$ gives

$\int a(j)x(j) + \int a(j)\delta x(j)dj$ and

$\frac{\partial A}{\partial x_i} = a_i$ ought to go over to $\frac{\partial A}{\partial x(i)} = a(i)$.

The appropriate form is $\frac{\delta x(j)}{\delta x(i)} = \delta(i - j)$

The analogue is then

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad \begin{aligned} \delta_{ij} &= 1 \qquad i = j \\ &= 0 \qquad i \neq j \end{aligned}$$

$$\frac{\delta x(i)}{\delta x(j)} = \delta(i-j) \qquad \begin{aligned} \delta(i-j) &= 0 \quad i \neq j \\ &= \infty \quad i = j \end{aligned}$$

but in such a way that

$$\int \delta(i-j)\, d_j = 1.$$

Our previous definition of L's equation now becomes

$$\frac{\delta}{\delta q(t)} \int L \, d\tau$$

$$= \int \delta(t-\tau) \frac{\partial L}{\partial q} + \int \frac{\delta \dot{q}(\tau)}{\delta q(t)} \frac{\partial L}{\partial \dot{q}}$$

$$+ \int \frac{\delta \ddot{q}(\tau)}{\delta q(t)} \frac{\delta L}{\delta \ddot{q}} + \ldots \quad .$$

But $\displaystyle \frac{\delta \dot{q}(\tau)}{\delta q(t)} = \frac{d}{d\tau} \frac{\delta q(\tau)}{\delta q(t)} = \frac{d}{dt} \delta(t - \tau)$

$$\frac{\delta \ddot{q}(\tau)}{\delta q(t)} = \frac{d^2}{d\tau^2} \frac{\delta q(\tau)}{\delta q(t)} = \frac{d^2}{d\tau^2} \delta(t - \tau).$$

The rule with δ functions is always to convert any integral into

$$\int_{-\infty}^{\infty} \delta(t - \tau)\phi(\tau)\, d\tau = \phi(t)$$

and one does this (as before) by integration by parts $\displaystyle \int \dot{\delta}(\tau-t) \frac{\partial L}{\partial \dot{q}} = -\int \delta(t-\tau) \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$

$$= -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$$

So that L's equations are

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} - \ldots = 0.$$

Example: a dynamical system has

$$L = (\ddot{q}^2)^n \phi(q)$$

what are its equations of motion.

<u>Conservation Laws</u> For most examples L is not a function of time, and if we are discussing some basic physical system this says that the law of physics involved is the same whenever we study it.

$$\frac{dL}{dt} = \Sigma \frac{\partial L}{\partial q_i} \dot{q}_i + \Sigma \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i;$$

$$= \Sigma \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \Sigma \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

$$= \frac{d}{dt} \Sigma \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

$\therefore$ $\Sigma \dot{q} \frac{\partial L}{\partial \dot{q}} - L$ = constant, the Jacobi integral, as above.

The invariance under time displacement of the equations of motion implies conservation of energy.

The result is general: any invariance leads to a conservation. The simple cases are time, above, displacement in space $r \to r + \varepsilon$

$$\delta L = \Sigma \frac{\partial L}{\partial r} \delta r = \varepsilon \Sigma \frac{\partial L}{\partial r}$$

But if laws are invariant for any ε, we must have

$$\Sigma_a \frac{\partial L}{\partial r_a} = 0 \quad \therefore \quad \frac{d}{dt} \Sigma_a \frac{\partial L}{\partial V_a} = 0 \quad , \text{ writing V for r}$$

$$\therefore \quad P = \Sigma_a \frac{\partial L}{\partial V_a} \text{ is conserved}$$

For particles $P = \Sigma_a m v_a$ conservation of momentum.

If the whole system is moved with a velocity $\underset{\to}{V}$, $V_a = V'_a + V$ amounts to a moving frame of reference. Then

$$P = \Sigma mv$$

$$= \Sigma mv' + V\Sigma m$$

$$P = P' + VM \qquad M = \Sigma m$$

$$V = \frac{P}{M} \text{ in a frame where } P' = 0$$

i.e. system has its centre of mass at rest where centre of mass $\underline{R} = \Sigma m_a \underline{r}_a / \Sigma m_a$

$$\text{Energy} = \tfrac{1}{2}\Sigma\, m_a v_a^2 + U$$

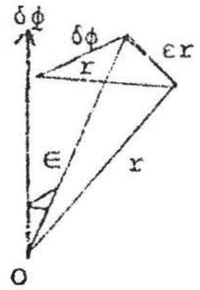
$$= \tfrac{1}{2}\Sigma\, m_a (v_a' + V)^2 + U$$

$$= \tfrac{1}{2}\Sigma\, m_a v_a'^2 + V\cdot \Sigma\, m_a v_a + U$$

$$E = E' + V\cdot P' + U.$$

A related theorem is Konig Theorem: the kinetic energy of a body can be separated into the kinetic energy which would obtain if all the mass were concentrated at the centre of mass, and the kinetic energy which would obtain if the centre of mass were fixed and the body rotated about it.

Angular momentum Conservation law follows from the isotropy of space i.e. laws of nature are invariant on rotation.



consider a rotation $\delta\phi$, also considered as a vector $\underline{\delta\phi}$. O arbitrary origin

$$\delta\underline{r} = \delta\underline{\phi} \times \underline{r}.$$

so that $\delta\underline{v} = \delta\underline{\phi} \times \underline{v}$

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \underline{r}_a} \delta\underline{r}_a + \frac{\partial L}{\partial \underline{V}_a} \cdot \delta\underline{v}_a \right) = 0 \text{ if laws invariant}$$

$$\therefore \sum_a (\dot{\underline{p}}_a \cdot \delta\underline{\phi}_a \times \underline{r}_a + \underline{p}_a \cdot \delta\underline{\phi} \times v_a) = 0$$

$$\text{or } \delta\phi\cdot \frac{d}{dt} \Sigma\, r_a \times p_a = 0$$

$$\therefore \underline{M} = \sum_a \underline{r}_a \times \underline{p}_a \text{ is conserved}$$

M is Angular momentum.

Change of origin $M = M' + a \times p$.

In frame of reference moving with V, $M = \Sigma\, m_a r_a \times v_a = \Sigma\, \underline{m}_a\, r_a \times \underline{v}'_a$

$$+ \Sigma\, \underline{m}_a \underline{r}_a \times \underline{V}_a$$

$M = M' + \mu \underline{R} \times \underline{V}$ or $M = M' + R \times P$ in the case where one system has its C.M. at rest i.e.

M = Intrinsic ang.mom + ang.mom due to motion as whole.

In a central field one can take the centre of the field as origin when

$$L = \tfrac{1}{2}\, \Sigma\, m_a(\dot{r}_a^2 + r_a^2 \sin^2\theta_a \dot{\phi}_a^2 + r_a^2 \dot{\theta}_a^2) - \Sigma\, U(r_a)$$

and $\underline{M}$ is conserved along any axis thro that centre.

<u>Examples</u> 1) a homogeneous field exists in the z direction, prove that M_z is conserved (irrespective of origin)

2) What are components of M in cylindrical coordinates

$$M_x = m(r\dot{z} - z\dot{r})\sin\phi - mrz\dot{\phi}\cos\phi$$

$$M_y = -m(r\dot{z} - z\dot{r})\cos\phi - mz\dot{\phi}\sin\phi$$

$$M_z = mr^2\dot{\phi}$$

$$M^2 = m^2 r^2 \dot{\phi}^2 (r^2 + z^2) + m^2 (r\dot{z} - z\dot{r})^2$$

3) In polar coordinates

$$M_x = -mr^2(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi)$$

$$M_y = mr^2(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi)$$

$$M_z = mr^2\dot{\phi}\sin^2\theta$$

$$M^2 = m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

<u>Virial Theorem</u>: <u>Scaling</u>

A scaling argument is based on the idea that in certain physical situations a change $r \to \alpha r$ for all coordinates can be absorbed by a redefinition of the constants in an equation in a non trivial way.

There are many scaling <u>hypotheses</u> in physics, but in mechanics the process is applied as a rigorous result for some specially simple cases.

Suppose $U(\alpha r_1, \alpha r_2 \, . \, \alpha r_n) = \alpha^k U(r_1 \ldots r_n)$

and change $t \to \beta t$. Then

$$L \to \frac{\alpha^2}{\beta^2} T - \alpha^k V$$

$$= \text{constant} \times L \text{ provided that}$$

$$\frac{\alpha^2}{\beta^2} = \alpha^k \qquad \text{or } \beta = \alpha^{1-k/2}$$

Now consider a closed system of particles (in box say in thermal equilibrium (but not necessarily so)) with such a potential and consider the time average of any quantity $f(t)$

$$\bar{f} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f(t)\,dt$$

If f contains a term $\frac{dF}{dt}$, this does not contribute provided F is bounded because $\int \frac{\partial F}{dt} dt = F(\tau)-F(0)$ and $\lim_{\tau\to\infty} \frac{1}{\tau} \int \frac{dF}{dt} \therefore \to 0$. Apply this to T.

$$2T = \Sigma\, p_a v_a = \frac{d}{dt} \Sigma\, p_a\, r_a - \Sigma\, r_a\, \dot{p}_a$$

$\dot{p}$ is replaced by $-\partial U/\partial r_a$ and 1st term gave no contribution by our result above, hence

$$2\bar{T} = \sum_a r_a \frac{\partial U}{\partial r_a} \quad \text{or} \quad \boxed{2\bar{T} = k\bar{U}}$$ vi :al theorem

Since $\bar{T} + \bar{U} = \bar{E} = E$ $\qquad \bar{U} = 2E/(k+2)$

$$\bar{T} = kE/(k+2)$$

This seems at first sight a great theorem, but in practice there is always a reason for it being useless.

Textbooks of mechanics normally contain chapters on special cases e.g. planetary theory, and on scattering theory, and on small oscillations. Analytical dynamics contributes very little to these which can all be solved by elementary methods.

We just note that for small oscillations, L can be expanded as a quadratic about a point where $\frac{\partial L}{\partial q} = 0$ and leaves $\frac{1}{2}\Sigma(a_{nm}\dot{q}_n\dot{q}_m + b_{nm}q_nq_m)$

so that $\underline{\underline{a}}\,\ddot{\underline{q}} = \underline{\underline{b}}\,\underline{q}$

if one writes the eigenvalues of the matrix $-\underline{\underline{a}}^{-1}\underline{\underline{b}}$ as ω_α^2, and uses the eigenvectors as coordinate system $\ddot{q}_\alpha = -\omega_\alpha^2 q_\alpha$ i.e. frequencies are $\omega^2 = \omega_\alpha^2$.

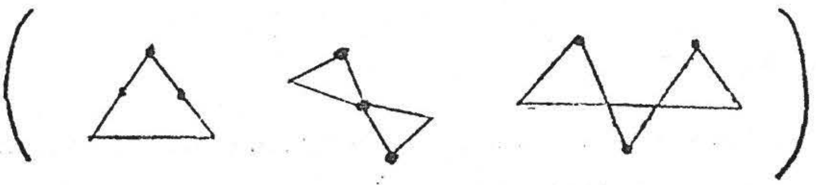
Interest centres on the degeneracies of the ω_α, and this is best resolved in highly symmetric systems (with high degeneracy) by group theory. A few simple examples are set:

<u>Examples</u> Small oscillations

1) Study the small oscillations of the hinged pendulum and the double pendulum (Exs 3,4) (L and L)

2) A light string of length 4 a is stretched under tension between fixed end points. Particles of masses m, $\frac{4m}{3}$, m are attached to points distant a, 2a, 3a, 4a from one end.

Find the normal modes



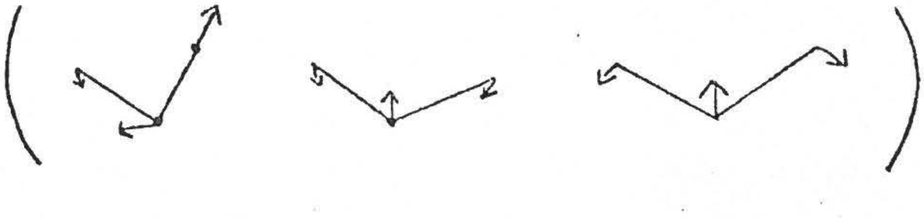
(Pars ChlX)

and solve the motion.

3) A heavy rod AB of mass M hangs in a horizontal position from two supports to which it is attached by vertical light strings each of length a attached to A and B. A particle C, mass m, hangs from A by a light string of length a and a similar particle D from B. Equilibrium is disturbed in the vertical plane.

Solve the motion and use it to illustrate the fact that two pendulum clocks hanging in the same way transfer their amplitudes so that one amplitude is large when the other is small, which situation reverses and is periodic. (Pars ChlX).

4) Solve the small oscillations of a triangular molecule H_2O where the potential is a function of the HO distances and $H\hat{O}H$ alone.



(Land L)

Friction produces irreversible terms in the equations of motion e.g. a damped oscillator

$$\nu\dot{x} + \ddot{x} + \omega_0^2 x = 0$$

has the term $\nu\dot{x}$ which $\to -\nu\dot{x}$ under the operation $t \to -t$, as is physically to be expected. To incorporate friction into the Lagrangian formulism one can generalise to Rayleigh's dissipation function, or the Rayleighan. In the example above one writes $\dot{x}$ in $\nu\dot{x}$ as v and notes that

$$\frac{d}{dv}\, \tfrac{1}{2}\nu v^2 = \nu v$$

Thus formally, if one writes

$$R = L + F$$

$$L = L(x,\dot{x})$$

$$F = F(v)$$

the equation is recovered from $R = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2}m\omega^2 x^2 + \tfrac{1}{2}m\nu v^2$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial F}{\partial v} = 0$$

At this point one puts $v = \dot{x}$, but not before.

This procedure seems quite arbitrary at this stage, but Rayleigh showed it to be quite systematic, allowing ν to be a function of x (but not of $\dot{x}$).

For if we look at the rate of loss of energy

$$\frac{dE}{dt} = \frac{d}{dt}\left(\Sigma\, \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L\right)$$

$$= \Sigma\, \dot{x}_i \left(\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_i}\right] - \frac{\partial L}{\partial x_i}\right)$$

$$= -\Sigma\, \dot{x}_i \frac{\partial F}{\partial v_i} = -\Sigma\, \dot{x}_i\, \nu_{ij}(x)\dot{x}_j$$

$$= -2F$$

where we have generalised our example to several degrees of freedom and to a general frictional force on x_i of $\sum_j \nu_{ij}(x)\dot{x}_j$.

Brownian dynamics

An important case at a molecular level arises when frictional losses are balanced by a random force. The classic case is of a small sphere radius a in a viscous liquid buffeted by molecular collisions:

$$m\ddot{r} + 6\pi a\eta\dot{r} = f(t)$$

Let us be a bit more general and add a harmonic force, and look at it in one dimension to ease the algebra

$$\ddot{x} + \nu\dot{x} + \omega_0^2 x = f(t)$$

The force f(t) fluctuates in such a way that $\langle f(t)\rangle = 0$ where $\langle\ \rangle$ means average i.e.

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau f(t)\,dt = 0$$

but $\langle f(t)f(0)\rangle \; (=\langle f(t+\tau)f(\tau)\rangle)$

$$= \tfrac{1}{2}\, h(t)$$

One expects h(t) to be a decreasing function of t, and in the limit of a very fast decrease

$$h(t) = h\delta(t)$$

the force f(t) is called white noise.

If one fourier transforms

$$\langle f(t+\tau)f(\tau)\rangle = \tfrac{1}{2}h(t)$$

one finds $\langle f_\omega f_{\omega'}\rangle = \tfrac{1}{2}\, h_\omega\, \delta(\omega + \omega')$

$$\text{where } h(t) = \frac{1}{2\pi}\int h_\omega d\omega e^{-i\omega t}$$

and for white noise $h_\omega = h$ a constant, all frequencies equally present.

Then fourier transforming the whole equation

$$(-\omega^2 + \omega_0^2 + i\gamma\omega)x_\omega = f_\omega$$

$$\langle x_\omega x_{\omega'}\rangle = \tfrac{1}{2}\,\frac{h_w\,\delta(\omega+\omega')}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}$$

This enables us to work out the average behaviour of the buffeted oscillator

$$\langle (x(t) - x(0))^2 \rangle$$

$$= \frac{1}{(2\pi)^2} \left\langle \int (e^{i\omega t} - e^{i\omega o})\, x_\omega (e^{i\omega' t} - e^{i\omega' o})\, x_{\omega'}\, d\omega \right.$$

$$= \frac{1}{(2\pi)^2} \int \frac{\tfrac{1}{2} h_\omega \delta(\omega+\omega')}{(\omega^2-\omega_0^2)^2+\nu^2\omega^2} (e^{i\omega t}-1)(e^{i\omega' t)}-1)\, d\omega d\omega'$$

$$= \frac{1}{(2\pi)^2} \int \frac{h_\omega (1 - \cos\omega t)\, d\omega}{(\omega^2 - \omega_0^2)^2 + \nu^2 \omega^2}$$

Special cases: (a) $\omega_0^2 = 0$ $\qquad h_\omega = h$

$$(x(t) - x(0))^2 = \frac{h}{(2\pi)^2} \; 2 \int \frac{\sin^2(wt/2)\, d\omega}{\omega^2(\omega^2 + \omega^2)}$$

$$= \frac{2h/\nu^2}{(2\pi)^2} \left[\int \frac{\sin^2(\omega t/2)}{\omega^2} - \int \frac{\sin^2(\omega t/2)}{\omega^2 + \nu^2} \right]$$

$$= \frac{\pi h/\nu^2}{(2\pi)^2} \left(t - \frac{1}{\nu} (1 - e^{-\nu t}) \right)$$

$t \to \infty$
$$= \left(\frac{h}{4\pi\nu^2} \right) t$$

(b) if inertia is small then ω^2 in $\omega^2 - \omega_0^2$ can be ignored and

$$(x(t) - x(0))^2 = \frac{h}{(2\pi)^2} \int \frac{(1 - \cos wt)\, dw}{\omega_0^4 + \nu^2 \omega^2}$$

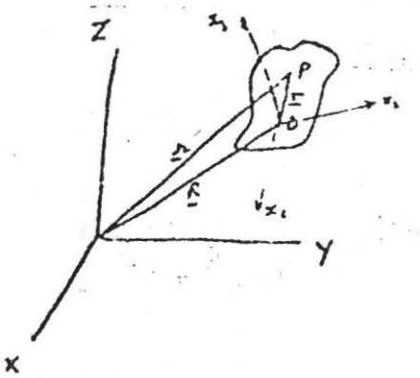
$$= \frac{h}{\omega_0^2\, \nu (2\pi)^2} \; \pi \left(1 - e^{-\frac{\omega_0^2}{\gamma} t} \right)$$

$t \to \infty$
$$= \frac{h}{4\pi\omega_0^2 \nu} .$$

In case (a) one has the Brownian random walk, $(x(t) - x(0))^2 \sim t$ in case (b) a buffeted oscillator where there is a constant average displacement of the oscillator.

Although the foundation of physics is the Lagrangian and Hamiltonian formalism of analytic dynamics, the random dynamics briefly alluded to above has far greater application in classical physics.

<u>Rotations</u> Analytical dynamics can offer something new in the study of spinning and rolling, but first we give a revision of that subject using the Lagrangian formalism. Consider the rotation of a rigid body, c.m. is O



$$d\underline{r} = d\underline{R} + d\underline{\phi} \times \underline{r}$$

$$\frac{d\underline{r}}{dt} = \underline{v}, \frac{d\underline{R}}{dt} = \underline{V}$$

$$\frac{d\phi}{dt} = \Omega$$

$$\underline{v} = \underline{V} + \underline{\Omega} \times \underline{r}$$

If we change origin by $\underline{r} = \underline{r}' + \underline{a}$

$$\underline{V}' = \underline{V} + \underline{\Omega} \times \underline{a}$$

$\Omega' = \Omega$, so there is an 'angular velocity' independent of the coordinate system.

$$T = \Sigma \tfrac{1}{2} m (V + \Omega \times v)^2$$

$$= \tfrac{1}{2}\mu V^2 + \tfrac{1}{2} \Sigma m (\Omega^2 r^2 - (\underline{\Omega} . \underline{r})^2)$$

$\mu = \Sigma m$, cross term vanishes. Define the inertia tensor

$$I_{ik} = \Sigma m (x_i^2 \delta_{ih} - x_i x_k)$$

$$T = \tfrac{1}{2}\mu V^2 + \tfrac{1}{2} I_{ik} \Omega_i \Omega_k - U$$

Principal axes of $I \to \tfrac{1}{2} \Sigma I_i \Omega_i^2 = T_{rot}$.

Angular momentum (L and L Ch VI)

$$M = \Sigma\, mr \times (\Omega \times r)$$

$$= \Sigma\, m\left[r^2\,\Omega - \underline{r}(\underline{r}.\Omega)\right]$$

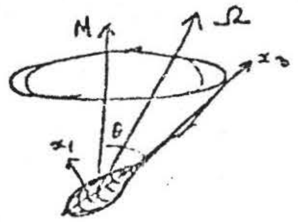
$\underline{M} = \underline{\underline{I}}\,\underline{\Omega}$, in prim axes $M_i = I_i\Omega_i$.

Special cases: sphere M = constant, $\underline{\Omega}$ = constant Rotator ($I_1 = I_2$, $I_3 = 0$)

$$\vec{M} = I\vec{\Omega} \quad \Omega \perp \text{ to axis of rotator}$$

Hence free rotation of rotator is uniform motion in plane about an axis ⊥ to the plane.

Symmetrical body $I_1 = I_2 \neq I_3$. One can chose $x_i x_2$ axes arbitrarily ∴ take x_2 ⊥ to plane containing constant $\underline{M}$ and instantaneous x_3 axis. $M_2 = 0$ and $\Omega_2 = 0$ thus M, Ω and axis of symmetry are always in one plane i.e. $v = \Omega \times r$ velocity of every point on the axis of the body is ⊥ to that plane i.e. axis rotates uniformly about $\underline{M}$ in circular cone: regular precession.



$\Omega_3 = M_3/I_3 = \frac{M}{I_3}\cos\theta$, Ω prec $\sin\theta = \Omega$,

Ω precession $= M/I_1$.

Equations of motion of rigid body

By summing e.g. motion of parts of body, the total momentum $P = \mu V$

μ total mass, V vel. of c. mass

$$\frac{dP}{dt} = F = \text{total force } \Sigma f$$

$$= -\frac{\partial U}{\partial \underline{R}} \quad U \text{ potential energy, } \underline{R} \text{ c.m.}$$

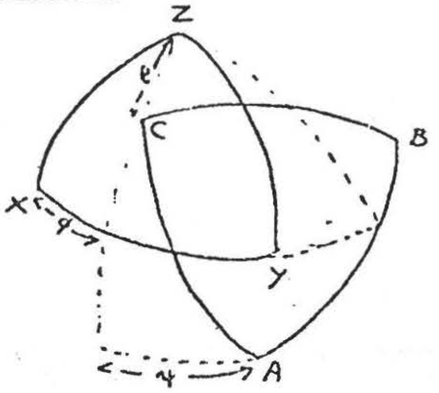
This result comes also directly from Lagranges eqs.

Similarly $\frac{dM}{dt} = K$

where $M = \Sigma\underline{r} \times \underline{p}$ and $K = \Sigma r \times \underline{f}$

$\underline{r} \times \underline{f}$ is moment of force and K the torque.

Eulerian angles Look at a rotation taking spherical triangle XYZ into ABC. The



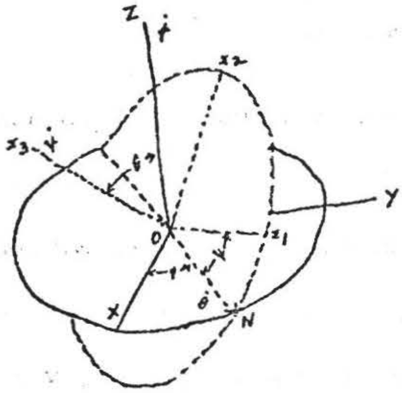
rotation is specified by three angles

θ, ϕ, ψ.

θ, ϕ usual polar coordinate angles,

and ψ the rotation about the polar axis.

Another diagram (one used by L and L) is



The moving plane $x_1 x_2$ intersects the fixed plane XY in ON the line of nodes.

Collecting components of angular velocity Ω along the moving axes

$$\Omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$
$$\Omega_2 = \dot{\phi} \sin\theta \cos\psi + \dot{\theta} \sin\psi$$
$$\Omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$$

If $\dot{\theta}$, $\dot{\phi}$ are zeros $\dot{\psi}$ is spin of body.

Kinetic energy $T_{rot} = \Sigma\, \tfrac{1}{2} I_i \Omega_i^2$

and for symmetrical body $I_1 = I_2$

$$T_{rot} = \tfrac{1}{2} I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

Since x_1, x_2 axes are arbitrary for symmetric body take x_1 to be ON line of nodes i.e. $\psi = 0$, then

$$\Omega_1 = \dot{\theta} \quad \Omega_2 = \dot{\phi} \sin\theta \quad \Omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$$

Euler's Equations

Simplest form of equations comes when one uses a moving coordinate system whose axes are principal axes of inertia. If $\underline{A}$ is a vector which does not change in the moving system, only the rotation changes it

$$\frac{dA}{dt} = \underline{\Omega} \times \underline{A}$$

in general one will have to add the change due to the moving system

$$\frac{dA}{dt} = \frac{d'A}{dt} + \underline{\Omega} \times \underline{A}$$

$$\frac{dP}{dt} + \Omega \times P = F, \quad \frac{d'M}{dt} + \Omega \times M = K$$

or $$\mu\left(\frac{dV_i}{dt} + (\Omega \times V)_i\right) = F_i$$

Using $x_1 x_2 x_3$ along principal axes $M_1 = I_1 \Omega_1$ etc.

$$I_1 \frac{d\Omega_1}{dt} + (I_3 - I_2)\Omega_2\Omega_3 = K_1 \text{ etc.}$$

and in free rotation

Not net torque $\underline{K}=0$

$$\frac{d\Omega_1}{dt} + (I_3 - I_2)\Omega_2\Omega_3/I_1 = 0 \text{ etc.}$$

Examples: $I_1 = I_2 \quad \dot{\Omega}_3 = 0 \quad \Omega_3 = \text{constant}$

$$\dot{\Omega}_1 = -\omega\Omega_2 \quad \dot{\Omega}_2 = \omega\Omega_1 \qquad \omega = \Omega_3(I_3 - I_1)/I_1$$

$$\Omega_1 = A\cos\omega t \qquad \Omega_2 = A\sin\omega t; \text{ leading to } \dot{\psi} = \Omega_3(1 - I_3/I_1).$$

Assymmetrical top

Suppose $I_3 > I_2 > I_1$

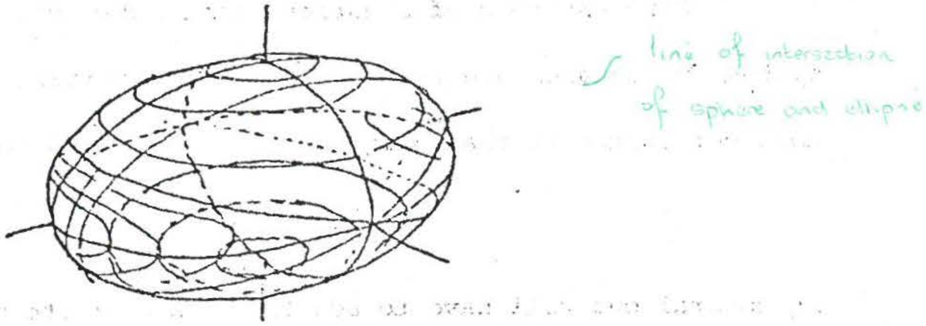
Then $\Sigma I_i \Omega_i^2 = 2E$ — Conservation of Energy

$\Sigma I_i^2 \Omega_i^2 = M^2$ — Conservation of Momentum

or $\Sigma \dfrac{M_i^2}{I_i} = 2E$ Ellipsoid

$\Sigma M_i^2 = M^2$ Sphere

Then vector $\underline{M}$ lies on line of intersection of these two surfaces



When $M \simeq M_1$ or $M \simeq M_3$ the intersection is a small closed curve giving the precessing locus and is stable. When near M_2 not stable and the axis wanders around.

One can eliminate say Ω_1 and Ω_3 from Euler's equation to give

$$\dot{\Omega}_2 = \sqrt{\left\{\left[(2EI_3 - M^2) - I_2(I_3 - I_2)\Omega_2^2\right]\left[(M^2 - 2EI_1) - (I_2(I_3 - I_1)\Omega_2^2\right]\right\}}$$
$$\times / I_2\sqrt{(I_1 I_3)}$$

i.e. $\dot{\Omega}_2 = \sqrt{(\alpha^2 - \beta^2\Omega_2^2)(\partial^2 - \partial^2\Omega_2^2)}$

or if $\tau = t\sqrt{(I_3 - I_2)(M^2 - 2EI_1)/(I_1 I_2 I_3)}$

and $s = \Omega_2\sqrt{(I_2(I_3 - I_2)/(2ET_3 - M^2)}$

and $k^2 = (I_2 - I_1)(2EI_3 - M^2))/(I_3 - I_2)(M^2 - 2EI_1)$ (<1)

$$\tau = \int^s \frac{ds}{\sqrt{(1 - s^2)(1 - k^2 s^2)}}$$

$s = \mathrm{sn}\tau$, Jacobian Elliptic functions.

Examples Rotation: Tops

1) A heavy symmetric top spins about a fixed point. Solve the motion in terms of the integral

$$t = \int \frac{d\theta}{\sqrt{2(E' - U_e(\theta))/I_1'}}$$

where $I' = I_1 + \mu l^2$

$$E' = E - \frac{M_3^2}{2I_3} - \mu g l \qquad U_e = \frac{(M_2 - M_3\cos\theta)^2}{2I_1' \sin^2\theta} - \mu g l(1 - \cos\theta)$$

2) Find the kinetic energy of a cylinder rolling on a plane.

3) A cylinder rolling inside another cylinder

4) A cone rolling on a plane

5) A rod moves on a smooth plane which rotates about a horizontal line with constant ang. vel. ω.

Show that the problem is separable when expressed in terms of (ξ,η) the c.g. G of the rod and θ the angle between the rod and $O\xi$. $O\eta$ is inclined at ωt below the horizontal.

$$(L = \tfrac{1}{2}(\dot\xi^2 + \dot\eta^2 + \omega^2\eta^2) + \tfrac{1}{2}k^2(\dot\theta^2 + \omega^2\sin^2\theta)$$
$$+ g\eta \sin\omega t$$

where Mk^2 is the moment of inertia about an axis through G $\perp$ to the rod.

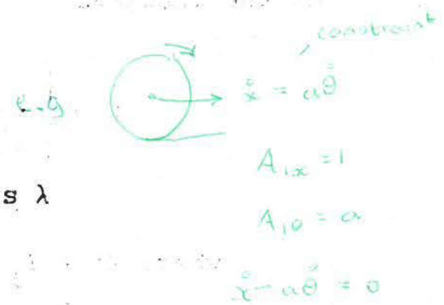
Solve the motion.

6) A penny rolls on a table making α with plane with its centre travelling in a circle radius b with speed $b\omega$. Show that $\{(2k + 1)\, b + ka\cos\alpha\}\omega^2 = g \cot\alpha$ where kMa^2 is value of 2 prin. mon. inertia, $2kMa^2_5$ an.

Dynamics with Constraints

Suppose there is a restriction e.g. rolling condition. It does no work and acts on q, $\dot q$ only (not explicitly on $\ddot q$). Say it is

$$\sum_s A_{rs}\,\dot q_s = 0 \qquad A = A(q)$$



Then Lagrange's/Euler's method is to introduce multipliers λ

$$S = \int L\,dt + \Sigma\,\lambda_r\,A_{rs}(q)\,\dot q_s$$

$$\delta S = 0 \rightarrow$$

$$0 = -\frac{d}{dt}\frac{\partial L}{\partial \dot q_r} + \frac{\partial L}{\partial q_r} + \Sigma\,\lambda_\alpha \frac{\partial A_{\alpha s}}{\partial q_r}\dot q_s - \frac{d}{dt}\Sigma\,\lambda_\alpha A_{\alpha r}(q_r)$$

But $\dfrac{d}{dt}\Sigma\lambda A = \Sigma\dot\lambda A + \Sigma\lambda \dfrac{\partial A}{\partial q}\dot q$ — cancel

So if we write $\dot\lambda = \xi$ one has

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_r} - \frac{\partial L}{\partial q_r} + \Sigma\xi_\alpha A_{\alpha r} = 0}$$

The ξ are now determined as in the calculus of variations by the constraints $\Sigma A\dot q = 0$. You will find many examples of the use of these equations in the text books. <u>But</u> you will see that ξ comes in and then goes out again and is like the reactions of Newtonian mechanics (indeed ξ's <u>are</u> reactions, keeping the system following the constraints.) It is natural to ask if there is a method of going directly to the equations of motion, doing to the Lagrange - Euler equation what Lagrange did to Newton. This can be done in the Gauss-Hertz principle of Least Curvature, and (grandest of all analytic dynamical equations) the Gibbs-Appell equations. This is not done by Landau and Lips. but is in Whittaker, and Pars. I follow Pars.

Consider a simple (indeed trivial since it is soluble i.e. integrable i.e. "holonomic") constraint $a\dot x + b\dot y = 0$ for a particle on a line in a plane.

The simple kinematics of a particle in a plane allows any values for $\dot{x}_i$, $\dot{y}_i$ and for $\ddot{x}_i$, $\ddot{y}_i$ in unconstrained motion. We can consider a displacement δx_i called a <u>virtual</u> displacement which satisfies the constraint (and therefore does no work) but is a deviation from the true motion.

Now look at the equations of motion derived earlier

$$m \ddot{x}_r = X_r + X'_r$$

where X_r are forces present and X'_r comes from the constraint

$$X'_r = \Sigma \xi_\alpha A_{\alpha r}$$

For a virtual displacement δx_r, by its definition it satisfies $\Sigma A_{rs} \delta_{x_s} = 0$

(x_s are the q's of the initial development)

Hence $\Sigma X'_2 \, \delta x_r = 0$

and $\sum_r \, (m_r \ddot{x}_r - X_r) \delta x_r = 0$

Another version of the equation $\Sigma A \delta x = 0$ comes when we consider the system with a $\Delta \dot{x}_r$ difference in velocity from the true velocity

($\delta x = (\Delta \dot{x})t$ (some time)

$$\Sigma \, A_{rs} \Delta \dot{x}_s = 0$$

and $\quad \Sigma (m_r \ddot{x}_r - X_r) \; \Delta \dot{x}_r = 0.$

One can extend this argument to accelerations for if $\Sigma A_{rs} \dot{x}_s = 0$, by differentiating

$$\Sigma (A_{rs} \ddot{x}_s + \frac{dA_{rs}}{dt} \dot{x}_s) = 0$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \Sigma \, \dot{x}_i \frac{\partial}{\partial x_i}$$

Now consider two motions with different accelerations but the same velocity. For this to be possible

$$\Sigma (A_{rs} (\ddot{x}_s + \Delta \ddot{x}_s) + \frac{dA_{rs}}{dt} \dot{x}_s) = 0$$

$$\therefore \quad \Sigma A_{rs} \Delta \ddot{x}_s = 0$$

and $\quad \Sigma (m_r \ddot{x}_r - X_r) \Delta \ddot{x}_r = 0.$

Now consider the 'Curvature' to use Hertz' term (though the principle was introduced by Gauss)

$$C = \tfrac{1}{2} \Sigma\, m_r (\ddot{x}_r - \frac{X_r}{m_r})^2$$

and consider a variation in the <u>acceleration</u> <u>alone</u>

$$\Delta C = \tfrac{1}{2} \Sigma\, m\, (\Delta\ddot{x})^2 + \Sigma\, (m\ddot{x} - X)\Delta\ddot{x}$$

$$0 \qquad\qquad 0$$

$$\therefore \Delta C > 0$$

and $\delta C = 0$ for the true solution of the motion.

<u>Example</u> Atwood's machine

$$C = \tfrac{1}{2} \left\{ M(f-g)^2 + m(-f-g)^2 \right\}$$

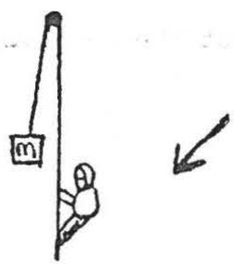
$$(M = m) \quad C = \tfrac{1}{2} \left\{ (M+m) f - (M-m) g \right\}^2 + 2Mmg$$

$$\delta C = 0$$

$$f = \frac{M - m}{M + m}\, g$$

equally from $\frac{\partial C}{\partial f} = 0$.

<u>Example</u> <u>Atwood's monkey</u>



monkey mass M

climbs up string

at rate $\dot{\phi}$ along string

$\phi(0) = \dot\phi(0) = 0$ $\quad z$ = ht of monkey

ζ = ht of M

$$C = \tfrac12\{m(\ddot z + g)^2 + M(\ddot\zeta + g)^2\}$$

$$z + \zeta = \phi$$

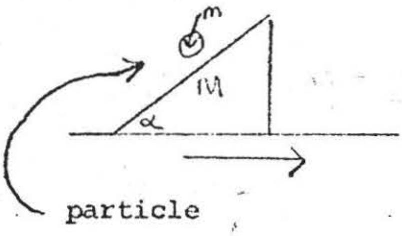
$$\therefore \quad C = \tfrac12\{M(\ddot z + g)^2 + M(\ddot\phi - \ddot z + g)^2\}$$

$$\frac{\partial C}{\partial \ddot z} = 0$$

$$(M + m)\ddot z = M\ddot\phi + (M - m)g$$

$$z = \frac{M}{M+m}\phi + \tfrac12(M - m)gt^2$$

Example: Particle on wedge



all sliding smoothly

wedge acc. with f

particle acc. with f' relative to wedge

$$C = \tfrac12 Mf^2 + \tfrac12 m\{(f'\cos\alpha - f)^2 + (f'\sin\alpha - g)^2\}$$

$$\frac{\partial C}{\partial f} = \frac{\partial C}{\partial f'} = 0$$

$$\frac{f}{m\cos\alpha} = \frac{f'}{M+m} = \frac{g\sin\alpha}{M+m\sin^2\alpha}$$

These 3 problems are tiresome otherwise.

Gibbs-Appell Equations

are a generalisation of Gauss-Hertz. Consider a system of particles masses m_k cartesian coordinates x_k. Consider the system usefully described by n coordinates q_i which are constrained by

$$\sum_s A_{rs}\dot q_s = 0$$

Define the Gibbsian $G = \tfrac12 \Sigma m_k \ddot x_k^2$ since the q are related to the x in some formula (which if explicit q = q(x) then q is called a Lagrangian coordinate, but if it involves $\dot q$ $\dot x$ and is not integrable to q = q(x) is called a quasi coordinate), one can write G interms of $q_1 \ldots q_n$ and the $\dot q$'s and $\ddot q$'s. The constraint allows us to write m of the velocities $\dot q_1 \ldots \dot q_n$ in terms of the others. Let these others be called ℓ: $\ell_1 \ldots \ell_{n-m}$, so now $\ddot q_1 \ldots \ddot q_n$ can be written in terms of

$\ddot{\ell}_1 \ldots \ddot{\ell}_{n-m}$, $\dot{\ell}_1 \ldots \dot{\ell}_{n-m}$ and $q_1 \ldots q_n$, so G can be written in terms of these variables $\ddot{p}_i$, $\dot{p}_i$, q_j.

Now consider the work done by the external forces in a virtual displacement (note the constraint reactions X' do no work, but the X's will).

In terms of the ℓ's, the work done will be

$$\Sigma \mathcal{L}_r \, \delta\ell_r \quad (= \Sigma X_r \, \delta x_r)$$

but different no. of r's

Consider

$$\Delta(G - \Sigma L_r \ddot{\ell}_r)$$

$$= \tfrac{1}{2} \Sigma m_r (\ddot{x}_r + \Delta\ddot{x}_r)^2 - \tfrac{1}{2}\Sigma m_r \ddot{x}_r^2$$

$$- \Sigma L_s \Delta\ddot{\ell}_s$$

$$= \tfrac{1}{2} \Sigma m(\Delta\ddot{x})^2 + (\Sigma m\ddot{x}\Delta\ddot{x} - \Sigma L \, \Delta\ddot{\ell})$$

$$- \tfrac{1}{2} \Sigma m(\Delta\ddot{x})^2$$

and $\Sigma m\ddot{x}\Delta\ddot{x} - \Sigma L \, \Delta\ddot{\ell} = 0$ as we now prove.

The result follows from the fact that if the ℓ are functions of the x in virtual displacement the equation $\Sigma(m\ddot{x} - X)\Delta\ddot{x} = 0$

implies $\quad \Sigma m\ddot{x}\Delta\ddot{x} = \Sigma X\Delta\ddot{x} = \Sigma L \, \Delta\ddot{\ell}$

for the two terms on right are both rates of doing work by the external forces

[Differentiate $\Sigma m\ddot{x}\Delta\dot{x} = \Sigma x\Delta\dot{x} = \Sigma \;\; \Delta\dot{\ell}$]

Hence $\Delta G = \Delta\Sigma L \ddot{\ell}$

$$\boxed{\frac{\partial G}{\partial \ddot{\ell}_r} = L_r} \quad \longleftarrow \quad \text{Gibbs Appell eqs.}$$

<u>Examples</u> motion in polar coordinates in plane. Convenient coordinates are

$r^2 = x^2 + y^2$ and $d\ell = x dy - y dx$

$$= r^2 d\theta$$

After straightforward algebra

$$G = \tfrac{1}{2} m \left\{ \left(\ddot{r} - \frac{\dot{\ell}^2}{r^3} \right)^2 + \frac{\ddot{\ell}^2}{r^2} \right\}.$$

If radial and transverse forces are R , S the work done in virtual displacement is

$$R\,\delta r + \frac{S}{r}\,\delta\ell$$

Gibbs-Appell eq. are

$$\frac{\partial G}{\partial \ddot{r}} = R \qquad \frac{\partial G}{\partial \ddot{\ell}} = \frac{S}{r}$$

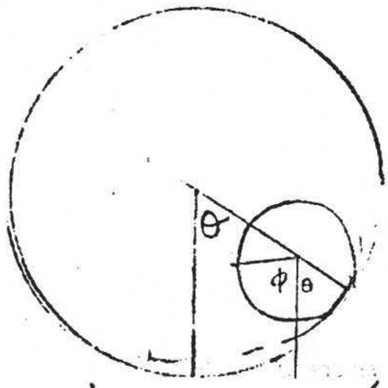
$$m\left(\ddot{r} - \frac{\dot{\ell}^2}{r^3} \right) = R \qquad \ddot{r} - r\dot{\theta}^2 = R/m$$

$$m\ddot{\ell} = rS \qquad m\frac{d}{dt}(r^2\dot{\theta}) = rS$$

the well known result.

Example (set earlier by L. equation)

Cylinder rolls inside another cylinder



Acceleration of c. of g. is f

thro' cross section of small cylinder

$$G = \tfrac{1}{2} M f^2 + \tfrac{1}{2} \int dm (r^2 \ddot{\theta}^2 + r^2 \dot{\theta}^4)$$

only the ¨ terms matter in G and these are

$$\mathfrak{S} \subset \tfrac{1}{2} M f^2 + \tfrac{1}{2} I\ddot{\theta}^2$$

$a(\theta + \phi) = b\theta \qquad a\phi = C\theta \qquad c = b - a$

$$\therefore \mathfrak{S} = \tfrac{1}{2} M(c^2\ddot{\partial}^2 + c^2\dot{\theta}^4) + \tfrac{1}{2}(\tfrac{1}{2}Ma^2)\ddot{\phi}^2$$

↓

drop as no ..

$$\mathfrak{S} = \frac{3}{4} Mc^2\ddot{\theta}^2$$

Work done by virtual displacement is

$$Mg\,\delta(c\cos\theta) = -Mg\,c\sin\theta\,\delta\theta$$

$$\therefore \frac{\partial\mathfrak{S}}{\partial\ddot{\theta}} = -Mg\,c\sin\theta$$

$$\ddot{\theta} = -\frac{2}{3}\frac{g}{c}\sin\theta. \text{ Pendulum.}$$

<u>Example</u> Physicists' roulette

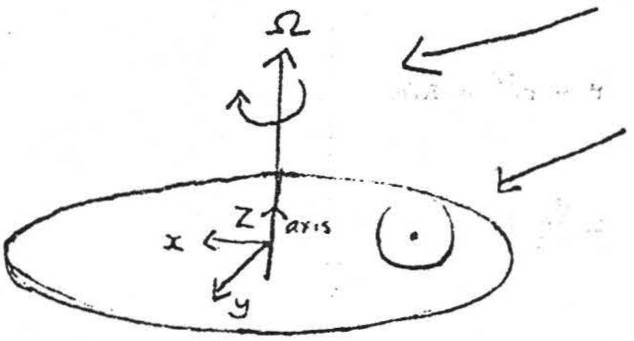


table rotates with $\Omega(t)$

sphere rolls on table

c.g. is $G = (X,Y)$

Let $\underline{\omega}$ be angular velocity of sphere

Rolling means $\qquad \dot{X} - a\omega_2 = \Omega y$

$\qquad \dot{Y} + a\omega_1 = \Omega x$

Use coordinates $X, Y, \ell_1, \ell_2, \ell_3$

$$\dot{\ell}_1 = \omega_1 \quad \dot{\ell}_2 = \omega_2 \quad \dot{\ell}_3 = \omega_3$$

(these ℓ's are quasi coordinates, one can't integrate them out)

$$2\mathfrak{S} = M(\ddot{X}^2 + \ddot{Y}^2) + A(\ddot{\ell}_1^2 + \ddot{\ell}_2^2 + \ddot{\ell}_3^2)$$

which must be written in terms of the correct number of degrees of freedom = 3;

use $\ddot{x}, \ddot{y}, \ddot{\ell}_3$.

$$a\ddot{\ell}_2 = \ddot{X} + \Omega\dot{Y} + \dot{\Omega}Y$$

$$a\ddot{\ell}_1 = -\ddot{y} + \Omega\dot{X} + \dot{\Omega}X$$

$$2\mathfrak{S} = M(\ddot{X}^2 + \ddot{Y}^2) + \frac{A}{a^2}(\ddot{X} + \Omega\dot{Y} + \dot{\Omega}Y)^2$$

$$+ \frac{A}{a^2}(\ddot{Y} - \Omega\dot{X} - \dot{\Omega}X)^2 + A\ddot{\ell}_3^2$$

work done in a virtual displacement

$$\delta\ell_1 = -\delta Y/a \qquad \delta\ell_2 = \delta X/a$$

Suppose force on centre of sphere is X, Y, Z and couple (P, Q, R) then

$$X\delta x + Y\delta y + P\delta\ell_1 + Q\delta\ell_2 + R\delta\ell_3$$

$$= (X + \frac{Q}{a})\delta X + (Y - \frac{P}{a})\delta y$$

$$+ R\delta\ell_3$$

Thus G-A eq. give

$$M\ddot{x} + \frac{A}{a^2}(\ddot{x} + \Omega\dot{y} + \dot{\Omega}y) = X + \frac{Q}{a}$$

$$M\ddot{y} + \frac{A}{a^2}(\ddot{y} - \Omega\dot{x} - \dot{\Omega}x) = Y - \frac{P}{a}$$

$$A\ddot{\ell}_3 = R.$$

A special case is Ω = constant when force is $M\xi$, $M\eta$, $M\zeta$ thro centre $\omega_3 = \ell_3$ = constant

$$(A + Ma^2)\ddot{x} + A\Omega\dot{y} = Ma^2\xi$$

$$(A + Ma^2)\ddot{y} - A\Omega\dot{x} = Ma^2\eta$$

Solid sphere has $A = \frac{2}{5}Ma^2$ and

$$\ddot{x} + \frac{2}{7}\Omega\dot{y} = \frac{5}{7}\xi$$

$$\ddot{y} - \frac{2}{7}\Omega\dot{x} = \frac{5}{7}\eta .$$

If turntable is at α to horizontal, ξ is down-hill coord

$$\xi = g\sin\alpha \ : \text{ if } K = \frac{2}{7}\Omega \quad \lambda = \frac{5}{7}\xi$$

put $x + iy = z$

$$\ddot{z} - ik\dot{z} = \lambda$$

$$z = z_0 + \frac{1}{k^2}(\lambda + ikw_0)(1 - e^{ikt}) + \frac{i\lambda}{k^2}(kt).$$

A trochoid.

Roulette in a storm

Let the plane now rotate with velocity Ω about vertical but be at an angle α to vertical. A sphere rolls on the plane under gravity. Solve the motion, (Solution given on page 209 of Pars §13.6.)

For supermen only A rough ellipsoid rolls and spins on a perfectly rough table. Obtain criteria for the stability of its spinning from the Gibbs Appell equations for its general motion. (Pars §13.15)

For geniuses Obtain criteria for the stability of a bicycle(Whipple Q.J. P and A maths 30 1899 312-48)

The Hamiltonian Formalism

Starting with Lagranges equations we introduce

$$p = \frac{\partial L}{\partial \dot{q}} \text{ and write } H = \Sigma p\dot{q} - L$$

This is $L(q\dot{q})$ but $\dot{q}$ can be replaced by q,p in the equations by solving $p = \partial L/\partial \dot{q}$. Then one has

$$\frac{\partial H}{\partial p} = \dot{q} + \underbrace{p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \dot{q}}{\partial p}\frac{\partial L}{\partial \dot{q}}}_{0}$$

and

$$\frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q}$$

$$= \frac{\partial \dot{q}}{\partial q}\cdot\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}$$

$$= -\dot{p}.$$

The time derivative of H gives

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \underbrace{\frac{\partial q}{\partial t}\frac{\partial H}{\partial q} + \frac{\partial p}{\partial t}\frac{\partial H}{\partial p}}_{0}$$

So if H is indepedent of time $\partial H/\partial t = 0$ and $\frac{dH}{dt} = 0$ H = E conserv of energy

The rate of change of any function Z(p,q) is

$$\dot{Z} = \frac{\partial Z}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial z}{\partial q}\frac{\partial q}{\partial t}$$

$$= -\frac{\partial Z}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial Z}{\partial q}\frac{\partial H}{\partial p}$$

This is often written

$$= [H, Z], \text{ Poisson bracket.}$$

Liouville's equation is conveniently expressed thus: let probability of finding p, q be

$$P(p, q, t) = \delta(p - P(t))\delta(q - Q(t))$$

where P(t), Q (t) are solutions of Ham.'s eq.

$\frac{\partial P}{\partial t} + [H, P] = 0$ where H is now written in terms of phase space coordinates p, q

$$\left(\frac{\partial}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}\right) P = 0$$

which is the familiar

$$\left(\frac{\partial}{\partial t} + \frac{\underline{v}\partial}{\partial \underline{r}} - \frac{\underline{F}}{m}\frac{\partial}{\partial \underline{v}}\right) P(r,v,t) = 0$$

Relativistic Formulation

One needs this to handle the e-m field. One way of making sure a theory is in accordance with (special) relativity is to make it covariant. This formalism recognises that in a general space one has to acknowledge the existence of two kinds of vector, the contravariant and covariant. (For full details see text books of relativity.) Write the vector

$$(\underset{\sim}{r},\ ct) = x^\mu \qquad \mu = 1,\ 2,\ 3;\ 4$$

and the vector $(\underset{\sim}{r},\ -ct) = x_\mu$

A scalar quantity has no free index, the central quantity $ds^2 = dx_\mu dx^\mu$ is an example. I one writes $ds^2 = g_{\mu r} dx^\mu dx^\nu$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1_1 & \\ & & -1 \end{pmatrix}$$

ds^2 is . (arc length)^2 and $g_{\mu\nu}$ is already familiar in 3D e.g. in polar coordinates

$g_{11} = 1, g_{22} = r^2, g_{33} = v^2 \sin^2\theta$. But in 3D alone one does not have to bother with the difference between x^μ and x_μ.

The metric tensor $g_{\mu\nu}$ raises and lowers suffices $x_\mu = g_{\mu\nu} x^\nu$

and g itself can have raised suffices:

$$g_{\mu\nu}\, g^{\nu\mu'} = \delta_\mu^{\mu'} \quad \text{kronecker delta.}$$

We just quote these results and also just quote the Lagrangian of the electromagnetic interaction.

Firstly take fixed field and ask for its L. Hamilton's prin. function has to be a scalar $\int \frac{1}{2} m\dot{x}^2 dt$ is not. Make it so by considering

$$- \int imc\, ds = - \int mc^2 \sqrt{\frac{v^2}{c^2} - 1}\; cdt$$

$$= - \int mc^2 \sqrt{1 - \frac{v^2}{c^2}}\; dt$$

$$= - \int mc^2 dt + \tfrac{1}{2} \int mv^2 dt + O\left(\frac{1}{c^2}\right)$$

Next e.m. potential $e\int \phi(r)\,dt\,d^3r$ gives the force $e\underset{\sim}{E}$ and

$$\int A.\dot{x}\; dt\; d^3r \text{ gives } \frac{e}{c}\,(v \times H)$$

From this follow Maxwell's equations and the full Lagrangian is

$$\sum_{\text{particles}} \int mc^2 \sqrt{1 - \frac{v^2}{c^2}}\; dt \quad \longrightarrow \quad \Sigma e \int \phi d^3 dt + \Sigma \frac{e}{c} \int A.\dot{r}dt$$

$$+ \sum_{\text{changes}} \int e_i\, A_\mu dx^\mu$$

$$+ \frac{1}{4\pi} \int \tfrac{1}{2} F_{\mu\nu}.F^{\mu\nu}$$

$$\frac{1}{4\pi} \int \tfrac{1}{2}(E^2 - H^2)\, d^3r dt.$$

The Hamiltonian is interesting for one finds that for a particle in a field

$$p = \frac{\partial L}{\partial \dot{q}} = mv - \frac{e}{c}\,\underline{A}$$

and

$$H = \tfrac{1}{2} mv^2$$

$$= \frac{(p + \frac{e}{c} A)^2}{2m}$$

$H = \frac{1}{2}mv^2$ reflects fact that mag. field does no work, but you can't get H's equations unless you write H in term of p.

$$H \text{ for the E. M field is } \frac{1}{4\pi}\int \tfrac{1}{2}(E^2 + H^2)d^3r.$$

where $H = \text{curl } \underline{A}$

Just as (dx, cdt) is a four vector, one has $(\frac{\underline{A}}{c}, \phi)$ is a four vector.

One can compress both the $\underline{E}$ and $\underline{v}\times\underline{H}$ terms into $\frac{e}{c}\int A_\mu dx^\mu$

(Example: work out in detail that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = e(\underline{E} + \frac{v\times H}{c})$$

when L is $\int A_\mu dx^\mu$ $\left(= e\int \phi d^3r dt + e\int \frac{A}{c}\cdot \dot{r}\, d^3r dt\right)$

There remains L for the field itself. One can combine $\underline{E}$, and $\underline{H}$ into a 4 x 4 antisymmetric tensor with

H being F_{12} F_{23} F_{31}

E being F_{14} F_{24} F_{34}

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and the scalar is $F_{\mu\nu}F^{\mu\nu}$.

Tease this out and show it equals

$$\frac{1}{8\pi}\int (E^2 - H^2)d^3r dt.$$

Continuous fields

We have taken it that there is no problem in varying a field as against a particle, but it is worth spelling it out.

If one has a particle $x(t)$ in a field $\phi(x(t))$ the Lagrangian contains

$$\int \phi(x(t))dt$$

and Lagranges equations $+ \left.\frac{\partial\phi}{\partial x}\right|_{x = x(t)}$.

A way to write this is to introduce a density function $\rho(x,t) = \delta(x - x(t))$

and write $\int \phi(r)\rho(x)dx$ or in 3D, $\phi(r)\rho(r)d^3r$

ρ naturally generalizes to the 4 current vector $j_\mu = (j_x\ j_y\ j_z\ c\rho)$

where $j_x = \dot{x}(t)\delta(x - X(t))\delta(y - Y(t)\delta(z - Z(t))$

$$j_\mu = (j, c\rho). \text{ The Lagrangian term is } \int A_\mu j^\mu d^4x$$

Now consider the part of $F_{\mu\nu}F^{\mu\nu}$ containing ϕ alone. It is

$$\int [-(\nabla\phi)^2 + \frac{\dot{\phi}^2}{c^2}]\, d^3rdt + e\int\phi(r,t)\rho(r,t)$$

$\phi \to \phi + \delta\phi$ gives

$$S \to S + \int \delta\phi(r,t)\left[\frac{1}{4\pi}\left[\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi\right] - e\rho\right] d^3rdt$$

so that L's equation is

$$\nabla^2\phi = -4\pi\rho + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

and similarly for all the other Maxwell equations. Get clear in your mind the differenc between the coordinate of a charge $r(t)$, and a point in space where one measures a field $\phi(r \text{ in } t)$. Equally if we studied say sound waves, writing in terms of a density (or equally pressure) fluctuation

$$\frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2} - \nabla^2\rho = 0 \qquad \rho = \rho(r,t)$$

comes from $\int L\,dt = \text{constant} \times \int \left[\frac{\dot\rho^2}{c^2} - (\nabla\rho)^2\right] d^3r dt$

Linear wave equations can be regarded as assemblies of harmonic oscillators.

Suppose we study $\phi(r,t)$ in a box, then

$$\phi(r,t) = \phi_0 + \sum_{n,m,\ell} \cos\frac{\pi n x}{L}\cos\frac{\pi m y}{L}\cos\frac{\pi \ell z}{L}\,\phi_{n,m,\ell}$$

if we use a cosine fourier series. It is often useful to use cyclic conditions when one can employ the complex notation

$$\phi(r,t) = \phi_0 + \sum_{\underline{n}} e^{2\pi i\frac{\underline{n}.\underline{r}}{L}}\phi_{\underline{n}}$$

(The space is now based on 2π rather than π and the number of physical states will be the same.) In the limit of a large box

$$\phi(rt) \to \underset{\substack{\downarrow \\ 0 \text{ as } \frac{1}{v}\int\phi}}{\phi_0} + \int d^3n e^{2\pi i\frac{\underline{n}.\underline{r}}{L}}\phi(\underline{n})$$

$$= \frac{1}{(2\pi)^3}\int d^3k\, e^{ikr}\,\phi_k$$

where $\phi_{\underline{k}} = V\ \phi(n)$ (V: volume $= L^3$) $\qquad (2\pi)^3\frac{d^3n}{L^3} = d^3k$

$$\phi_k = \int e^{-i\underline{k}.\underline{r}} d^3r$$

The Lagrangian

$$\frac{1}{8\pi}\int\left[(\nabla\phi)^2 - \frac{\dot{\phi}^2}{c^2}\right] d^3r dt$$

becomes

$$\frac{1}{(2\pi)^3}\frac{(2\pi)^3}{(2\omega)^3}\frac{1}{8\pi}V\int\left(k^2 - \frac{w^2}{c^2}\right)|\phi_{\underline{k}\underline{\omega}}|^2 d^3k d\omega$$

or $\int(k^2\phi_k\phi_{-k} - \frac{1}{c^2}\dot{\phi}_k\dot{\phi}_{-k})d^3k\, dt.$

It is often useful to recognize that with $\phi(r,t)$ real, ϕ_{-k} is ϕ_k e.g.: for radiation with

$$H = \text{curl}\ \underline{A}$$

and

$$E = -\frac{1}{c}\frac{\partial A}{\partial t}$$

if one writes [see Landau and Lifschitz Q. theory of fields Chapter 4; but beware they simplify some things]

$$\underline{A} = \sum_k (a_k e^{ikr} + a_k^* e^{-ikr})$$

$$\underline{E} = -\frac{1}{c}\sum_{\underline{k}}(k\, a_k e^{ikr} - k\, a_k^x e^{-ikr})$$

?

$$\underline{H} = i\sum_{\underline{k}}(k \times a_k e^{ikr} - k \times a_k^* e^{-ikr})$$

If one writes $Q_k = \sqrt{\frac{V}{4\pi c^2}}(a_k + a_k^*)$

$$P_k = -i\, c^2k^2\sqrt{\frac{V}{4\pi c^2}}(a_k - a_k^*)$$

The Hamiltonian becomes

$$H = \Sigma \tfrac{1}{2}(P_k^2 + Q_k^2)$$

and $\frac{\partial H}{\partial Q} = -\dot{P}$ $\quad \frac{\partial H}{\partial p} = \dot{Q}$

is $\quad \ddot{Q}_k - c^2k^2Q_k = 0$ the wave equation.

$$\underline{E} = 2\sqrt{\frac{\pi}{V}}\,\Sigma\, ck\,(Q_{\underline{k}} \sin \underline{k}.\underline{r} + P_k \cos kr)$$

$$\underline{H} = 2\sqrt{\frac{\pi}{V}}\,\Sigma\,\frac{1}{k}\left\{ck(k{\times}Q_k) \sin k\underline{r} + (k{\times}P_k) \cos kr\right\}.$$

Thus wave motion ≡ assembly of harmonic oscillators

The quantum mechanics stemming from a wave Hamiltonian therefore has integer energy levels and corresponds to an assembly of photons, phonons, electrons, mesons etc. etc.

<u>Hamilton-Jacobi Theory</u> (Following L&L §43)

$$S = \int_0^t L\,dt \quad \text{proceed as we did in the beginning}$$

by parts, two terms

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\,\delta q\right]_0^t + \int_0^t \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) \partial q t$$

← 0 by L's eqs →

$$\delta S = \Sigma\, p\delta q$$

or $\quad \frac{\partial S}{\partial q} = p$

Directly one has $\frac{dS}{dt} = L$

so $$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \Sigma \frac{\partial S}{\partial q} \dot{q}$$

$$= \frac{\partial S}{\partial t} + \Sigma p\dot{q}$$

or $\frac{\partial S}{\partial t} = - H$

i.e. $$dS = \Sigma pdq - Hdt$$

If one now writes $S = \int(\Sigma pdq - Hdt)$, $\delta S = 0$
gives Hamilton's equations.

Since $\frac{\partial S}{\partial t} + H(p, q, t) = 0$

and $p = \frac{\partial S}{\partial q}$ one gets

$$\frac{\partial S}{\partial t} + H(q_1 \ldots q_s; \frac{\partial S}{\partial q_1} \ldots \frac{\partial S}{\partial q_s}; t) = 0$$

The whole of analytic dynamics is here expressed as a partial differential equation, and it is employed in various complicated orbit problems. Schrödinger had this equation in mind when he introduced his equation.

Put $$\psi = e^{-iS/\hbar} \qquad \frac{\partial \psi}{\partial q} = - \frac{i}{\hbar} \frac{\partial S}{\partial q} \psi$$

$$\frac{\partial \psi}{\partial t} = - \frac{i}{\hbar} \frac{\partial S}{\partial t} \psi$$

$$= + \frac{i}{\hbar} H(q; \frac{i\hbar}{\psi} \frac{\partial \psi}{\partial q}) \psi$$

This is the classical version of Schrödinger's eq., $\hbar$ is only a scale parameter, and this equation is of course nothing but a manipulation of H-J eq.

Quantum mechanics has :

$$\frac{\partial\psi}{\partial t} = i/\hbar\, H(q,\, i\hbar \frac{\partial}{\partial q})\,\psi \text{ which really is different.}$$

Schrödinger's equation

$$\frac{\partial\psi}{\partial t} - \frac{i}{\hbar}\, H(q,\, i\hbar \frac{\partial}{\partial q})\,\psi = 0$$

the Hamiltonian version of qu. mech. There is a Lagrangian version, noted by Dirac and exploited by Feynman which says that

$$\psi(q,t) = \int e^{-iS/\hbar}\,(\delta q)\,\psi(q',t')$$

where (δq) means integrate over all paths starting at q',t' ending at q,t, weighted with $e^{iS/\hbar}$, $\hbar$ now non trivial.

To prove this is equivalent to the Sch. eq. Solve the Schrödinger equation for a very short time interval. We can do this by saying over small time interval q is almost a constant, so if we fourier transform $\frac{\partial}{\partial q}$ by taking

$$\psi(q,t) = \int G(q\, q_1\, tt_1)\,\psi(q_1 t_1)\,dq_1$$

$$t \simeq t_1$$

G is equally $G(\frac{q+q_1}{2}, \frac{q-q_1}{2};\, t\, t_1)$

and one can approximate

$$G\left(\frac{q+q_1}{2}, \frac{q-q_1}{2}, t - t_1\right)$$

by putting $\frac{q+q_1}{2} \sim q$ and fourier transform on $q - q_1$

$$G\Big(\overbrace{\left\{\frac{q+q_1}{2}\right\}}^{\sim q}, \frac{q-q_1}{2}, t-t_1\Big) = \int dp\, e^{ip(q-q_1)} G\Big(\overbrace{\frac{q+q_1}{2}}^{\sim q}, p, t-t_1\Big)$$

$$\left(\frac{\partial}{\partial t} - H(q,p)\right) G(qp\; t-t_1) = \delta(t-t_1)$$

$$G = e^{\frac{i}{\hbar}(t-t_1)H(p,q)}$$

Break up tt' into a large number of little intervals $t_1 t_2$ .... at each stage have $q_1 q_2$ ... $p_1 p_2$ ... and drift this into $q(t)$ $p(t)$

$$\psi(q,t) = \int \pi\, dq_1 dq_2 \,..\, \pi\, dp_1 dp_2 \,...\, e^{\frac{i}{\hbar}\sum_i H(p_i,q_i)(t_i - t_{i+1}) + \sum p_i (q_i - q_{i+1}) i\hbar}$$

$$\to \int \begin{matrix} d \text{ path in } q \\ d \text{ path in } p \end{matrix}\; e^{\frac{i}{\hbar}\int_{t'}^{t} (p\dot{q} - H)dt}\, \psi(q't')$$

In particular if H is $\frac{p^2}{2m} + U(q)$ one can 'complete the square' to integrate out p

$$e^{-\frac{i}{\hbar}\int\left[\left(p\dot{q} - \frac{p^2}{2m}\right) + U(q)\right]dt}$$

$$e^{-\frac{i}{\hbar}\int\left(p - \frac{\dot{q}}{m}\right)^2 - \frac{i}{\hbar}\int\left(\frac{m\dot{q}^2}{2} - U(q)\right)}.$$

$p \to p + \dot{q}/m$ drops out since $\int dp$ no longer contains the motion

$$\psi(q,t) = \int e^{-\frac{i}{\hbar}\int L(q\dot{q})dt} \quad \psi(q't)\ (\delta q)$$

all paths from q't' to qt.

This discussion is far to brief to be understandable on its own; further details in modern q.m books or Feyman and Hibbs. The important point is that Q. Mech. also has both Hamiltonian $\frac{\partial\psi}{\partial t} = [H,\psi]$

and Lagrangian $\frac{\partial\psi}{\partial\psi'} = e^{iS/\hbar}$

formulations.

<u>Manpertius principle</u> is known as principle of least time. I have never found this useful and so put it in only for completeness

$$\delta S = - H\ \delta\ t$$

but H = E for conservative system

$$\therefore \quad \delta S = - E\delta t$$

$$S = \int_0^t \Sigma\ p\ \dot{q}\ dt - Et$$

← sometimes called action, sometimes abbreviated action →

Since $\delta S = -E\delta t$

$$\delta\int \Sigma p\dot{q}\ dt = 0$$

Now $p = \frac{\partial}{\partial\dot{q}} L(q\dot{q})$ and $E(q,\dot{q}) = E$ constant. Hence if we write dt in terms of q and $\dot{q}$, one has p in terms of q and dq, with E as parameter and a new variational principle. For example if

$$L = \tfrac{1}{2} m\dot{q}^2 - U(q) \qquad p = m\dot{q}$$

$$E = \tfrac{1}{2} m\dot{q}^2 + U \qquad dt = \sqrt{\frac{1}{2(E-U}}\ dq$$

$$\Sigma p\dot{q} = \frac{\dot{q}^2}{m}$$

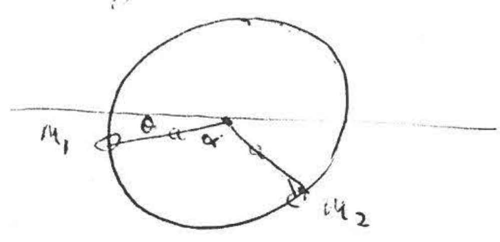
$$\int \Sigma p\dot{q}\ dt = \int \sqrt{2(E-U)}\ dq$$

$$\text{i.e. } \delta \int \sqrt{2m(E-U)}\ dq = 0.$$

Analytical Dynamics Solutions to Examples

Page 5

1)



$$T = \tfrac{1}{2} m_1 a^2 \dot{\theta}^2 + \tfrac{1}{2} m_2 a^2 (\dot{\theta} + \dot{\alpha})^2 \qquad (\dot{\alpha} \to 0)$$

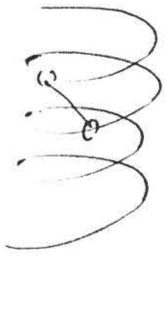
$$V = -\left(a m_1 g \sin\theta + m_2 a g \sin(\theta + \alpha) \right)$$

$$L = T - V$$

$$-(m_1 + m_2) a^2 \ddot{\theta} + a m_1 g \cos\theta + m_2 a g \cos(\theta + \alpha) = 0$$

$$\ddot{\theta} = \frac{g}{a} \left(\frac{m_1 \cos\theta + m_2 \cos(\theta + \alpha)}{m_1 + m_2} \right)$$

2)



$$T = \tfrac{1}{2} m_1 \left(b^2 \dot{\theta}_1^2 + a^2 \dot{\theta}_1^2 \right) + \tfrac{1}{2} m_2 (a^2 + b^2) \dot{\theta}_2^2$$

$$V = -g m_1 b \theta_1 - g m_2 b \theta_2$$

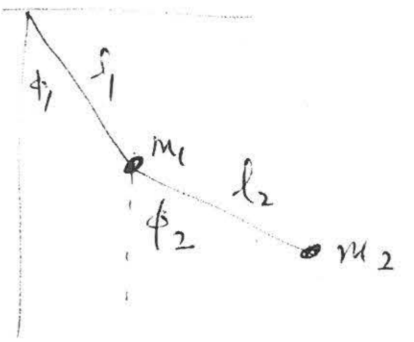
and $\theta_2 = \theta_1 + \alpha$

$$\therefore L = \tfrac{1}{2} (m_1 + m_2)(a^2 + b^2) \dot{\theta}^2 + g b (m_1 + m_2) \theta + g m_2 b \alpha \quad (\text{constant})$$

$$\ddot{\theta} = \frac{b g}{a^2 + b^2}$$

$$\ddot{z} = \frac{1}{1 + \left(\frac{a}{b}\right)^2} g \qquad \text{Similarly } \ddot{x}, \ddot{y}.$$

3) Coplanar double pendulum



$$T_1 = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$$

$$T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2} m_2 \left[l_1 \dot{\phi}_1^2 + l_2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \right]$$

$$V = -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$

Hence $L = T - V$,

$$= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_1$$
$$- V$$

Eq motion

$$-(m_1 + m_2) l_1^2 \ddot{\phi}_1 - \frac{d}{dt}\left(m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_2\right)$$
$$+ \left(-m_2 l_1 l_2 \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2\right)$$
$$- m_1 g l_1 \sin\phi_1 - m_2 g l_1 \sin\phi_1 = 0$$

$$(m_1 + m_2) l_1^2 \ddot{\phi}_1 - m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \ddot{\phi}_2$$
$$+ m_2 l_1 l_2 \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$-(m_1 + m_2) g l_1 \sin\phi_1 = 0$$

Similarly for ϕ_2.

4) Follows immediately from diagram.

5) $$L = \frac{1}{2}\sum \dot{\underline{q}}\, \underline{\underline{a}}(q)\, \dot{\underline{q}} - V(\underline{q})$$

$\underline{\underline{a}}$ must be symmetric.

$$\frac{d}{dt}\left(\underline{\underline{a}}(q)\dot{q}\right) + \frac{\partial V}{\partial \underline{q}} - \frac{1}{2}\dot{\underline{q}}\frac{\partial \underline{\underline{a}}}{\partial \underline{q}}\dot{\underline{q}} = 0$$

$$\frac{\partial q}{\partial t}\frac{\partial \underline{\underline{a}}}{\partial q}\frac{\partial q}{\partial t} + \underline{\underline{a}}\frac{\partial^2 q}{\partial t^2} + \frac{\partial V}{\partial \underline{q}} - \frac{1}{2}\dot{\underline{q}}\frac{\partial \underline{\underline{a}}}{\partial \underline{q}}\dot{\underline{q}}$$

Put all indices in

$$\frac{\partial q_\alpha}{\partial t}\frac{\partial a_{ij}}{\partial q_\alpha}\frac{\partial q_j}{\partial t} + a_{ij}\frac{\partial^2 q_j}{\partial t^2} + \frac{\partial V}{\partial q_i} = 0$$
$$- \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial q_i}\frac{\partial q_\alpha}{\partial t}\frac{\partial q_\beta}{\partial t}$$

$$a_{ij}\ddot{q}_j + \begin{bmatrix} jk \\ i \end{bmatrix}\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial q_i} = 0$$

where relabelling indices and symmetrizing

$$\begin{bmatrix} jk \\ i \end{bmatrix} = \frac{1}{2}\left[\frac{\partial a_{ki}}{\partial q_j} + \frac{\partial a_{ij}}{\partial q_k} - \frac{\partial a_{jk}}{\partial q_i}\right]$$

Christoffel Symbol.

Page 8

$$L = (\ddot{q}^2)^n \phi(q) \quad \left(= \ddot{q}^{2n}\phi(q) \text{ but if } q \text{ were vector one would need the } (\ddot{\underline{q}}^2)^n \text{ form} \right)$$

$$S = \int dt\, (\ddot{q}^2)^n \phi(q)$$

$$\frac{\delta S}{\delta q(\tau)} = \int dt\, 2n\,\ddot{q}(t)\,\ddot{\delta}(t-\tau)\left(\ddot{q}^2(t)\right)^{n-1}\phi(q)$$

$$+ \int dt\, (\ddot{q}^2)^n\, \delta(t-\tau)\, \frac{\partial \phi}{\partial q}$$

$$= \frac{d^2}{dt^2}\left(2n\,\ddot{q}\,\ddot{q}^{2n}\phi \right)$$

$$+ \ddot{q}^{2n}\left(\frac{\partial\phi}{\partial q}\right)$$

$$= 0$$

Page 10

1) $\dfrac{d\underline{M}}{dt} = \sum \underline{r} \times \underline{f}$ force $\underline{f}$ acts at point $\underline{r}$ of body

Homogeneous field $\underline{f} = e\underline{E}$ or $m\underline{g}$ etc

If $\underline{E}$ is $(0,0,E_z)$

$$\frac{dM_z}{dt} = \sum \underline{r} \times (0,0,E_z) = 0$$

Other examples 2) 3) Straight forward algebra (See Landau + Lifshitz page 109)

1) Double pendulum (see notes) ... of the solutions
Small oscillations

$$L = \tfrac{1}{2}(m_1+m_2)l_1^2\dot\phi_1^2 + \tfrac{1}{2}m_2 l_2\dot\phi_2^2 + m_2 l_1 l_2\dot\phi_1\dot\phi_2$$
$$-\tfrac{1}{2}(m_1+m_2)g l_1\phi_1^2 - \tfrac{1}{2}m_2 g l_2\phi_2^2.$$

Eq. of motion

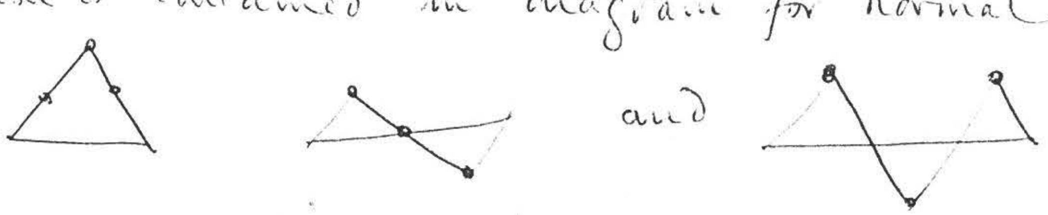
$$(m_1+m_2)l_1\ddot\phi_1 + m_2 l_2\ddot\phi_2 + (m_1+m_2)g\phi_1 = 0$$
$$l_1\ddot\phi_1 + l_2\ddot\phi_2 + g\phi_2 = 0$$

now try $\phi_1 = A_1 e^{i\omega t}$ $\quad\phi_2 = A_2 e^{i\omega t}$

Determinant gives
$$\begin{vmatrix} (m_1+m_2)(g-l_1\omega^2) & -\omega^2 m_2 l_2 \\ -l_1\omega^2 & g-l_2\omega^2 \end{vmatrix} = 0$$

$$\omega^2 = \frac{g}{2m_1 l_1 l_2}\Big\{(m_1+m_2)(l_1+l_2) \pm \sqrt{(m_1+m_2)}\sqrt{(m_1+m_2)(l_1+l_2)^2 - 4m_1 l_1 l_2}\Big\}.$$

2) Solution is contained in diagram for normal modes



3) Let supporting string make θ with vertical & pendulums make φ, ψ. Then

$$L = \tfrac{1}{2}Ma^2\dot\theta^2 + \tfrac{1}{2}ma^2(\dot\theta+\dot\varphi)^2 + \tfrac{1}{2}ma^2(\dot\theta+\dot\psi)^2$$
$$-\tfrac{1}{2}Mga\theta^2 - \tfrac{1}{2}mga(\theta^2+\varphi^2) - \tfrac{1}{2}mga(\theta^2+\psi^2)$$

$$\therefore\ 2k^2\ddot\theta + \ddot\varphi + \ddot\psi + 2k^2 n^2\theta = 0 \qquad k^2 = \frac{M}{2m}+1 \quad n^2 = \frac{g}{a}$$
$$\ddot\theta + \ddot\varphi + n^2\varphi = 0$$
$$\ddot\theta + \ddot\psi + n^2\psi = 0$$

Hence $\ddot\varphi - \ddot\psi + n^2(\varphi-\psi) = 0$ and $\varphi - \psi$ is a normal coordinate.
Take sum of 2$^{\text{nd}}$ eq and 3$^{\text{rd}}$ multiply by k^2 and add or subtract from first:

$$(k+1)(2h\ddot{\theta} + \ddot{\varphi} + \ddot{\psi}) + kn^2(2h\theta + \varphi + \psi) = 0$$

$$(k-1)(2h\ddot{\theta} - \ddot{\varphi} - \ddot{\psi}) + kn^2(2h\theta - \varphi - \psi) = 0$$

gives other two.

One can now do straightforward algebra to show that when M/m is large (k large)

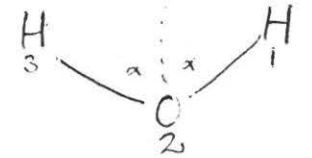
$$\varphi = \alpha \cos \nu t \cos(n-\nu)t$$

$$\psi = \alpha \sin \nu t \sin(n-\nu)t$$

The oscillations transfer from one to the other + back again etc..

when φ, ψ have close periods differing by small ν.

(Further details Pars page 132)

4) 

Displacements $\underline{u}_1\ \underline{u}_2\ \underline{u}_3$ related by

$$m_H(x_1 + x_3) + m_0 x_2 = 0$$

$$m_H(y_1 + y_3) + m_0 y_2 = 0$$

$$(y_1 - y_3)\sin\alpha - (x_1 + x_3)\cos\alpha = 0$$

Change in HO and OH are δl_1 and δl_2 say

$$\delta l_1 = (x_1 - x_2)\sin\alpha + (y_1 - y_2)\cos\alpha$$

$$\delta l_2 = -(x_3 - x_2)\sin\alpha + (y_3 - y_2)\cos\alpha$$

Angle changes by

$$\delta = \frac{1}{l}\left[(x_1 - x_2)\cos\alpha - (y_1 - y_2)\sin\alpha\right] + \frac{1}{l}\left[-(x_3 - x_2)\cos\alpha - (y_3 - y_2)\sin\alpha\right]$$

Lagrangian is $\frac{1}{2} m_H (\dot{u}_1^2 + \dot{u}_3^2) + \frac{1}{2} m_0 \dot{u}_2^2 - \frac{1}{2} k_1(\delta l_1^2 + \delta l_2^2) - \frac{1}{2} k_2 l^2 \delta^2$

Normal coords $Q = x_1 + x_3$, $q_1 = x_1 - x_3$, $q_2 = y_1 + y_3$

Put $\mu = 2m_H + m_0$ then

$$L = \frac{1}{2} m_H \left(\frac{2m_H}{m_0} + \frac{1}{\sin^2\alpha}\right)\dot{Q}^2 + \frac{1}{4} m_H \dot{q}_1^2 + \frac{\mu m_H}{4m_0}\dot{q}_2^2 - \frac{1}{4}k_1 Q^2\left(\frac{2m_H}{m_0} + \frac{1}{\sin^2\alpha}\right)\left(1 + \frac{2m_H \sin^2\alpha}{m_0}\right) - \frac{1}{4}q_1^2(k_1\sin^2\alpha + 2k_2\cos^2\alpha)$$

$$-\frac{1}{4} q_2^2 \frac{\mu^2}{m_0^2}(k_1\cos^2\alpha + 2k_2\sin^2\alpha) + q_1 q_2 \frac{\mu}{\ }(2k_2 - k_1)\sin\alpha\cos\alpha$$

$$\omega_Q = \sqrt{\left[\frac{k_1}{m_H}\left(1 + \frac{2m_H}{m_O}\sin^2\alpha\right)\right]}$$

others are

$$\omega^4 - \omega^2\left[\frac{k_1}{m_H}\left(1 + \frac{2m_H}{m_O}\cos^2\alpha\right) + \frac{2k_2}{m_H}\left(1 + \frac{2m_H}{m_O}\sin^2\alpha\right)\right] + \frac{2\mu k_1 k_2}{m_H^2 m_O} = 0.$$

Picture of modes given in notes.

<u>Page 21</u>. Top: symmetrical with fixed point mass μ, l c.m. to fixed point.

$$L = \tfrac{1}{2}(I_1 + \mu l^2)(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \tfrac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - \mu g l\cos\theta$$

ψ, ϕ are cyclic coords $\therefore$ their mom. coords conserved

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = \text{constant} = M_3$$

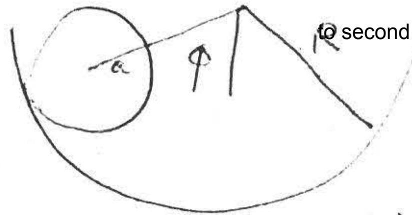
$$p_\phi = [\underbrace{(I_1 + \mu l^2)}_{I_1'}\sin^2\theta + I_3\cos^2\theta]\dot\phi + I_3\dot\psi\cos\theta = \text{constant} = M_z$$

$$\text{Energy} = \tfrac{1}{2}I_1'(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \tfrac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 + \mu g l\cos\theta = \text{constant} = E.$$

Now eliminate $\dot\phi, \dot\psi$ to give

$$E' = E - \frac{M_3^2}{2I_3} - \mu g l = \tfrac{1}{2}I_1'\dot\theta^2 + \underbrace{\frac{(M_z - M_3\cos\theta)^2}{2I_1'\sin^2\theta} - \mu g l(1-\cos\theta)}_{U_{eff}}$$

Hence $t = \displaystyle\int \frac{d\theta}{\sqrt{2(E' - U_{eff}(\theta))/I_1'}}$ (Elliptic integral)

3) 

$$T = \frac{1}{2}\mu(R-a)^2\dot\phi^2 + \frac{1}{2}I_3(R-a)^2\dot\phi^2/a^2$$
$$= \frac{3}{4}\mu(R-a)^2\dot\phi^2 \quad \left(\text{Question 2 is } R\to\infty \;\; R\dot\phi = a\dot q = v\right)$$

4) $$T = \frac{1}{2}\mu a^2\dot\theta^2\cos^2\alpha + \frac{1}{2}I_1\dot\theta^2\cos^2\alpha + \frac{1}{2}I_3\dot\theta^2\frac{\cos^4\alpha}{\sin^2\alpha}$$

$$= 3\mu h^2\dot\theta^2(1+5\cos^2\alpha)/40$$

h ht of cone, θ angle between fixed line & line of contact.

$$\left(I_1 = \frac{3}{20}\mu\left(R^2+\frac{1}{4}h^2\right) \quad I_3 = \frac{3}{10}\mu R^2 \quad \text{by integration through cone}\right)$$

5) $$L = \frac{1}{2}(\dot\xi^2+\dot\eta^2+\omega^2\eta^2) + \frac{1}{2}k^2(\dot\theta^2+\omega^2\sin^2\theta) + g\eta\sin\omega t$$

directly. This is separable

$\ddot\xi = 0 \qquad \ddot\eta - \omega^2\eta = g\sin\omega t \qquad \ddot\theta = \omega^2\cos\theta\sin\theta$

$\dot\xi$ = constant

$$\eta = \alpha\cosh\omega t + \frac{\beta}{\omega}\sinh\omega t + \frac{g}{2\omega^2}(\sinh\omega t - \sin\omega t)$$

$2\ddot\theta = 2\omega^2\sin 2\theta \quad \therefore$ simple pendulum.

6) Rolling penny. C.G is $(\xi, \eta, a\sin\theta)$ (~ Cartesian), orientation of penny (θ, φ, ψ) (Polar)

$$L = \frac{1}{2}M(\dot\xi^2+\dot\eta^2+a^2\cos^2\theta\,\dot\theta^2) + \frac{1}{2}A(\dot\theta^2+\dot\varphi^2\sin^2\theta)$$
$$+\frac{1}{2}C(\dot\psi+\dot\varphi\cos\theta)^2 - Mga\sin\theta \qquad (C=2A)$$

Rolling Constraints: $d\xi\cos\varphi + d\eta\sin\varphi - a\,d\theta\sin\theta = 0$ — ⊥ to instantaneous roll direction

$-d\xi\sin\varphi + d\eta\cos\varphi + a\,d\psi + a\,d\varphi\cos\theta = 0$ — parallel to instantaneous roll direction

Equations of motion

Force eqns on disc

$$M\ddot{\xi} = \lambda\cos\varphi - \mu\sin\varphi$$
$$M\ddot{\eta} = \lambda\sin\varphi + \mu\cos\varphi$$

λ, μ two Lagrange multipliers

$$\frac{d}{dt}\left(Ma^2\cos^2\theta\,\dot\theta + A\dot\theta\right) = \left(-Ma^2\cos\theta\sin\theta\,\dot\theta^2 + A\cos\theta\sin\theta\,\dot\varphi^2 - C\omega_3\dot\varphi\sin\theta - Mga\cos\theta - \lambda a\sin\theta\right)$$

$p_\theta = \frac{\partial L}{\partial\dot\theta}$ $\left[\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta}\right]$ $\frac{\partial L}{\partial\theta}$ + Excn force

$$\frac{d}{dt}\left(A\sin^2\theta\,\dot\varphi + C\cos\theta\,\omega_3\right) = \mu a\cos\theta$$

$[\dot\phi = \dot\varphi]$

$$\frac{d}{dt}\left(C\omega_3\right) = \mu a$$

where $\omega_3 = \dot\psi + \dot\varphi\cos\theta$.

5 eq. motion + 2 of constraint for 7 variables

Now follows a lot of algebra which gives

$$\ddot\xi\cos\varphi + \ddot\eta\sin\varphi + (-\dot\xi\sin\varphi + \dot\eta\cos\varphi)\dot\varphi = a(\sin\theta\,\ddot\theta + \cos\theta\,\dot\theta^2)$$
$$-\ddot\xi\sin\varphi + \ddot\eta\cos\varphi - (\dot\xi\cos\varphi + \dot\eta\sin\varphi)\dot\varphi = -a\dot\omega_3$$
$$\lambda = Ma(\sin\theta\,\ddot\theta + \cos\theta\,\dot\theta^2 + \omega_3\dot\varphi)$$
$$\mu = Ma(\sin\theta\,\dot\theta\dot\varphi - \dot\omega_3)$$

giving

$$(A + Ma^2)\ddot\theta = A\dot\varphi^2\cos\theta\sin\theta - (C + Ma^2)\omega_3\dot\varphi\sin\theta - Mga\cos\theta$$
$$(C + Ma^2)\dot\omega_3 = Ma^2\dot\theta\dot\varphi\sin\theta$$
$$\frac{d}{dt}\left(A\dot\varphi\sin^2\theta\right) = C\omega_3\dot\theta\sin\theta.$$

If $p = \cos\theta$ and $A = kMa^2$ $C = 2kMa^2$

$$(2k+1)\,d\omega_3/dp = -\dot\varphi \qquad \frac{d}{dp}\left((1-p^2)\dot\varphi\right) = -2\omega_3$$

$$\frac{d}{dp}\left((1-p^2)\frac{d\omega_3}{dp}\right) - \frac{2}{2k+1}\omega_3 = 0 \quad \text{and finally elim } \omega_3$$

if $\zeta = (1-p^2)\dot\varphi$ $\quad (1-p^2)\,d^2\zeta/dp^2 = \frac{2}{2k+1}\zeta.$

In steady motion, angle α centre moves with $b\omega$ where $\omega = \dot\varphi$ (from $a\dot\omega_3 = -b\dot\omega$)

$$((2k+1)b + ka\cos\alpha)\,\omega^2 = g\cot\alpha. \qquad \text{Q.E.D.}$$