31st Jan. 1994

Disorder and Quantum Chaos

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Particle in Disordered Potential

Why one particle? No interaction

Mean field (Hartree - Fock approx.)

Quasi-particles (Fermi - Liquid Theory)

What is the meaning of Disorder?

Consider electron in crystal => band structure

quasi-momentum p (because of translational invariance)

dispersion law $E_n(p) = \frac{p^2}{2m}$ if free $\hat{p} = -i\nabla$

$E(\hat{p})\psi = E(p)\psi$ Schrödinger equation

In general there can be defects, impurities, vacancies etc.

$$[E(\hat{p}) + V(r)]\psi = E\psi$$

What are the properties of this Schrödinger equation?

Metals

has itinerant electrons.

- Alloy - mix different components

  compositional disorder

- Impurities

  Vacancies

  Dislocations ...

Disorder from Statistical Mechanics point of view can be

Quenched or Annealed

Frozen

change with time e.g. H in metal

can be slow or fast but it is still assumed classical

Easily controlled is the content and concentration of impurities

— macroscopic parameters

Leads to the notion of Ensemble - whole set of systems with the same macroscopic parameters

Statistical Approach → average over ensemble (possible realisations of $V(r)$)

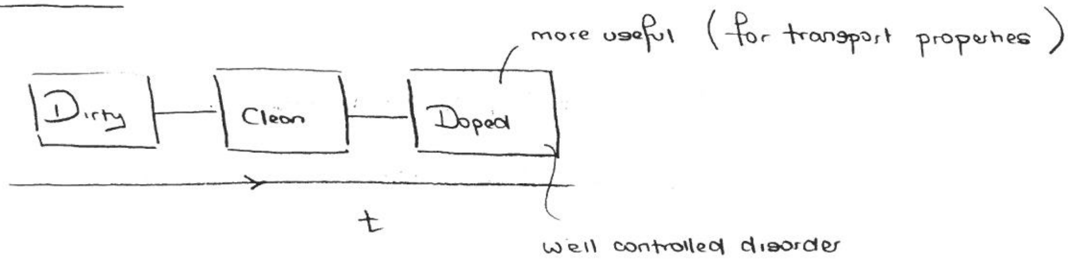
This type of averaging looks complicated but it leads to a great simplification if

we order things correctly.

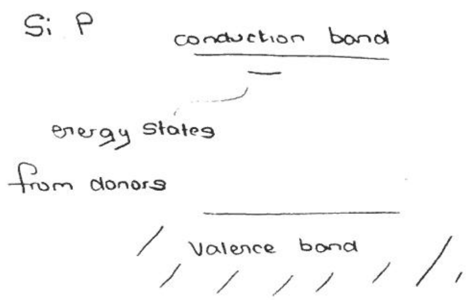
Statistical Properties of $V(r)$?

⇒ Different models of disorder

Semi-Conductors



Impurity states in semi-conductor



Donors dramatically increase conductance

(Acceptors can add close to valence band)

Single impurity — Low energy

Large radii

$$H_0 = \underbrace{-\frac{1}{2m_0}\nabla^2}_{\text{free electron}} + \underbrace{U(r)}_{\text{periodic}} + \underbrace{V(r)}_{\text{Single impurity potential}}$$

$$U(\underline{r} + \underline{N}) = U(\underline{r})$$

$$V(r) = \frac{e^2}{\kappa r}$$ κ is dielectric constant

$$V = 0 \Rightarrow \psi = \varphi_{np} = u_{np} e^{ipr} \frac{1}{\sqrt{v_0}}$$ Bloch theorem

volume of elimentary cell

$$\psi(r) = \sum \varphi_{np} B_{np}$$

$$(E_n(p) - E) B_n(p) + \sum V^{np}_{n'p'} B_{n'p'}$$

$$V^{np}_{n'p'} = \frac{1}{v_0} \int dr \, u_{np} u_{n'p'} V(r) e^{-ir(p-p')}$$

Assuming large and low energy state, p p' small gives largest contribution

$$V^{np}_{n'p'} \simeq \underbrace{\int \frac{dr}{v_0} U_{n0} U_{n'0}}_{\delta_{nn'}} \int dr \, V(r) e^{-ir(p-p')}$$

This approximation breaks down for deep levels in semi-conductors

Neglect different bands

- focus on closest

$$\left(\frac{p^2}{2m} - E \right) B_n(p) = \sum_{p'} V(p,p') B_{np'}$$

at bottom of band, with effective mass m

$$V(p-p') = \frac{4\pi e^2}{\kappa (p-p')^2}$$

After f.t.

$$\left(-\frac{\nabla^2}{2m} - \frac{e^2}{\kappa r} \right) F(r) = E F(r) \qquad F(r) = \frac{1}{\sqrt{v_0}} \sum_p B_n(p) e^{ipr}$$

Schrödinger equation for Hydrogen atom

$$E_j = -\frac{me^4}{2\kappa^2 j^2 \hbar^2} = \frac{-13.6\,\text{eV}}{j^2}\left(\frac{m}{m_0 \kappa^2}\right)$$

for GaAs, $\kappa = 12.5$

$m = 0.06\, m_0$

$$E_1 = 5.67\ \text{meV} \simeq 60\,\text{K}$$

For $T > 60\,\text{K}$ all electrons in conductance band

and $T \ll 60\,\text{K}$ all in donor impurity state

$$F(r) = \sqrt{\pi a^3}\ \frac{e^{-r/a}}{r}$$

Effective Bohr radius, $a = 0.53\,\text{Å}\left(\frac{\kappa m_0}{m}\right)$

100 Å in GaAs

much larger than lattice constant – justifies the

effective mass approximation

For many impurities, are there any collective effects at low temperatures?

Impurity Band

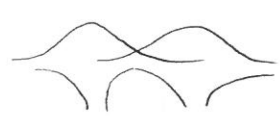
Single impurity $\rightarrow$ localised state at impurity centre

Many impurities? $\rightarrow$ create impurity band

for $N_i \rightarrow 0$ $\quad$ $N_i V$ degenerate states

concentration

Finite N_i – states overlap



Overlapping splits degeneracies

Density of States - # states per energy interval and unit volume

$$\nu(\varepsilon) = \frac{1}{\mathcal{U}} \sum_{\alpha} \delta(\varepsilon - E_{\alpha})$$

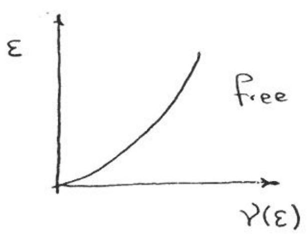
eigenergies

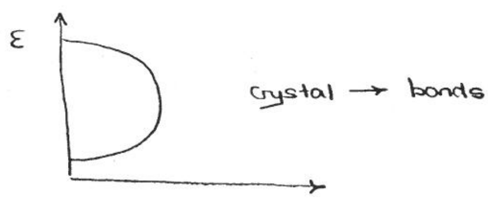
Crystal $\nu(\varepsilon) = 2 \sum_{n} \int \frac{dp}{(2\pi)^3} \, \delta(\varepsilon - p^2/2m)$

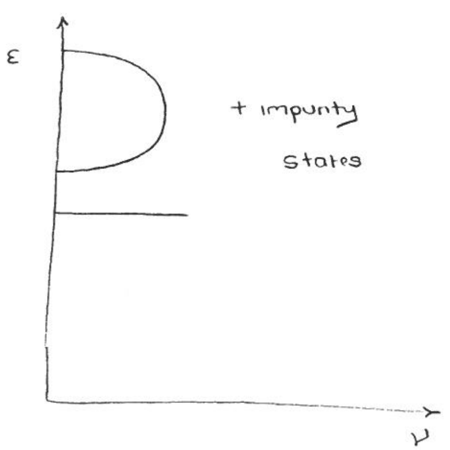
spin degeneracy

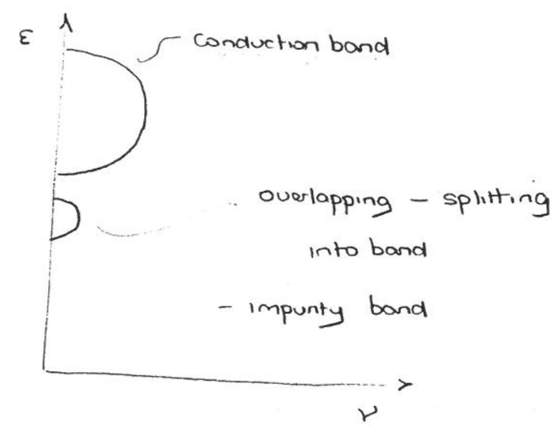
$E_n(p)$ in general

$$= 2 \, \frac{m p_{\varepsilon}}{2\pi^2} = \frac{2}{2\pi^2} \sqrt{2m^3 \varepsilon}$$









From point of view of DOS there is an impurity band but is it extended or localised

Parameter - Concentration of impurities × Bohr radius^3 $N_i a^3$

if $N_i a^3 \ll 1$, overlapping is small

$N_i a^3 \gg 1$ strong overlap

For simplicity – suppose impurities form a superlattice

$$\Psi_\alpha(r) = \sum \varphi(r - r_i)\, A_{\alpha i} \qquad \sum_i |A_{\alpha i}|^2 = 1$$

Fu

$$E_\alpha = \sum_i E^{(i)} |A_{\alpha i}|^2 + \sum_{ii'} I_{ii'} A_{\alpha i} A_{\alpha i'}$$

$$H = \sum_i E^{(i)} a_i a_i^+ + \frac{1}{2} \sum_{ii} I_{ii'} (a_i a_{i'}^+ + a_i^+ a_{i'})$$

2nd quantised , a_i^+ creates impurity state

$N_i a^3 \ll 1$, $I_{ii'} \sim e^{-r_{ii'}/a}$

$\longrightarrow$ only nearest neighbours coupled

$$E_p = -2I \left[\cos p_x b + \cos p_y b + \cos p_z b \right]$$

tight-binding model

b – superlattice constant

$$m_{eff} = \frac{1}{2b^2 I} \sim \frac{1}{2b^2} e^{b/a}$$

Bandwidth $V_b = 12 I \sim e^{-b/a}$ – very narrow impurity band

If 1 electron/impurity , band is half-filled and metallic

But this metallic state cannot survive because

1/ band is exponentially narrow and high – very long tunneling time

$\rightarrow$ dephasing

2/ Interaction between electrons – if on-site repulsion $e^2/\kappa a$

get two bands  $\updownarrow\ e^2/\kappa a$ Mott transition

3) disorder

(See Shklovskii & Efros

Electronic Properties of Doped Semi-conductors (Springer))

Models of disorder

Refs. J M Ziman Models of disorder (CUP 1979)

I M Lifshitz, S A Gredeskul, L A Pastur, Introduction to the theory of disordered systems (Wiley 1988)

NF Mott and E A Davis, Electronic processes in non-crystalline materials (OUP 1979)

Lattice Models of Disorder

1) Random on-site energies $E^{(0)} \longrightarrow E_i$ Anderson-Model

2) Random exchange integral $I_{ii'}$ Lifshitz Model

no parameter ($E^{(0)} = 0$ and then just magnitude of I)

Anderson model has well defined dimensionless parameter

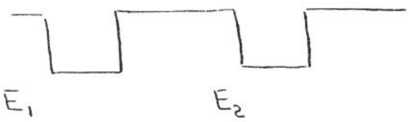
Suppose E_i independent of each other

Probability distribution is box $P(E_i) = \begin{cases} \frac{1}{W} & -\frac{W}{2} < E_i < \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$

Dimensionless parameter is I/W

Two-Well potential

a) nearly identical wells

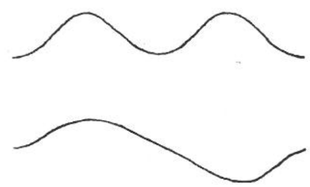


$\varepsilon = |E_1 - E_2| \ll I$

Two quantum states

$$\Psi_I = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2)$$

$$\Psi_{II} = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2)$$



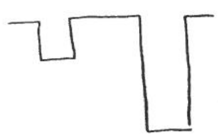
equal probability to find particle in either well

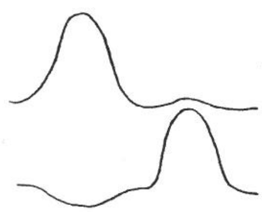
b) different wells $\quad \varepsilon \gg I$

$$\Psi_I = c_1 \varphi_1 + c_2 \varphi_2$$

$$\Psi_{II} = c_2 \varphi_1 - c_1 \varphi_2$$

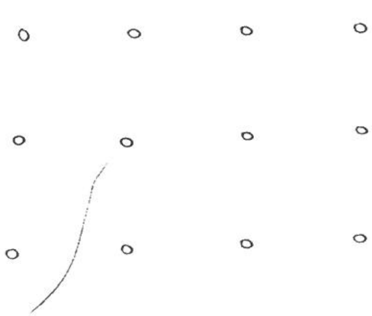
$$\frac{c_2}{c_1} \simeq \frac{I}{\varepsilon}$$





particles localized in one well or another

Lattice



Take site, look at neighbours. If $|E - E'| < \Delta$ then connect sites

If Δ small, got isolated connected sites

For larger Δ there will be clusters $\quad \Delta/W < x_c$ - finite clusters

At certain value of Δ $\quad \Delta/W > x_c$ - ∞ cluster (percolation)

for $\quad \varepsilon = |\varepsilon_1 - \varepsilon_2| \ll I \quad$ resonance

$\varepsilon \gg I \quad$ non-resonance

$\Delta \leftarrow I$ percolation on resonance sites $\quad \frac{I}{W} > x_c$

It is conjectured that this transition does occur in Anderson model although x_c should be altered from the percolation value. - the transition is thought to be sharp even in the quantum case.

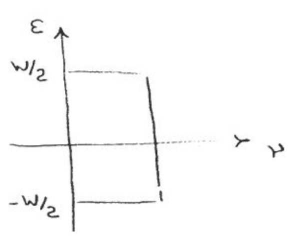
A second argument was given by Anderson.

Suppose $\rho(t=0, r) = \delta(r)$

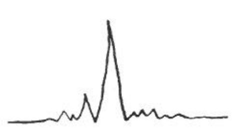
What is the probability of return? Locator expansion

$\rho(t, r=0)$? - $\rho(t=\infty, r=0) \neq 0$ Anderson criterion for localisation

$I = 0$ Only single site states

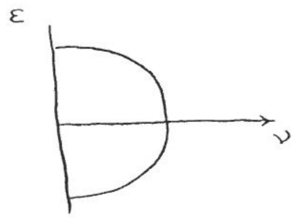


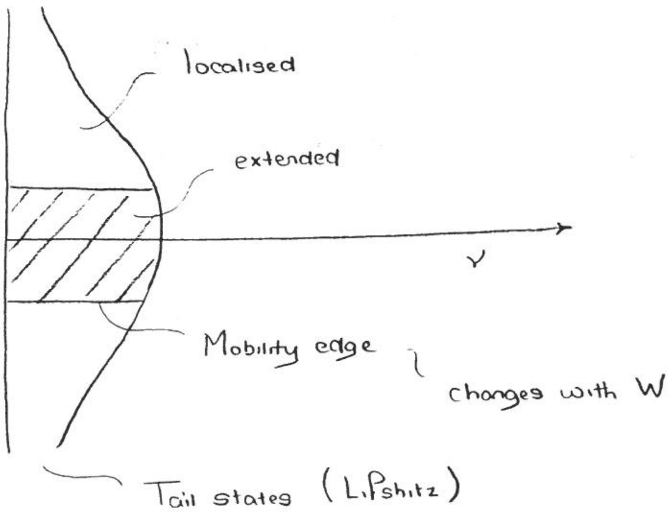
$I \ll W$ All states are still localised



$I \gg W$ there are extended states

$W = 0$





Tails at $I \ll W$

Take two neighbours

$$\varepsilon = |E_1 - E_2| = (\varepsilon_0^2 + I^2)^{1/2}$$

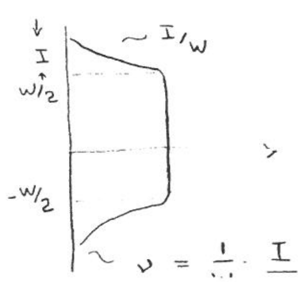
Normally $\varepsilon_0 \sim W$

$$\varepsilon = \varepsilon_0 \left(1 + O\left(I/W\right)^2\right)$$

Sometimes, with some probability, two sites will be close in energy, $\varepsilon_0 \lesssim I$

$$\varepsilon \sim \varepsilon_0 + O(I)$$

$$p(\varepsilon_0 \sim I) \sim \frac{I}{W}$$



In fact tail states can extend further but it is not universal – it depends on model

Anderson Model

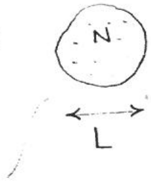
Suppose all $E_i = -W/2$

$\Rightarrow$ End of spectrum, $\varepsilon_c = -W/2 - 12I$, $E > \varepsilon_c$

What is $\nu(E \to \varepsilon_c)$? – find by method of optimal fluctuations

For low energy we require whole region with anomalously small energy

$N \gg 1$



$$N = \left(\frac{L}{b}\right)^3$$

$$E_i + \frac{W}{2} \sim I$$

like potential well,

$$E = \varepsilon_c + O\left(\frac{1}{2mL^2}\right)$$

from quantum zero-point energy

$$P(N) \sim \left(\frac{I}{W}\right)^N$$

Prob. of states to be in width I

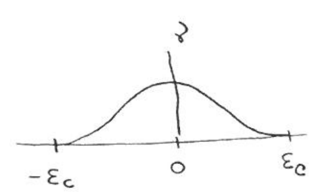
from edge of spectrum $\varepsilon = E - \varepsilon_c \sim I N^{-2/3}$

$$N(\varepsilon) = \left(\frac{\varepsilon}{I}\right)^{-3/2}$$

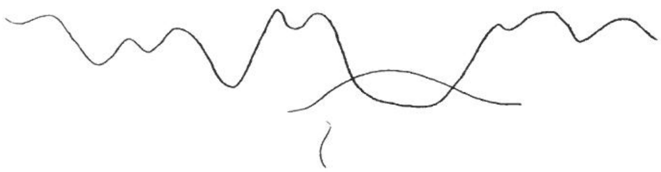
$$dN = -\frac{3}{2} \frac{\varepsilon^{-5/2}}{I^{-3/2}} \, d\varepsilon$$

$$P(\varepsilon) \sim \left(\frac{I}{W}\right)^{(I/\varepsilon)^{3/2}} \left(\frac{I}{\varepsilon}\right)^{5/2} \frac{1}{I}$$

$$= \exp\left[-\left(\frac{I}{\varepsilon}\right)^{3/2} \ln\left(\frac{W}{I}\right)\right]$$



$$\nu(\varepsilon) = N_i \, P(\varepsilon)$$



Tail states are strongly localised

In Kronig-Penny model can put nodes at δ fn potentials and get extended states but usual situation leads to localisation with distance from centre.

Extended States

$W = 0$ Perfect crystal, Bloch States

$W \ll I$ Each site can scatter.

Scattering cross-section S

$$S \simeq \left(\frac{\overbrace{\delta E}^{W}}{\underbrace{E}_{\sim I}}\, b\right)^2 \sim \left(\frac{W}{I}\, b\right)^2$$

$$b \sim N_i^{-1/3}$$

Mean Free Path

$$l = \underbrace{\frac{1}{N_i\, S}}_{\text{from theory of gases}} \sim b\left(\frac{I}{W}\right)^2 \gg b$$

For small W, $l \gg b \sim \frac{1}{k_F}$

wavelength of particle

$$(k_F l)^{-1} \ll 1 \quad \text{(Small parameter)}$$

$$k_F l \sim \left(\frac{I}{W}\right)^2$$

Parameters $\quad N_i a^3 \;,\; I/W \;,\; k_F l$

all are connected and have similar meaning

Near mobility edge $\quad k_F l \sim 1$

7th Feb

Continuous Models of Disorder

$$-\frac{1}{2}\nabla^2 \psi_\alpha(r) + V(r)\psi_\alpha(r) = E_\alpha \psi_\alpha(r) \qquad (\hbar = 1 \;, m = 1)$$

1) Identical impurities located randomly

$$V(r_i) = \sum_{r_i} u(\underbrace{r - r_i}_{\text{random}})$$

e.g. Impurity donor states

$$u_i(r) = \frac{e^2}{\kappa r}$$

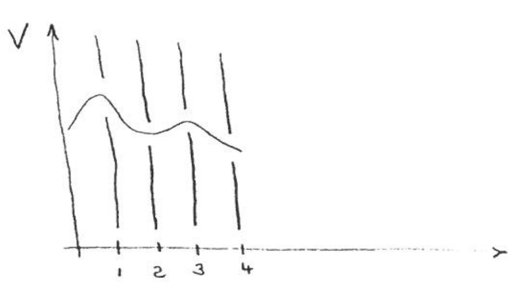
2) General Case – distribution function of $V(r)$

$$P\{V(r)\} \text{ - functional}$$

$P\{V(r)\}\mathcal{D}V(r)$ is the probability for $V(r)$ to be in the interval $\mathcal{D}V(r)$

$$\mathcal{D}V(r) = \lim_{N \to \infty} \prod_{j=1}^{N} dV(r_j)$$

functional



Suppose a property is also a functional of $V(r)$

$$A\{V(r)\}$$

Ensemble average

$$\langle A\{V(r)\}\rangle \equiv \frac{1}{Z}\int \mathcal{D}V(r)\, P\{V(r)\}\, A\{V(r)\}$$

Random potential: Assume i) $\langle V(r)\rangle = 0$ (otherwise shift total energy so that it is)

ii) $\langle V(r)V(r')\rangle = W(r,r')$

Assume a stationary random potential so $W = W(r-r')$, independent of $r+r'$

iii) $\langle V(r_1)\ldots V(r_N)\rangle$ remains the same when $r_i \to r_i + R$ (translational inv.)

Z is normalisation factor, $\langle 1\rangle = 1$

$$Z = \int \mathcal{D}V(r)\, P\{V(r)\}$$

Gaussian Potentials

$$P\{V(r)\} = \exp\left[-\frac{1}{2}\int dr\,dr'\, V(r)K(r-r')V(r')\right]$$

$$\int K(r-r_1)W(r_1-r')\,dr_1 = \delta(r-r')$$

K and W connected trivially in ~~Fourier transform~~

For Gaussian potential

$$\langle V(r_1)\ldots V(r_N)\rangle = \sum P_{r_1,r_2\ldots r_N}^{r^{(1)},r^{(2)},\ldots r^{(N)}} \prod_{n=1} W(r^{(2n-1)} - r^{(2n)})$$

Extremal Path

$X(\tau)$ $\qquad \delta S = S(X + \delta x(\tau)) - S(X) = \int_{-\tau_0/2}^{\tau_0/2} d\tau\, \delta x(\tau) \left(- \frac{d^2 X}{d\tau^2} + V'(x) \right)$

$$\frac{d^2X}{d\tau^2} = V'(x)$$ Newton's equation in the Potential $-V$

To determine X we need to use method of Steepest descents and sum over trajectories close to the extremal path

$x(\tau)$ – generic trajectory

$$x(\tau) = X(\tau) + \sum_n c_n x_n(\tau)$$

$x_n(\tau)$ orthonomal set of functions

$$\int_{-\tau_0/2}^{\tau_0/2} x_n(\tau) x_m(\tau)\, d\tau = \delta_{nm} \qquad x_n(\pm \tau_0/2) = 0$$

$$\mathcal{D}x(\tau) = \prod_n \frac{dc_n}{\sqrt{2\pi}}$$

Assume that c_n are small and expand $S(x(\tau))$ up to quadratic terms

$$S(X + \delta x) = S_0 + \int_{-\tau_0/2}^{\tau_0/2} d\tau\, \delta x \left[-\frac{1}{2} \frac{d^2}{d\tau^2} + \frac{1}{2} V'' \right] \delta x$$

X extremal $\longrightarrow$ no linear terms

$$-\frac{d^2 x_n}{d\tau^2} + V''(X) x_n = \varepsilon_n x_n$$

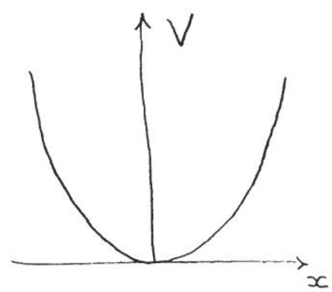
$$S = S_0 + \frac{1}{2} \sum_n c_n^2 \varepsilon_n$$

$\int dc_n$ independent set of Gaussian integrals

If X extremal all $\varepsilon_n \geq 0$ so $S \geqslant S_0$. Let us leave $\varepsilon = 0$ for a while!

$$\langle x_i | e^{-H\tau_0} | x_f \rangle = \frac{1}{Z} e^{-S_0} \prod_n \frac{1}{\sqrt{\varepsilon_n}}$$

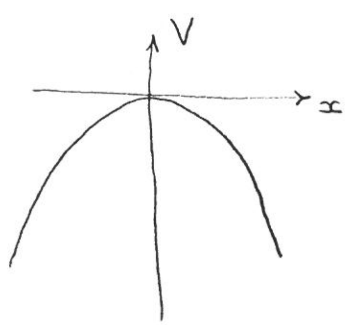
$$= \frac{1}{Z} e^{-S_0} \left[\det \left(-\frac{d^2}{d\tau^2} + V''(X) \right) \right]^{-1/2}$$



$V''(x=0) = m\omega^2$

$V \sim \dfrac{m\omega^2 x^2}{2}$

boundary conditions $x_i = x_f = 0$



$X(\tau) = 0$ is the only classical solution

$$S_0 = S(X) = 0$$

So $\langle 0 | e^{-H\tau_0} | 0 \rangle = \dfrac{1}{Z} \left\{ \det \left(-\dfrac{d^2}{d\tau^2} + \omega^2 \right) \right\}^{-1/2}$

$\sim (1 + \text{corrections})$

$$-\frac{d^2 x_n}{d\tau^2} + \omega^2 x_n = \varepsilon_n x_n$$

$x(\pm \tau_0/2) = 0 \quad \Rightarrow \quad \varepsilon_n = \omega^2 + \dfrac{\pi^2 n^2}{\tau_0^2} \qquad n = 1, 2, 3 \ldots$

$$\frac{1}{Z}\{\det\}^{-1/2} = \Big\{\underbrace{\frac{1}{Z}\Big(\prod \frac{\pi^2 n^2}{\tau_0^2}\Big)^{-1/2}}_{\text{determinant for a free particle}}\Big\} \times \Big\{\prod_n \Big(1 + \frac{\omega^2\tau_0^2}{\pi^2 n^2}\Big)\Big\}^{-1/2}$$

$$\frac{1}{Z}\prod\Big(\frac{\pi^2 n^2}{\tau_0^2}\Big)^{-1/2} = \langle 0| e^{-\frac{p^2}{2}\tau_0} |0\rangle$$

$$= \sum_n |\langle p_n | x = 0\rangle|^2 \, e^{-p_n^2 \tau_0/2}$$

$$= \int \frac{dp}{2\pi} e^{-p^2\tau_0/2} = \frac{1}{\sqrt{2\pi\tau_0}}$$

(Note that both Z and $\prod$ vanish but ratio does not)

$$\prod_n \Big(1 + \frac{\omega^2\tau_0^2}{\pi^2 n^2}\Big)^{-1/2} \quad \text{using} \quad \pi y \prod_{n=1}^{\infty}\Big(1 + \frac{y^2}{n^2}\Big) = \sinh \pi y$$

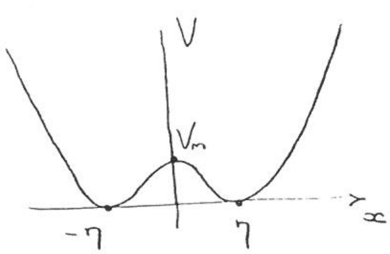
$$\langle 0| e^{-H\tau_0} |0\rangle \equiv G(0, -\tau_0/2;\, 0, \tau_0/2)$$

$$= \sqrt{\frac{\omega}{2\pi \sinh \omega\tau_0}}$$ Green function of Harmonic Oscillator

When $\tau_0 \to \infty$

$$= \sqrt{\frac{\omega}{\pi}}\, e^{-\omega\tau_0/2}\Big(\underbrace{1}_{\text{ground state}} + \underbrace{\tfrac{1}{2} e^{-2\omega\tau_0}}_{n=2} + \cdots \Big)$$

Tunneling



$$V(x) = \lambda(x^2 - \eta^2)^2$$

$$8\lambda\eta^2 = \omega^2 \quad , \quad V_m = \lambda\eta^4 = \frac{\omega^4}{64\lambda}$$

$\lambda \to 0$
$V_m \to \infty$ $\quad \Rightarrow$ two identical wells

Nawely

If potential is very high, particle will be trapped in one well or another

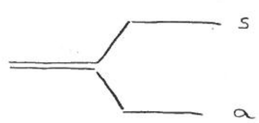
$$\langle x_0 \rangle = \begin{cases} \eta\,(1 + \text{corrections}) \\ -\eta\,(1 + \text{corrections}) \end{cases}$$

Broken Symmetry $\quad x \to -x$

$$E = \omega/2$$

but this is wrong because particles tunnel

$\rightarrow$ symmetric and asymmetric states



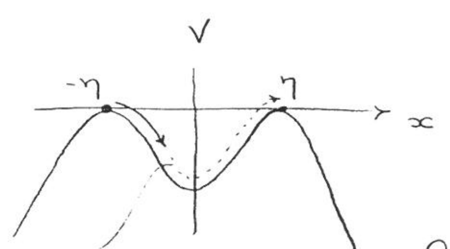
Symmetry is not Broken $\quad \langle x \rangle_0 = 0$

$$E_{0,1} = \frac{\omega}{2} \pm \frac{\omega}{2}\sqrt{\frac{2\omega^3}{\pi\lambda}}\; e^{-\omega^2/12\lambda} \qquad \text{exact solution}$$

Let us try to obtain formula by previous approach — Note that essential singularity prevents perturbation theory in λ

$$\langle -\eta|\, e^{-H\tau_0}\, |\eta\rangle = ? \qquad \text{No classical solution}$$

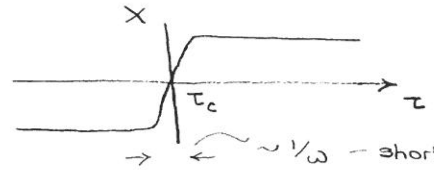
In Euclidean case



non-trivial solution with finite action

$$\frac{d^2X}{d\tau^2} = V'(X)$$

$$X(\tau) = \eta \tanh \frac{\omega(\tau-\tau_c)}{2}$$

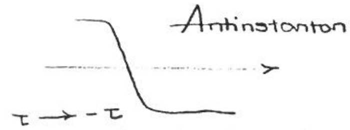


$\to \leftarrow \sim 1/\omega$ - short time under barrier

$\forall\ \tau_c$

(origin of zero mode solutions)

INSTANTON



Energy of Solution

$$E = \frac{\dot{x}^2}{2} - V(x) = 0$$

$$S_0 = S(X) = \int_{-\infty}^{\infty} d\tau\ \dot{x}^2 = \frac{\omega^2}{12\lambda}$$

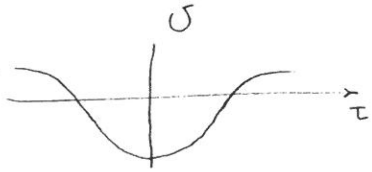
$$\langle -\eta | e^{-\tau_0 H} | \eta \rangle = \frac{1}{Z}\left[\det\left(-\frac{d^2}{d\tau^2} + \omega^2\right)\right]^{-1/2}$$

$$\times \left[\frac{\det\left(-\frac{d^2}{d\tau^2} + V''(x)\right)}{\det\left(-\frac{d^2}{d\tau^2} + \omega^2\right)}\right]^{-1/2} e^{-S_0}\left(1 + \text{corrections}\right)$$

$$-\frac{d^2}{d\tau^2}x_n(\tau) + \left(\omega^2 - \frac{3}{2}\omega^2 \frac{1}{\cosh \omega\tau/2}\right)x_n(\tau) = \varepsilon_n x_n(\tau)$$

looks like Schrödinger Equation (See Landau + Lifshitz)

Feb. 16th



a) two bound states $\varepsilon_0 = 0,\ \varepsilon_1 = \frac{3\omega^2}{4}$

$$x_0 = \sqrt{\frac{3\omega}{8}}\ \frac{1}{\cosh \omega\tau/2}$$

Zero energy of ground state is not accidental

Zero Mode

Zero Mode

Comes from 'translational invariance' of τ_c

$$S[X(\tau,\tau_c)] = S_0 = S[X(\tau,\tau_c+\delta\tau_c)]$$

$$x_0(\tau) = \frac{dX(\tau,\tau_c)}{d\tau_c}\left(\frac{-1}{\sqrt{S_0}}\right) = -\frac{dX}{d\tau}\left(-\frac{1}{\sqrt{S_0}}\right) \qquad \text{zero mode}$$

For tail states, note potential V which holds localised state has freedom of choice for position

Integrate over τ_c?

Recall $\delta(x(\tau)) = \sum_n c_n x_n(\tau) \qquad \int \frac{dc_n}{\sqrt{2\pi}} \propto \frac{1}{\sqrt{\varepsilon_n}}$

$$c_0 \rightarrow c_0 + \delta c_0 \Rightarrow x_0(\tau) \rightarrow x_0 + \delta c_0 x_0 \qquad \delta c_0 = \sqrt{S_0}\,\delta\tau_c$$

$$\delta x = \frac{dX}{d\tau_c}\delta\tau_c = -\sqrt{S_0}\,x_0\,\delta\tau_c \qquad dc_0 = \sqrt{S_0}\,d\tau_c$$

Correct normalisation because $S_0 = -\int d\tau\, \dot{x}^2$

$$\langle\eta|e^{-H\tau_0}|\eta\rangle_1 = \begin{pmatrix}\text{harmonic}\\ \text{oscillator}\end{pmatrix}\sqrt{\frac{S_0}{2\pi}}\,\omega\, d\tau_c \times \left\{\frac{\det'\left(-\frac{d^2}{d\tau^2}+V''\right)}{\omega^{-2}\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)}\right\}^{-1/2}$$

1 instanton contribution

det' is det without zero mode

must determine this for other bound state and excited states

ε_1 brings to { } a factor of 3/4

One Particle Tunneling	Tails in DOS
Imaginery time τ	Coordinate $\vec{r}$
Trajectories $x(\tau)$	Potential $V(\vec{r})$ (given realisation of random potential)
Time interval τ_0	Volume of system
Path integral $\int \mathcal{D}x(\tau)$	Functional Integral $\int \mathcal{D}V(r)$
Action $S = \int_{-\tau_0/2}^{\tau_0/2} d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right]$	$S = \int dx\,dx'\, V(x) K(x-x') V(x') - \ln G_E(x,x'\vert V)$, $K \simeq \frac{1}{\gamma}\delta(x-x')$
Extremal Path $X(\tau)$	Extremal realisation of Potential $V_0(r)$
$\frac{d^2X}{d\tau^2} = V'(X)$	$V_0(r) = \frac{\gamma \Psi_0^2}{E-E_0}$ $-\frac{1}{2}\nabla^2\Psi_0 - \gamma\frac{\Psi_0^3}{E_0-E} = E_0\Psi_0(r)$
Centre of the Instanton τ_c ; $X(\tau-\tau_c)$	z ; $V_0(r-z)$
Size of the instanton $\frac{1}{\omega}$	$\xi \propto \frac{1}{\sqrt{E}}$
Parameter ; small instanton density e^{-S_0}	$S_0 \propto \frac{E}{\gamma}^{2-d/2}$, d is the number of dimensions
$S_0 = S(X(\tau)) = \frac{\omega^3}{12\lambda}$	
Zero mode	$V(r) = \overset{\sim\text{optimal}}{V_0} + \frac{\Psi_0^2}{E-E_0} u_n(r)$
ε_n: $-\frac{d^2x_n}{d\tau^2} + V''(X)x_n = \varepsilon_n x_n$	$\left(-\frac{1}{2}\nabla^2 - E + V_0\right)u_n = \frac{\gamma\varepsilon_n}{E_0-E}\Psi_0^2 u_n$ $-\frac{\gamma\varepsilon_n}{E_0-E}\Psi_0 \int dr\, \Psi_0^3 u_n$? can be solved in 1d

$$\langle \rho(E) \rangle = \frac{(E_0 - E)^{-1/2}}{(2\pi\gamma)^{(1+d)/2}} \left(\frac{b}{2}\right)^d \gamma^2 \frac{2a}{b^2}$$

$$\times \left[\prod_{n=2}^{\infty} \left(1 - \frac{2}{\varepsilon_n}\right)\right]^{-1/2} \exp\left[-\frac{1}{2\gamma^2 a^2} + 1\right]$$

$$\frac{1}{a^2} = \int V_0^2 \, dr \qquad \frac{1}{b^2} = \int dr \, (\nabla V)^2$$

All steps in instanton approach are standard – only luck to solve equation.

Always need large energy – but its non-perturbative

Basic Reminder of Green Functions (See Feynman and Hibbs)

$G^{EX}(r, t; r', t')$ – the probability amplitude (EX = exact)

$$\langle r | e^{iH(t'-t)} | r' \rangle$$

$$G^{EX}(r,t;r',t') = \sum_\alpha \psi_\alpha^*(r)\psi_\alpha(r') e^{iE_\alpha(t'-t)}$$

$$G_E^{EX}(r,r') = \sum_\alpha \frac{\psi_\alpha^*(r)\psi_\alpha(r')}{E - E_\alpha}$$

Properties

1) Free Particle $\qquad \alpha \to \vec{p}, \quad \sum_\alpha \to \int \frac{d\vec{p}}{(2\pi)^d} \equiv \int (dp)$

$$\langle r | e^{-i\hat{p}^2/2m\,(t-t')} | r' \rangle = \int (dp) \langle r | p \rangle e^{-ip^2(t-t')/2m} \langle p | r' \rangle$$

$$= \int (dp)\, e^{ip^2(t'-t)/2m}\, e^{i\vec{p}.(\vec{r}-\vec{r}')}$$

Gaussian integral can be evaluated

$$G_0(r,t\,;r',t') = \left[\frac{m}{2\pi i(t'-t)\hbar}\right]^{d/2} e^{im(r-r')^2/2\hbar\,\cdot\,(t'-t)}$$

Units : $G \sim \frac{1}{\sqrt{\text{Volume}}}$ ~ probability amplitude , $\int dr \to$ probability

2/ One-dimensional Harmonic Oscillator

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$$

$$G(0,0\,;x,t) = \left[\frac{\omega}{2\pi i \sin\omega t}\right]^{1/2} \exp\left[\frac{i\omega}{2\sin\omega t}\left(x^2\cos\omega t\right)\right]$$

See instanton story

Disordered Systems $V(r)$

Averaged G^{EX} ?

$$G^{EX}(r,r',t) = \frac{1}{Z_r}\int \mathcal{D}r(\tilde{t})\exp\left\{iS\left[r(\tilde{t})\,|\,V(r)\right]\right\}$$

$$S[r(\tilde{t})] = \int_0^t d\tilde{t}\left[\frac{\dot{r}^2}{2m} - V(r)\right]$$

Averaging $P(V(r))$

$$G(r-r',t) = \frac{1}{Z_r}\int \mathcal{D}r(\tilde{t})\int \mathcal{D}V(r)\,P[V(r)]\,e^{iS}$$

$$\left\langle e^{-i\int_0^t V(r(\tilde{t}))\, d\tilde{t}} \right\rangle = ?$$

Assume potential is Gaussian $\Rightarrow$ $\int V d\tilde{t}$ is normally distributed

$$\left[\quad \langle e^{i\phi} \rangle = e^{-\langle \phi^2 \rangle /2} \, e^{i\langle \phi \rangle} \quad \right]$$

$$\left\langle \left[\int_0^t d\tilde{t}\, V(r(\tilde{t})) \right]^2 \right\rangle = \int d\tilde{t}_1\, d\tilde{t}_2 \left\langle V(r(t_1))\, V(r(t_2)) \right\rangle$$

$$\langle V(r_1) V(r_2) \rangle = W(r_1 - r_2) = \gamma\, \delta(r_1 - r_2)$$

$$= \int d\tilde{t}_1\, d\tilde{t}_2\; W(r(t_1) - r(t_2))$$

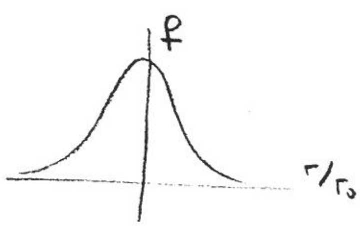
$$t_+ = (t_1 + t_2)/2$$

$$t_- = t_1 - t_2$$

$$= \int_0^t dt_+ \int_{-t_+}^{t_+} dt_-\; W(r(t_1) - r(t_2))$$

Further evaluation is impossible without approximation

Let $w(r)$ have a size r_0, $\quad W(r) = \dfrac{\gamma}{r_0^{\,d}}\, f\!\left(\dfrac{r}{r_0}\right)$



Assume classical trajectory on scale r_0, $\quad \vec{r}(t) = \vec{v}\, t$

$$t_- \ll t_+ \quad \rightarrow \quad \int_0^t dt_+ \int_0^\infty dt_- \quad \frac{2}{v} W(r)\, dr$$

1d integral over trajectory

$$= \frac{t}{2\tau}\,, \quad \frac{1}{2\tau} = \frac{2}{v} \int_0^\infty W(r)\, dr$$

So $\left\langle \left[\int_0^t d\tilde{t}\, V(r(\tilde{t})) \right]^2 \right\rangle = \frac{t}{2\tau}$

$$G(r,t) = G_0(r,t)\, e^{-t/2\tau}$$, always correct for Energy large enough

$\frac{1}{2\tau} = \frac{\gamma}{V r_0^{\,d-1}}$ ~ divergent when $r_0 \to 0$ for $d > 1$

~ in fact this correct for $r_0 \gg \frac{1}{p} = \lambda_F$

otherwise $\frac{1}{\tau} \propto \frac{\gamma p^{d-1}}{V}$, $r_0 < \frac{1}{p} = \lambda_F$

All this meaningful only if $t > 0$ – measured after particle is created

After a time $t > \tau$ ~ mean free time $\quad G(r,t) \to 0$

Average over potential → mixes phases and $\langle G \rangle$ decays

Perturbation Theory

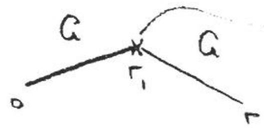
$$\frac{\delta G^{EX}(0,0;r,t)}{\delta V(r_1)} = \int_0^t G^{EX}(0,0;r_1,t_1)\, G^{EX}(r_1,t_1;r,t)\, dt_1$$

functional derivative

F.T.

$$\frac{\delta G_E^{EX}(0,r)}{\delta V(r_1)} = G_E^{EX}(0,r_1)\, G_E^{EX}(r_1,r)$$

~ can be seen by writing G in terms of Ψ and making perturbation

G  , δG  δV

G^{EX}  , G_0 , $\times \; V(r_1)$

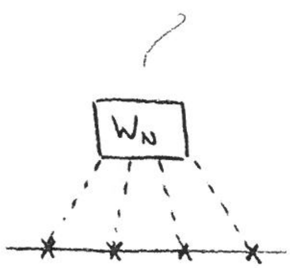
$G^{EX} =$ 

$\int$ over r_1 (and t_1)

Average a) $\langle V(r) \rangle = 0$

b) $\langle V(r_1) V(r_2) \rangle = W(r_1 - r_2)$

c) $\langle V(r_1) \ldots V(r_N) \rangle_c = W_N \{r_i\}$



General Case

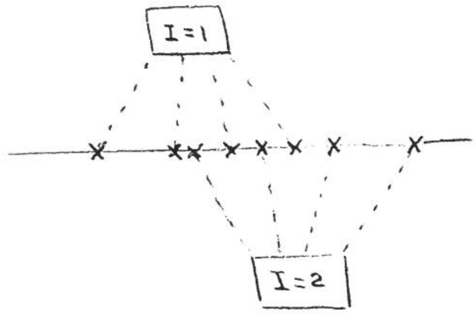
Gaussian Potential , $W_N = 0$ for $N > 2$ 

$G = \sum$ 

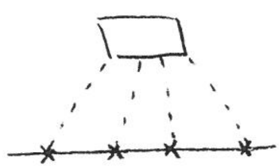
i)  ii)  $= W(r_1 - r_2)$

and iii) integrate over r_i

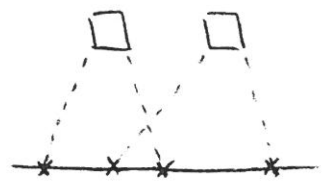
Randomly Located Impurities $\sum_i u(r-r_i)$



Are these important?



vs



$\sim N_i$ $\qquad\qquad$ $\sim N_i^2$

impurity concentration

Limit $N_i \to \infty$ $\qquad$ $N_i u^2 \to$ constant

$u \to 0$

can neglect first in favour of second $\longrightarrow$ equivalent to Gaussian diagrammatics

(this was also shown earlier)

The high order terms can be important for localised states

Classical Formula

for cross-section S of scattering impurity

Mean free path $\quad l = \dfrac{1}{N_i S}$

Time of free path $\quad \tau = l/v = \dfrac{1}{N_i S v}$

Born Approximation

$$u(r-r_i) = u_0\,\delta(r-r_i)$$

$$S = 2\pi u_0 \int (dp)\,\delta\left(\frac{p^2}{2m} - E\right) = 2\pi u^2 \rho$$

$$\overset{\shortparallel}{\rho(E)}$$ - density of states

$$\rho(E) \overset{d=3}{=} \frac{p^2}{2\pi v}$$

For $r_0 \gg \frac{1}{p}$ $\quad S \sim r_0^{\,d-1}$

$$\tau \propto \frac{1}{W_0 V r_0^{\,d-1}}$$

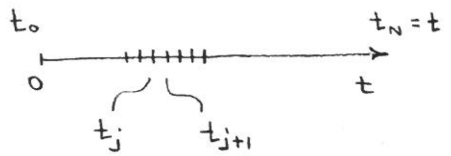
$$G(x_i, 0; x_f, t) = \langle x_f | e^{-iHt} | x_i \rangle$$
$$= \frac{1}{Z} \int \mathcal{D}x(\tilde{t})\, e^{iS[x(\tilde{t})]}$$

$$S[x(\tilde{t})] = \int_0^t d\tilde{t}\, \mathcal{L}[x(\tilde{t})] \quad , \quad \mathcal{L} = \frac{m\dot{x}^2}{2} - V(x)$$

What is

$$Z = ? \quad , \quad Z(V) = \text{const.} \quad (\text{indep. of } V)$$

Path integral as a limit of definite integrals



$$\delta = t_{j+1} - t_j \quad , \quad t = N\delta$$

$$x_j = x(t_j) \quad , \quad x_0 = x_i \, , \quad x_N = x_f$$

$$\int \mathcal{D}x(\tilde{t})\, F(x(\tilde{t})) \overset{?}{=} \lim_{\substack{N \to \infty \\ \delta \to 0}} \int \prod_{j=1}^{N-1} dx_j \; F\{x_j\}$$

Is limit well defined ?

No integral over x_i, x_f

(What are the units even?)

Need normalisation factor $1/A^N$

$$\int \mathcal{D}x(\tilde{t})\, F(x(\tilde{t})) = \lim_{\substack{N \to \infty \\ \delta \to 0}} \frac{1}{A} \int \prod_{j=1}^{N-1} \frac{dx_j}{A} \; F\{x_j\} \quad , \quad A = \sqrt{\frac{2\pi i \delta}{m}} \quad 1d$$

1) Determine A

2) Derive Schrödinger Equation

$$\Psi(x_2, t_2) = \int G(x_2, t_2; x_1, t_1)\, \Psi(x_1, t_1)\, dx_1\, dt_1$$

$$\psi(x, t+\delta) = \int dy \, \frac{1}{A} \exp\left[i\delta \mathcal{L}\left(\dot{x} = (x-y)/\delta, \tilde{x} = (x+y)/2\right)\right] \psi(y,t)$$

$$= \int_{-\infty}^{\infty} dy \, \frac{1}{A} \left\{ \exp\left[i \frac{m(x-y)^2}{2\delta} - i\delta V\left(\frac{x+y}{2}, t\right)\right] \right\} \psi(y,t)$$

Only $x-y$ small will contribute by stationery phase

$$\frac{m(x-y)^2}{2\delta} \sim 1 \ (\hbar)$$

$$|x-y| \sim \sqrt{2\delta/m}$$

$$y = x + \eta$$

To second order in δ

$$\psi(x,t) + \delta \frac{\partial \psi}{\partial t} = \int_{-\infty}^{\infty} d\eta \, \frac{1}{A} \, e^{im\eta^2/2\delta} \left[1 - i\delta V(x,t)\right]$$

$$\times \left\{ \psi(x,t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2}\eta^2 \frac{\partial^2 \psi}{\partial x^2} \right\}$$

1) Zeroth order

$$\frac{1}{A} \int_{-\infty}^{\infty} d\eta \, e^{im\eta^2/2\delta} = 1 = \frac{1}{A}\sqrt{\frac{2\pi i \delta}{m}} \quad \text{(as advertised)}$$

So $\mathcal{Z}$ does not depend on V, because V is always multiplied by δ

2) Note $\displaystyle\int_{-\infty}^{\infty} \eta \, d\eta \, e^{im\eta^2/2\delta} = 0$, $\displaystyle\frac{1}{A}\int \eta^2 d\eta \, e^{im\eta^2/2\delta} = \frac{i\delta}{m}$

first order

$$\delta \frac{\partial \psi}{\partial t} = i\delta \left[\frac{1}{2m} \frac{\partial^2 \psi}{\partial x^2} - V(x)\psi \right]$$

$$i \frac{\partial \psi}{\partial t} = H\psi \qquad \text{Schrödinger Equation}$$

Recall
$$G(r,t) = \langle G^{EX}(R+r, t; R, 0)\rangle$$
$$= G_0(r,t)\, e^{-t/2\tau}$$

$$G_0(r,t) = \left(\frac{m}{2\pi i t}\right)^{d/2} \exp\left[\frac{imr^2}{2t}\right]$$

(Correct for any statistics because there is only one particle)

τ – time of mean free path

$l = \tau v$ – mean free path

$l = \dfrac{1}{N_i S}$

concentration, cross-section

$\dfrac{1}{\tau} \sim \dfrac{N_i v}{r_0^{d-1}}$

$\dfrac{1}{2\tau} = \dfrac{2}{v}\int W(r)\,dr \quad , \quad \langle V(r_1)V(r_2)\rangle = W(r_1 - r_2)$

1d integral

r_0 – characteristic scale

$\dfrac{1}{2\tau} \sim \dfrac{1}{r_0^{d-1}}$ valid when $r_0 p \gg \hbar$

Impurities with concentration N_i and δ-like potential

$$u(r-r_i) = u_0\,\delta(r-r_i)$$

$\left.\begin{array}{l} N_i \to \infty \\ \delta \to 0 \end{array}\right\} \Rightarrow$ Born Approximation

Density of States

$$S = 2\pi u_0^2 \int (dp)\, \delta\left(\frac{p^2}{2m} - E\right) = 2\pi u_0^2\, \nu(E)$$

Large E, $\quad \nu(E) = \dfrac{p_E^2}{2\pi v_E} \propto \sqrt{E} \qquad d=3$

$$\frac{1}{\tau} \propto N_i u_0^2 \nu$$

Perturbation Theory based on Diagram Technique

a) r,t ———— r',t' $\quad G_0(r',t'; r,t)$

b) $\underset{r_1}{\times}\cdots\cdots\underset{r_2}{\times}$ $\quad W(r_1 - r_2) = \gamma\,\delta(r_1 - r_2)$

c) $\int \underbrace{dr_1 dt}$

intermediate

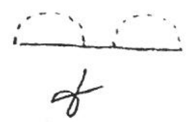
d) r,t ═══ r',t' $\quad G(r',t'; r,t)$

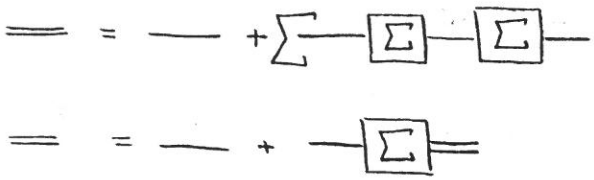
Dyson Equation

Self-energy operator

Σ - sum of all irreducable diagrams

but not





Dyson Equation

$G \quad G_0 \quad G_0 \quad G$

$$G(r,t) = G_0(r,t) + \int_0^t dt_1 \int_{t_1}^t dt_2 \int dr_1\, dr_2\; \underbrace{G_0(r_1,t_1)}_{>0}\, \underbrace{\Sigma(r_2-r_1, t_2-t_1)}_{>0}\, \underbrace{G(r-r_2, t-t_2)}_{>0}$$

For fourier transforms need analytical structure of G at all times

Factor $\quad \Theta(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

$$G^R(r,t) = G(r,t)\,\Theta(t)$$ Retarded Green function

$$\left(i\frac{\partial}{\partial t} - H\right) G^R(r,t;r',t') = \underbrace{i\delta(r-r')\,\delta(t-t')}_{\text{given by } \Theta \text{ function}}$$

This is close to definition found in field theory

$$t \xrightarrow{FT} E \qquad r \to p$$

$$\Sigma^R(r,t) = \Sigma(r,t)\,\theta(t)$$

$$G^R(p,E) = G_0^R(p,E) + G_0^R(p,E)\,\Sigma^R(p,E)\,G^R(p,E)$$

$$\left[G^R(p,E)\right]^{-1} = \left[G_0^R(p,E)\right]^{-1} - \Sigma^R(p,E)$$

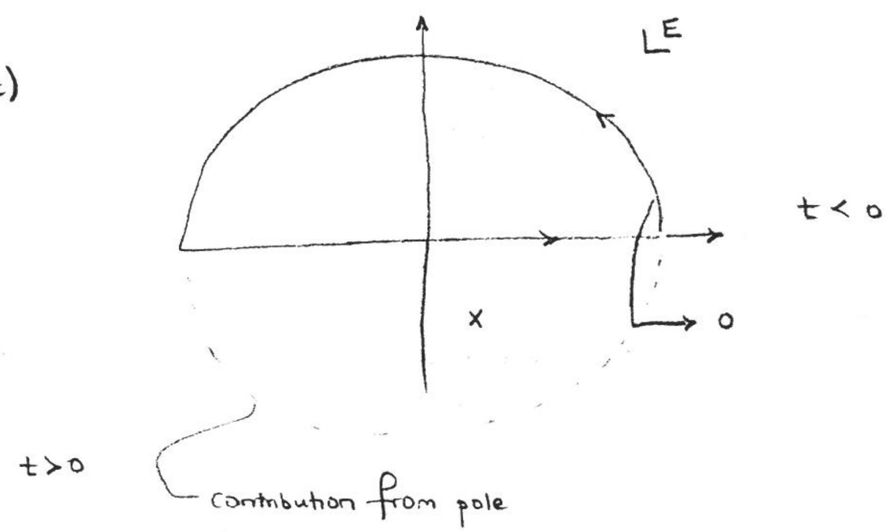
In principle we can build quantum mechanics out of purely retarded functions

$$G^R(p,E) = \int dr\,dt\; e^{ipr - iEt}\, G^R(r,t)$$

$$= \int dr\,dt\; e^{ipr - iEt} \int (dp_1)\, e^{ip_1^2 t/2 - ip_1 r - t/2\tau}$$

$$= \frac{1}{E - p^2/2m + i/2\tau}$$

$$G^R(E) \longrightarrow G^R(t)$$



$$G^R(p,E) = \frac{1}{E - p^2/2m - \Sigma^R(p,E)}$$

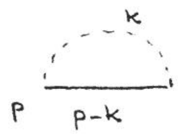
Should find a) $\mathrm{Im}\,\Sigma^R = -\frac{1}{2\tau}$

b) $\mathrm{Re}\,\Sigma^R$ irrelevant

E is large

Let us see if we can get this from diagrams

first order



$$= \Sigma_1^R(p,E) = \int dk\, w(k)\, G_0^R(p-k,E)$$

quenched random potential so no energy dependence – formalism applies to time-dependent case providing potential still classical

a) $$\mathrm{Im}\,\Sigma_1^R(p,E) = \int dk\, \underbrace{w(k)}_{\gamma \text{ for white noise}}\, \mathrm{Im}\, G_0^R(p-k,E)$$

$$\mathrm{Im}\, G_0^R = -\pi\, \delta(E - p^2/2m)$$

$$\mathrm{Im}\,\Sigma_1^R(p,E) = -\pi\gamma \int (dp)\, \delta(E - p^2/2m) = \underbrace{-\pi\gamma\nu = -\frac{1}{2\tau}}_{\text{equation for time of m.f.p at first order}}$$

$$\tau = \frac{1}{2\pi\nu\gamma}$$

depends smoothly on energy

$$\gamma \int (dk) \frac{1}{\left(E - \frac{k^2}{2m} + i0\right)^2}$$

$\nu\, d(k^2/2m)$

b) $\mathrm{Re}\,\Sigma_1^R(p,E)$ formally diverges – $\dfrac{d\,\mathrm{Re}\,\Sigma_1^R}{dE} \sim O\left(\dfrac{1}{E\tau}\right)$

can neglect if $E\tau \gg 1/\hbar$

Small parameter, $\dfrac{1}{E\tau} \ll \hbar$ Same parameter as in

$\dfrac{1}{pL} \ll \hbar$ Anderson Model $\dfrac{W}{I} \ll 1$

$\tau, L \propto \left(W/I\right)^2$

states are extended (This condition will be used very often)

If not satisfied then states are localised

28/2/94

Recall

$$G^R_{\text{Exact}}(r_f, t_f; r_i, t_i) = G(r_f, t_f; r_i, t_i)\,\Theta(t_f - t_i)$$

Average $G^R(r,t) = G_0^R(r,t)\, e^{-t/2\tau}$

$$G_0^R(r,t) = \left(\frac{m}{2\pi i t}\right)^{d/2} e^{imr^2/2t}\,\Theta(t) \qquad \text{GF of a free particle}$$

$$G_0^R(p,E) = \frac{1}{E - p^2/2m + i0}$$

$$G^R(p,E) = \frac{1}{E - p^2/2m + i/2\tau}$$

No singularity at $\operatorname{Im} E > 0$

Self-energy operator Σ^R

$$\Sigma_1 = \text{- - -} = \int (dk)\, G_0(k,E)\, \underbrace{W(\vec{k})}_{\gamma} = \gamma \int (dk)\, G_0(k,E)$$

① $$1/\tau \propto \gamma\nu \propto \gamma\, \underbrace{(mE)^{d/2}}_{p^d} \frac{1}{E} \propto \gamma E^{d/2-1} m^{d/2}$$

$$E\tau \propto \frac{1}{\gamma} E^{2-d/2} m^{-d/2}, \quad E > 0$$

② Now, recall that instanton density $\ll 1 \propto e^{-S_0}$ in earlier calculation of tail states

$$S_0 \propto \frac{1}{\gamma} |E|^{2-d/2} m^{-d/2} \gg 1$$

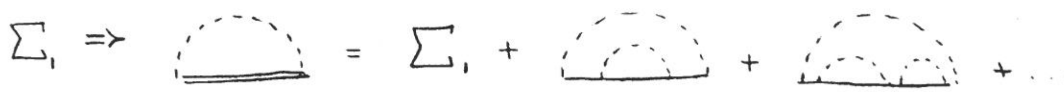
In both cases there is a semi-classical approximation (although in the second case there is no perturbation theory because of essential singularity)

wavelength $\ll l$, (and expansion $(E\tau)^{-1} \gg \hbar^{-1}$)

Note $\hbar$ has units so real expansion should involve a dimensionless parameter

Note both limits, ① and ②, although opposite extremes (nearly free and nearly localised) involve the same semi-classical parameter

Beyond Perturbation Theory in γ



but not 

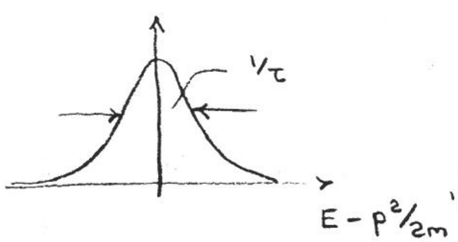
leads to change in $\nu(E) = \frac{-1}{\pi}\int (dp)\, G^R(p,E)$

$$\equiv \frac{-1}{\pi} G(r=0,E)$$

$$\left(\text{Now} \quad \mathrm{Im}\,\frac{1}{x+i0} = -\delta(x) \right)$$

$$\nu(E) = \frac{1}{\pi}\int (dp)\, \frac{(2\tau)^{-1}}{\left(E-p^2/2m\right)^2 + 1/4\tau^2}$$

Lorentzian instead of δ function



if $E \gg \frac{1}{\tau}$ $\quad \nu = \nu_0\left(1 + O\left(\frac{1}{E\tau}\right)\right)$

1) Qualitatively, the broadening of the singularity is correct – √ singularities are smeared out,

2) Quantitatively wrong for $E \lesssim 1/\tau$

Ask Boris about Orbach rule

<u>Disorder is irrelevant for $\nu(E)$</u>

No singularities at the mobility edge

2D Green function in a random magnetic field

(B.L. Altshuler and L.B. Ioffe. Phys.Rev.Lett. **69** ,2979 (1992))

Consider a motion of a particle in two dimensions $\mathbf{r} = (x,y)$ in a quenched magnetic field $\mathbf{B}(\mathbf{r})$. The Hamiltonian of this particle is

$$H = (2m)^{-1} [\mathbf{p} - \mathbf{a}(\mathbf{r})]^2 \qquad (1)$$

where $\mathbf{a}(\mathbf{r})$ is the vector potential of the field $\mathbf{B}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{a}(\mathbf{r}) \qquad (2)$$

Suppose that $\mathbf{B}(\mathbf{r})$ is Gaussian and δ - correlated random field:

$$\langle \mathbf{B}(\mathbf{r})\mathbf{B}(\mathbf{r}')\rangle = \Gamma\ \delta(\mathbf{r}-\mathbf{r}') \qquad (3)$$

1) Construct the diagram technique analogous to the case of random potential.

2) Calculate the self energy operator in the lowest order in Γ. Show that this contribution diverges. Is it infrared or ultraviolet divergence ?

3) In class it was shown that the path integral for the gauge invariant Greens function can be written as

$$\int D\,\mathbf{r}(t)\ \exp\{i(m/2) \int [d\mathbf{r}(t)/dt]^2\, dt - (\Gamma/2)A[\mathbf{r}(t)] \qquad (4)$$

where $A[\mathbf{r}(t)]$ is the directed area between the classical trajectory and the trajectory $\mathbf{r}(t)$. Writing $A[\mathbf{r}(t)]$ as

$$A[\mathbf{r}(t)] = \int y\, dx \qquad (5)$$

where x is the coordinate along the classical trajectory and y is the deviation, evaluate the path integral over $x(t)$. When Eq.(5) is valid ?

4) Show that the remaining integral over y describes a one-dimensional particle in a imaginary linear potential. Write down the corresponding Shrodinger equation and try to find an explicit form of the Green function

Advanced GF

Suppose we know $\psi(r, t=0)$, can we determine $\psi(r, t<0)$

Difference : $\Theta(-t)$, $\rightarrow \exp\left[+ t/2\tau\right]$

$$G^A(p,E) = \frac{1}{E - p^2/2m - i/2\tau}$$

$$t \leftrightarrow -t \quad , \quad G^R \leftrightarrow G^A$$

$$G^A(p,E) = G^R(p,E)^*$$

What do we learn from average GF ? Not much – for $t < \tau$ it behaves like free particle ; for $t > \tau$ it gets scattered somewhere.

Particle in a Random Magnetic Field – 2d Case (See problem sheet)

$$\mathcal{H} = \frac{1}{2m}(p - \vec{a})^2 \qquad \vec{a} \text{ Vector potential}$$

$$\vec{B} = \nabla \wedge \vec{a}$$

Assume random magnetic field is static, white-noise, Gaussian

$$\langle B(r) B(r') \rangle = \Gamma \, \delta(r - r')$$

Exercise in GF and Path integrals

Objective $G^R(r,t)$

$$\vec{q} = \vec{r} \quad , \quad \text{velocity} \quad \dot{\vec{r}} = \frac{\partial H}{\partial p} = \frac{p - \vec{a}}{m}$$

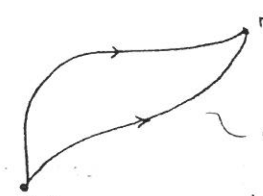
Lagrangian $\mathcal{L} = p\dot{q} - H = \frac{p^2 - a^2}{2m}$

$$= \frac{m\dot{r}^2}{2} + \dot{\vec{r}}.\vec{a}$$

Path Integral

$$G^R(r,t;r',t') = \Theta(t-t') \int_{r(t')=r'}^{r(t)=r} \mathcal{D}r(\tilde{t}) \exp\left[\frac{im}{2} \int_{t'}^{t} d\tilde{t} \left(\frac{dr}{dt}\right)^2 + i \int d\tilde{t}\, \frac{d\vec{r}}{d\tilde{t}} . \vec{a} \right]$$

What about Gauge Invariance ? – change $\vec{a}$ and we change G^R

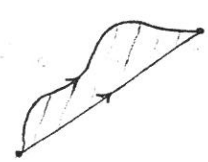


each path acquires a non-invariant phase,

but phase difference is

So $\quad \times \exp\left[-i \int_{t'}^{t} d\tilde{t}\, \vec{a}(r_0(\tilde{t}))\, \frac{dr_0}{d\tilde{t}}(\tilde{t}) \right]$

Choice of $r_0(t)$ is a choice of gauge – straight line is our choice



Green function depends on choice of Gauge – unlike observable !

$$\int_{t'}^{t} d\tilde{t} \left[\frac{dr}{d\tilde{t}} a(r) - \frac{dr_0}{d\tilde{t}} a(r_0) \right] = \oint d\vec{l}\, \vec{a}(\vec{l}) = \text{flux} = \int B . d^2A$$

$$\left(\text{Recall} \quad \langle e^{i\phi} \rangle = e^{-\langle \phi^2 \rangle /2} \right)$$

$$\int B(r_1) B(r_2)\, d^2r_1\, d^2r_2 = \Gamma \int d^2 r_1 = \Gamma A(r(\tilde{t}))$$

δ correlated disorder

of field

Objective

$$G^R(r,t) = \int_{r(0)=0}^{r(t)=r} \mathcal{D}r(\tilde{t}) \exp\left[i\frac{m}{2}\int \dot{r}^2 d\tilde{t} - \frac{1}{2}\Gamma A(r(\tilde{t}))\right]$$

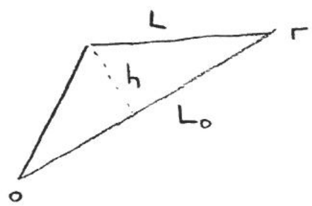
Estimate area $A(r(\tilde{t}))$

Typical trajectory : phase difference with classical trajectory $\lesssim 1$

$$\phi = \int \vec{p}\cdot d\vec{r} = pL$$

Length of trajectory

$$\delta L \lesssim \frac{1}{p} = \lambda$$



$$L = (L_0^2 + 4h^2)^{1/2}$$
$$= L_0 + \frac{2h^2}{L_0} + \cdots$$

$$\delta L \sim \lambda \sim \frac{2h^2}{L_0}$$

$$h \sim (L_0\lambda)^{1/2}$$

$$A \sim L_0 h = L_0^{3/2}\lambda^{1/2}$$

$$\langle \delta\phi^2 \rangle \sim \Gamma L_0^{3/2}\lambda^{1/2}$$

Mean free Path L_ϕ

length at which B field will dominate phase difference

$$\delta\phi(l_\phi) \sim 1 \quad , \quad \Gamma l_\phi^{3/2}\lambda^{1/2} \sim 1$$

$$l_\phi \sim \frac{1}{(\Gamma^2\lambda)^{1/3}} \quad , \quad \frac{1}{\tau_\phi} = v\lambda^{1/3}\Gamma^{2/3}$$

1) No perturbation theory in Γ (c.f. random potential)

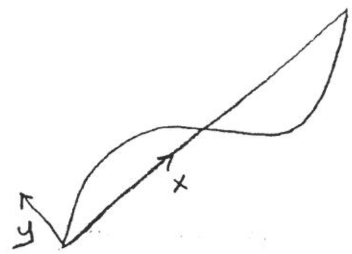
leads to divergencies (exercise)

2) Exact calculation shows $\quad (Ai'(\zeta_n) = 0)$

$$G(p,E) = \frac{\pi}{2}\frac{1}{\sqrt{\tau_\phi}}\sum_n \frac{(-i)^{2/3}}{\left(p^2/2m - E + (-i)^{2/3}\tau_\phi^{-1}\zeta_n\right)^{3/2}\zeta_n}$$

To derive G need to write

$$A = \int y\,dx$$



Valid if we can assume no .

More interesting - not the average amplitude, but the average probability.

$$|\langle x_f | e^{-iHt} | x_i \rangle|^2$$ can be observed (transition probability)

First problem

$$S_v[x(t)] = S_0[x(t)] + \int x(t)\,v(t)\,dt$$

$V(t)$ is some time-dependent (random) potential

2/3/94

Recall

QM + Disorder

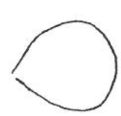
GF(Exact) $\longrightarrow$ Average GF

$\langle x_f | e^{iHt} | x_i \rangle$, G, G^R

Observable - Density of States

$$\nu(E) = -\frac{1}{\pi}\,\mathrm{Im}\,G^R(r,r;E)$$

gauge invariant



tadpole

Now let us think about transition probability

$$|\langle x_f | e^{-iHt} | x_i \rangle|^2$$

Problem

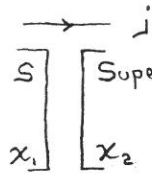
Action $S_{\xi}[x(t)] = S[x(t)] + \int_{t_i}^{t_f} x(t)\xi(t)\,dt$

Acts as external potential with $\xi(t)$ being a random force

$$S[x(t)] = \int_{t_i}^{t_f} dt \left[\frac{m\dot{x}^2}{2} - U(x) \right]$$

Josephson Junction

$x \to \varphi$ phase of the junction



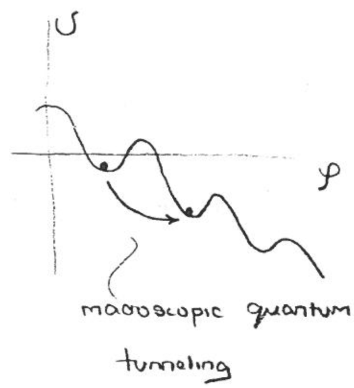
$\varphi = x_1 - x_2$

$U(\varphi) = -A\cos\varphi - j\varphi$

linear approx. to $\sin\varphi$

Mass $m = \frac{\hbar}{2e} C$ capacitance of junction

becomes problem of motion of particle in a potential



tunneling determines whether junction is resistive or superconducting — question is does ξ help or hinder tunneling ?

Now $|\langle x_f | e^{-iHt} | x_i \rangle|^2 = |G^R_{EX}(x_f, t_f; x_i, t_i)|^2$

$$= G^R_{EX}\, G^A_{EX}$$

$\Psi_i(x)$ - initial quantum state

$\Psi_f(x)$ - final quantum state

$$P[\psi_f, \psi_i] = \int dx_f \, dx_f' \, dx_i \, dx_i' \; \psi_f^*(x_f) \psi_f(x_f') \psi_i(x_i) \psi_i^*(x_i')$$

$$\times \underbrace{G^R_{EX}(f, i) \, G^A_{EX}(i', f')}_{J_{EX}(f, f'; i, i')} \Rightarrow$$

i ———— f

i' ———— f'

$$J_{EX} = \iint \mathcal{D}x_1(t) \, \mathcal{D}x_2(t) \exp\left[i S_\xi[x_1(t)] - i S_\xi[x_2(t)] \right]$$

$x_{1,2}(t_i) = x_i, x_i'$

$x_{1,2}(t_f) = x_f, x_f'$

because of complex conjugate

$$= \int \mathcal{D}x_1 \mathcal{D}x_2 \exp\left[i \left[S(x_1) - S(x_2) \right] + i \int dt \, \xi(t) \left[x_1(t) - x_2(t) \right] \right]$$

$\xi(t)$ - random force with probability distribution $P_\xi[\xi(t)]$

$$J \equiv \langle J_{EX} \rangle$$

$$= \int \mathcal{D}x_1 \mathcal{D}x_2 \exp\left[i S(x_1) - i S(x_2) \right] \; Q\left[x_1(t) - x_2(t) \right]$$

Influence Functional

(Generating function of P)

$$Q[x(t)] = \int e^{\int i \xi(t) x(t) \, dt} \; P_\xi[\xi] \, \mathcal{D}\xi$$

functional derivatives

give moments

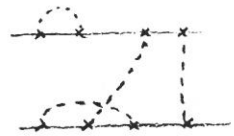
For Gaussian distribution of P_ξ, $\quad \langle e^{i\varphi} \rangle = e^{-\langle \varphi^2 \rangle / 2}$

$$Q = \exp\left[-\frac{1}{2} \iint dt \, dt' \left[x_1(t) - x_2(t) \right] \left[x_1(t') - x_2(t') \right] W(t-t') \right]$$

$$\langle \xi(t) \xi(t') \rangle = W(t-t')$$

J can not now be factorised after averaging

$$J \neq G^R \times G^A$$

$\Rightarrow$ 

$$J = \int \mathcal{D}x_1 \mathcal{D}x_2 \exp\left[i S(x_1) - i S(x_2) \right]$$

$$\times \exp\left\{ -\frac{1}{2} \iint dt\,dt' \left[x_1(t) - x_2(t)\right]\left[x_1(t') - x_2(t')\right] W(t-t') \right\}$$

(for discussion of influence functional see Chap. 12 of Feynman)

More general case : interaction of particle with enviroment

$$Q(x_1 - x_2) = Q(x_1, x_2) = \exp\left\{ i \frac{1}{2} \int dt\,dt' \left[x_1(t) - x_2(t)\right] \right.$$

$$\left. \times \left[x_1(t')\tilde{W}(t,t') - x_2(t')\tilde{W}^*(t,t') \right] \right\}$$

$\tilde{W}$ = real-random force case

$\tilde{W}$ = real + imaginary - enviroment

Assumptions :

1/ Stationary random process $\tilde{W}(t,t') = \tilde{W}(t-t')$

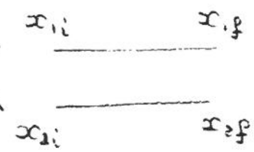
2/ $\tilde{\tilde{W}}$ = F.T. of $\tilde{W}$, $\tilde{\tilde{W}}$ real

$$\tilde{W}(t-t') = \underset{\text{even}}{W(t-t')} + i\,\underset{\text{odd}}{W_1(t-t')}$$

Final assumption is the main one,

$$3) \qquad W_1(t-t') = 2im\gamma_1\, \delta'(t-t')$$

derivative of δ function

$$J = \Big\langle$$ 

$$\Big\rangle = \int \mathcal{D}x_1 \mathcal{D}x_2 \exp\left[iS(x_1) - iS(x_2)\right]$$

$$\times \exp\left\{\left[-i\gamma_1 m \int_i^f dt \left[x_1(t) - x_2(t)\right]\left[\dot{x}_1(t) + \dot{x}_2(t)\right]\right]\right\}$$

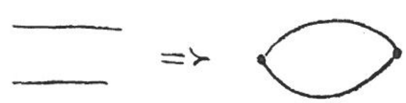
$$\times \exp\left\{-\frac{1}{2}\int dt\, dt' \left[x_1(t) - x_2(t)\right] W(t-t') \times \left[x_1(t') - x_2(t')\right]\right\}$$

Physical meaning of two terms:

Semiclassical Approach - Wigner transformation $\quad X = \dfrac{x_1 + x_2}{2}$

$$x_- = x_1 - x_2$$

Let us assume $\quad (x_-)_{if} = 0$



probability of getting from one point to another.

$$\mathcal{L}_{1,2} = \frac{\dot{x}_{1,2}^2 m}{2} - U(x_{1,2})$$

$$\int dt \left[\dot{x}_1^2 - \dot{x}_2^2\right] = 2\int dt\, \dot{X}\dot{x}_- = -2\int dt\, \ddot{X} x_-$$

$$U(x_{1,2}) = U(X \pm \tfrac{1}{2}x_-)$$

$$J(X_i, X_f) = \int \mathcal{D}\vec{x}(t) \exp\left\{ i\tilde{S}[\vec{x}] \right\}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \tilde{S} = \tilde{S}_1 + \tilde{S}_2$$

$$\tilde{S}_1 = -\int_i^f dt \left\{ x_- m\left(\ddot{X} + \gamma_1 \dot{X} \right) + \left(U(X + x_-/2) - U(X - x_-/2) \right) \right\}$$

$$\tilde{S}_2 = \frac{i}{2} \int dt\, dt'\, x_-(t) W(t-t') x_-(t')$$

So far these is no approximation

Semiclassics, U changes slowly in time and space

x_- is small

$$U(x_1) - U(x_2) = x_- U'(X)$$

When is this valid? Estimate x_-, or $\langle x_-^2(\Delta t) \rangle$

time interval

looking on $\tilde{S}_2$, $\langle x_-^2(\Delta t)\rangle \sim \dfrac{1}{W\Delta t} \ll x_0^2$ Condition for semi-classics

$$x_0^2 = \frac{U'''}{U'}$$

$$\tilde{S}_1 = -\int_i^f dt \left\{ x_- \left[m(\ddot{X} + \gamma_1 \dot{X}) + U'(X) \right] \right\}$$

This equivalent to the Langevin equation

$$m\ddot{X} = -m\gamma_1 \dot{X} - U'(X) + \xi(t)$$

viscous force

potential force

random force

$$\langle \xi(t)\xi(t') \rangle = W(t-t')$$

why δ' was taken for W_1 – otherwise get high order derivatives

Why are two problems equivalent ?

Dual description : $X(t)$ - Path , $P_x[X(t)]\,\mathcal{D}X$

$\xi(t)$ - force $P_\xi[\xi(t)]\,\mathcal{D}\xi$

Gaussian

$$P_\xi \mathcal{D}\xi = \exp\left[-\frac{1}{2}\int dt\,dt'\,\frac{\xi(t)\,\xi(t')}{W(t-t')}\right]\mathcal{D}\xi$$

$$\xi(t) = \xi[X(t)] \Rightarrow e^{i\tilde{S}[X]}\,\mathcal{D}\xi$$

(Think about integrating out x $\rightarrow$ P_ξ with $\xi = m\ddot{X} + m\gamma_1\dot{X} + U'(X)$)

$$P_\xi \mathcal{D}\xi = P_x \mathcal{D}X \underbrace{\frac{\mathcal{D}\xi}{\mathcal{D}X}}$$

Jacobian (must be equal to constant for identity)

$$\frac{\mathcal{D}\xi}{\mathcal{D}X} = \left[\frac{m}{\delta} + \frac{\gamma_1 m}{2}\right]^N$$

(See previous estimate of Z $0 \longrightarrow t$, $\delta = t/N$)

Josephson junction

$m \propto C$, $\gamma_1 = \dfrac{1}{CR}$

Normal resistence

W is strongly temperature dependent (δ correlated at very large T)

Lesson is that real problems require two coordinates

7/3/94

Recall what is interesting?

$$|\langle x_f t_f | x_i t_i \rangle|^2 = |\langle x_f | e^{-iHt} | x_i \rangle|^2$$

$$= \langle G^R_{EX}(f,i)\, G^A_{EX}(i\,f) \rangle$$

Recall Green functions in Field Theory (In Stat. Mech. diagram technique at T=0)

Two approaches: a) continuous field integrals

b) T - products

$a_j (a_j^+)$ annihilation (creation) operator of a particle in a state $|j\rangle$

$$a_j a_k^+ \pm a_k^+ a_j = \delta_{jk} \quad \begin{cases} + & \text{fermions} \\ - & \text{bosons} \end{cases}$$

$$a_j a_k \pm a_k a_j = 0$$

Field Operators

Particular set of wavefunctions $\quad |j\rangle = |x\rangle \qquad a_j = \Psi(x)$

In General case $\quad \hat{\Psi}(x) = \sum_j \varphi_j(x)\, a_j$

$\langle x | j \rangle$ wavefunction of state $|j\rangle$ in coordinate representation

$$\hat{\Psi}^+(x) = \sum_j \varphi_j^*(x)\, a_j^+$$

To introduce Green function we need Heisenberg representation

$$\hat{\Psi}(x) \longleftarrow \hat{\Psi}(x,t) \qquad i\frac{\partial \hat{\Psi}}{\partial t} = [H, \hat{\Psi}]$$

$$\hat{\psi}(x,t) = e^{iHt}\,\hat{\psi}(x)\,e^{-iHt}$$

Now have everything to define Green-function

$$G(x_1,t_1;x_2,t_2) = -i\langle \hat{T}\psi(x_1,t_1)\psi^+(x_2,t_2)\rangle$$

what is average and
what is $\hat{T}$?

Averaging: $\langle\ldots\rangle = \dfrac{\langle 0|\ldots|0\rangle}{\langle 0|0\rangle}$ Vacuum states

$\hat{T}$ is the operator of T-ordering

$$\hat{T}\hat{F}_1(t_1)\hat{F}(t_2) = \begin{cases} \hat{F}_1(t_1)\hat{F}_2(t_2) & t_1 > t_2 \\ \mp\hat{F}_2(t_2)\hat{F}_1(t_1) & t_2 > t_1 \quad \begin{cases} - \text{ fermions} \\ + \text{ bosons}\end{cases}\end{cases}$$

($t_1 = t_2$ is defined only by limit)

How can Green-function be used?

One particle operator

$$F^{(1)} = \sum_{ij} f^{(1)}_{ij}\, a_j^+ a_i = \int \psi^+_{(x)}\,\hat{f}^{(1)}\psi(x)\,dx$$

$$\langle F^{(1)}\rangle = \mp i\int dx\, f^{(1)}(x)\, G(x_1,t_1;x_2,t_2)\Big|_{\substack{x_1=x_2\\ t_1=t_2+i0}}$$

$$\hat{H} = \int dx\,\psi^+(x)\hat{H}_1(x)\,\psi(x)$$

$\hat{H}_1(x) = -\dfrac{1}{2m}\nabla^2 + V(x)$ is one example

$$i\frac{\partial G(1,2)}{\partial t_1} = H_1(x_1)\, G(1,2) + \delta(x_1 - x_2)\,\delta(t_1 - t_2)$$

exactly the Schrödinger equation written before to determine Green-function using

Feynman path integral

$$G = \begin{cases} G^R & t_1 > t_2 \\ G^A & t_1 < t_2 \end{cases}$$

This is the way field theorists generalise definition at $T = 0$ to write down perturbation theory.

Generating Functional – T-exponent

$$Z[\eta(x,t), \eta^*(x,t)] = \left\langle T \exp\left[i \int dt\, dx\, \hat{\psi}(x,t)\,\eta(x,t) + \hat{\psi}(x,t)^+ \eta^*(x,t) \right] \right\rangle$$

functional derivatives over η and η^* give products of G

$$\langle T \underbrace{\psi(1)}_{x_1 t_1} \ldots \psi(n)\, \psi^+(\bar{1}) \ldots \psi^+(\bar{n}) \rangle$$

$$= (-1)^n \frac{\delta^{2n}}{\delta\eta(1) \ldots \delta\eta(n)\, \delta\eta(\bar{1}) \ldots \delta(\bar{n})} Z$$

If $H = H_0 + H_1[\psi, \psi^+]$

$$Z[\eta, \eta^*] = \text{const} \left\langle \exp\left[-i \int dt\, H_1\left[\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\eta^*}\right] \right] \right\rangle Z_0[\eta, \eta^*]$$

goes into constant

Doubling of the Coordinates – necessary as seen before

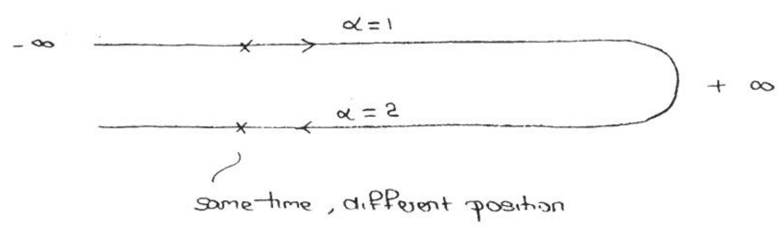
Keldysh Diagram Technique

A. Schmidt J. Low Temp Phys. 49, 609 (1982) - We will follow.

L.V. Keldysh Sov. Phys. JETP 20, 1018 (1965)

Idea

Instead of one contour in time, consider double



$$\vec{\psi}(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} \quad , \quad \vec{\psi}^{+}(t) = \begin{pmatrix} \psi_1^{+} \\ \psi_2^{+} \end{pmatrix}$$

$$\int_{-\infty}^{\infty} dt\, F[\psi, \psi^+] \longleftarrow \oint dt\, F[\vec{\psi}, \vec{\psi}^{+}]$$

$$\equiv \int_{-\infty}^{\infty} dt\, F[\psi_1, \psi_1^{+}] - \int_{-\infty}^{\infty} dt\, F[\psi_2, \psi_2^{+}]$$

This doubling allows us to introduce time ordering into Keldysh formalism

Crucial point is time ordering ; T_c

$$T_c\, \psi_\alpha(t)\psi_{\alpha'}^{+}(t') = \begin{array}{c|c} & \alpha \longrightarrow \\ \hline \alpha' \downarrow & \begin{pmatrix} \underset{\text{first later}}{T\psi_1(t)\psi_1^{+}(t')} & \psi_2(t)\psi_1^{+}(t') \\ \psi_2^{+}(t')\psi_1(t) & \underset{\text{first earlier}}{T^{-1}\psi_2^{+}(t')\psi_2(t)} \end{pmatrix} \end{array}$$

sign?

does not require T-ordering

Observables:

Density $\quad n(x,t) = \langle \psi(x,t)\, \psi^{+}(x,t) \rangle$

$$\vec{J}(x,t) = \langle \nabla\psi \cdot \psi^{+} - \psi \nabla \psi^{+} \rangle$$

Generating Functional

$$\vec{\eta}(x,t) = \begin{pmatrix} \eta_1(x,t) \\ \eta_2(x,t) \end{pmatrix}$$

$$Z[\vec{\eta},\vec{\eta}^*] = \left\langle \exp\left(i \int dt\, H_1 \left[\frac{\delta}{\delta \vec{\eta}}, \frac{\delta}{\delta \vec{\eta}^*} \right] \right) \right\rangle Z_0$$

Averaging: done not over ground state but over density matrix.

(Assume for now that we are in equilibrium)

$$\rho = \exp\left[\frac{\Omega - H}{T} \right] \quad , \; \Omega \text{ is the thermodynamic potential}$$

$$\underbrace{G_{\alpha\alpha'}(t,t')}_{\text{stop writing } x,\, x' \text{ for now}} = \frac{1}{Z} \frac{\delta^2 Z[\vec{\eta},\vec{\eta}^*]}{\delta \eta_\alpha(t)\, \delta \eta^*_{\alpha'}(t)} \Bigg|_{\eta=0} = \begin{pmatrix} \overbrace{G^c}^{\text{Causal G.F.}} & G^- \\ G^+ & \underbrace{\bar{G}^c}_{\text{anticausal}} \end{pmatrix}$$

In normal notation

$$G^c = i \langle T \psi(t)\, \psi^+(t') \rangle$$

$$G^- = -i \langle \psi(x,t)\, \psi^+(t') \rangle$$

$$G^+ = \pm \langle \psi^+(t')\, \psi(t) \rangle$$

What are those objects?

$$\bar{G}^c = -i \langle T^{-1} \psi(t)\, \psi^+(t') \rangle$$

Recall

$$a_j^+ a_j = \begin{cases} n_j & \text{fermions} \\ N_j & \text{bosons} \end{cases}$$

$$a_j a_j^+ = \begin{cases} 1 - n_j \\ N_j + 1 \end{cases}$$

Let us consider free particles to see what $G^{\pm}$ are in this case

$$x, t \longrightarrow p, E \begin{cases} \nearrow \varepsilon & \text{fermions} \\ \searrow \omega & \text{bosons} \end{cases}$$

Fermionic

$$G^+(p, E) = 2\pi i \, n_\varepsilon \, \delta\left(\varepsilon - \frac{p^2}{2m}\right)$$

density in Energy and momentum

$$n_\varepsilon = \frac{1}{e^{\varepsilon - \mu / T} + 1}$$

$$G^-(p, E) = -2\pi i \, (1 - n_\varepsilon) \, \delta\left(\varepsilon - \frac{p^2}{2m}\right)$$

Bosonic

$$D^+(p, \omega) = 2\pi i \, N_\omega \, \delta(\omega - \omega_k)$$

$$N_\omega = \frac{1}{e^{\omega / T} - 1}$$

$$D^-(p, \omega) = 2\pi i \, (1 + N_\omega) \, \delta(\omega - \omega_k)$$

$$G^c(t, t') = G^-(t, t')\,\theta(t - t') + G^+(t, t')\,\theta(t' - t)$$

$$\bar{G}^c(t, t') = G^-(t, t')\,\theta(t' - t) + G^+(t, t')\,\theta(t - t')$$

Write

$$G^c(t,t') = G^+(t,t') + \Theta(t-t')\left[G^- - G^+\right] = G^+ + G^A$$

$$\bar{G}^c(t,t') = G^- + G^R$$

To see this notice that

$$G^+ - G^- = \delta(\varepsilon - p^2/2m)\, 2\pi i = \underbrace{\frac{1}{\varepsilon - p^2/2m - i0}}_{G^R} - \underbrace{\frac{1}{\varepsilon - p^2/2m + i0}}_{G^A}$$

after F.T.

$$= \Theta(t-t')(G^+ - G^-) + \Theta(t'-t)(G^+ - G^-)$$

G^R, G^A carry no information about distribution function, just pure dynamics

Trick: Unitary Transformation

(in order to reduce the number of variables – because of dependencies)

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\vec{\xi}(t) = U\vec{\eta} \qquad \vec{\varphi} = U\vec{\psi}$$

$$G_{\beta\beta'} = U G_{\alpha\alpha'} U^{-1}$$

$$G_{11} = \frac{1}{2}\left(G^c - G^- - G^+ + \tilde{G}^c\right) = 0 \qquad G^+ + G^- = G^c + \bar{G}^c$$

$$G_{12} = \frac{1}{2}\left(G^c - G^+ + G^- - \bar{G}^c\right) = G^R$$

$$G_{21} = G^A$$

$$G_{22} = \frac{1}{2}\left(G^c + G^+ + G^- + \bar{G}^c\right) = \underbrace{G^K}_{\text{Keldysh G.F}} = \underbrace{F}_{\text{or F function}}$$

Keldysh GF $\quad F = G^+ + G^-$

$$= \begin{cases} 2n-1 \\ 2N+1 \end{cases} \times \left(G^R - G^A\right)$$

– This equation is true for equilibrum

$$G^R = (G^A)^*$$

Statistics and dynamics are now separated

Penalty is having to work with 2×2 matrices.

$$\hat{G}_{\beta\beta'} = \begin{pmatrix} 0 & G^A \\ G^R & F \end{pmatrix}$$

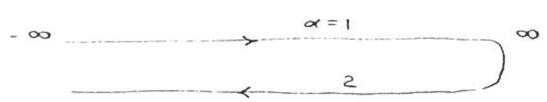
two independent functions

Now we are ready to derive the Langevin equation and find what corresponds to ψ in path integrals. Later we will attack non-equilibrum functions

9/3/94

Recall :

Double contour, one fields in one the other in the other



$$\vec{\psi} = \begin{matrix} \psi_1(t) \\ \psi_2(t) \end{matrix}$$

Time ordering along contour $\Rightarrow$ $\quad G_{\alpha\alpha'}(t,t') = \begin{pmatrix} G^c & G^- \\ G^+ & \tilde{G}^c \end{pmatrix}$

Not all components are independent

So we made a linear transformation or rotation

$$\vec{\varphi} = U\vec{\psi} \quad , \quad G \rightarrow UGU^{-1}$$

We found after rotation

$$\hat{G}_{\beta\beta'} = \begin{pmatrix} 0 & G^A \\ G^R & F \end{pmatrix} \qquad G^R = G^{A*}$$

Two green functions

(Bosonic Green functions

$$\hat{D}_{\beta\beta'} = \begin{pmatrix} 0 & D^A \\ D^R & D^K \end{pmatrix}$$

)

In equilibrium,

$$F_\varepsilon = (2n_\varepsilon - 1)(G^R - G^A) \quad , \quad 2n_\varepsilon - 1 = \tanh\left(\varepsilon - \mu/2T\right)$$

$$D^K_\omega = (2N_\omega + 1)(D^R - D^A) \quad , \quad 2N_\omega + 1 = \mathrm{cotanh}\left(\omega/2T\right)$$

Checked only for boso G.F but true in general

Functional Integral

Harmonic Oscillator with self-frequency ω_0

Temperature T

$$D^R_\omega = \left(D^A_\omega\right)^* = \frac{1}{\omega - \omega_0 + i0} + \frac{1}{\omega + \omega_0 + i0}$$

in equilibrium

$$= \frac{2\omega}{\omega^2 - \omega_0^2 + i0}$$

$$D^K = 4\pi i\, \omega\, (2N_\omega + 1)\, \delta(\omega^2 - \omega_0^2)$$

(using above relations)

Generating functional

$$Z_h[\zeta(t), \zeta^*(t)] = \exp\left[-\frac{i}{2}(\vec{\zeta} \,|\, \hat{D} \,|\, \vec{\zeta}\,)\right]$$ (after integrating out fields)

$$(\vec{\zeta} \,|\, \hat{D} \,|\, \vec{\zeta}\,) = \int dt\,dt'\, \zeta_\beta(t)\, D_{\beta\beta'}(t,t')\, \zeta^*_{\beta'}(t')$$

Introduce Action

$$\underbrace{S_h}_{\text{harmonic oscill}}[\vec{\varphi}(t), \underbrace{\vec{\varphi}(t)^*}_{\text{field not operator}}]$$

$$S_h = \frac{1}{2}(\vec{\varphi} \,|\, D^{-1} \,|\, \vec{\varphi})$$

$$Z_h[\zeta, \zeta^*] = \int D\varphi D\varphi^* \exp\left[\frac{i}{2} S[\varphi, \varphi^*] + \mathrm{Re}\int dt\, \zeta\varphi\right]$$

Integral is Gaussian and we get expression of generating function

$$D^{-1} = \begin{pmatrix} \dfrac{D^K}{D^R D^A} & (D^R)^{-1} \\ (D^A)^{-1} & 0 \end{pmatrix}$$

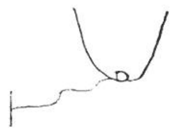
Substitute for harmonic oscillator

where did N_ω go — in D^+ ?

$$D^{-1} = \frac{1}{2\omega}\begin{pmatrix} 4i0\omega \coth \omega/2T & \omega^2 - \omega_0^2 + 2i0\omega \\ \omega^2 - \omega_0^2 - 2i0\omega & 0 \end{pmatrix}$$

Now, what if harmonic oscillator is connected to environment

Oscillator connected to string



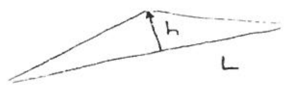
$x(t)$ - coordinate of oscillator

z - coordinate along string

$y(z)$ - displacement of string

Assume $\quad y(0,t) = x(t)$

and string is infinite



$\delta L \sim \frac{h^2}{L} \xrightarrow[L\to\infty]{} 0 \qquad$ no static force

Suppose string has mass per unit length ρ

σ is a tension in string

$c = \sqrt{\frac{\sigma}{\rho}}$ sound velocity

$\hat{y} = \sum_k \frac{1}{(2L\rho\,\omega_k)^{1/2}}\,\phi_k$ — normal modes — corresponds to h.o. with frequency ω_k

$\hat{\psi} = \sqrt{2m\omega_0}\,\hat{x} \qquad \hat{\Psi} = \sqrt{2m\omega_0}\; y(0)$

Generating functional

$$Z_s = \left\langle T_c \exp\left[i\int dt\; \mathbb{H}(t)\Psi(t) \right] \right\rangle_{\text{Before rotation}}$$

form string

capital η

$$= \exp\left[-\frac{i}{2}\left(\mathbb{Z}\,|R|\,\mathbb{Z} \right) \right]_{\text{after rotation}}, \qquad \mathbb{Z} = U\mathbb{H}$$

capital ζ

$$R(\omega) = \int \frac{dk}{2\pi} \frac{m\omega_0}{\rho c k} D_h(\omega, \omega_k)$$

mass from normalisation of coords in terms of h.o.

$$= \frac{i m \omega}{\rho c} \begin{pmatrix} 0 & \frac{1}{\omega - i0} \\ \frac{1}{\omega + i0} & \frac{2\omega}{T^2} \coth \frac{\omega}{2T} \end{pmatrix}$$

like h.o. with zero self-frequency

$$S_s[\Phi] = \frac{1}{2}(\Phi | R^{-1} | \Phi)$$

exactly as GF to functional integral

Multi-dim. system → 1d system — corresponding to 0 point — all other d.o.f. integrated out

If string and oscillator strongly correlated we must set $\Phi = \varphi$

$$S_{h+s} = (\varphi | D_h^{-1} + R^{-1} | \varphi) \equiv (\varphi | D_{h+s}^{-1} | \varphi)$$

$$D_{h+s}^{-1} = \frac{1}{2\omega_0} \begin{pmatrix} i\gamma\omega \coth \omega/2T & \omega^2 - \omega_0^2 - i\gamma\omega \\ \omega^2 - \omega_0^2 + i\gamma\omega & 0 \end{pmatrix}, \quad \gamma = \frac{2\rho c}{m}$$

Need to substitute action into functional integral

D_{h+s}^{-1} looks like free oscillator but with friction γ — energy is not conserved

To get Langevin equation and learn meaning of field operators we proceed,

$$S_{s+h} = \int dt\, dt' \left\{ \varphi_1 D_{11}^{-1} \varphi_1 + \varphi_1 D_{12}^{-1} \varphi_2 + \varphi_2 D_{21}^{-1} \varphi_1 \right\}$$

(fields are real)

$$= S_1 + S_2$$

$\omega \to \partial/\partial t$ for time representation

$$S_1 = -\frac{1}{\omega_0}\int dt\, \varphi_1(t)\left[\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_0^2\right]\varphi_2(t)$$

$$S_2 = \frac{i}{2\omega_0}\int dt\,dt'\, \varphi_1(t)W(t-t')\varphi_1(t')$$

$$W(\omega) = \frac{\gamma\omega}{m}\coth\frac{\omega}{2T} = \begin{cases} \dfrac{2T\gamma}{m}, & \omega \to 0 \\[2ex] \dfrac{\gamma|\omega|}{\omega_0}, & T \to 0 \end{cases} \qquad W(t) \propto \delta(t)$$

$$|\langle x_f| e^{-iHt}|x_i\rangle|^2 = J(x_f, x_f; x_i, x_i)$$

$$= \int \mathcal{D}\vec{x}(t)\, \exp\left\{iS[\vec{x}(t)]\right\}$$

Recall

$x_- = x_1 - x_2$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$X = \dfrac{x_1 + x_2}{2}$

$\vec{x}' = \begin{pmatrix} x_- \\ X \end{pmatrix}$

Wigner Transformation

like Keldysh rotation

$x(t) \longleftrightarrow \varphi(t)$

$$S = S_1 + S_2$$

$$S_1 = -\int_{t_i}^{t_f} dt\, \Big\{x_-\, m\left[\ddot{X} + \gamma\dot{X}\right] + \left[U(x_1) - U(x_2)\right]\Big\}$$

$$S_2 = \frac{i}{2}\int_i^f dt\,dt'\, x_-(t)\,W(t-t')\,x_-(t)$$

But $\varphi_{1,2} = \dfrac{\psi_1 \mp \psi_2}{\sqrt{2}}$

So expressions are equivalent if we substitute

a) $X = \dfrac{\varphi_2}{2\sqrt{m\omega_0}}$, $\quad x_- = \dfrac{\varphi_1}{\sqrt{m\omega_0}}$ $\qquad$ c) $\gamma = \dfrac{2\rho c}{m}$

b) $U(X \pm x/2) \longrightarrow \dfrac{\omega_0^2}{2} m\left(\varphi_2 \pm \varphi_1/2\right)^{1/2}$ $\qquad$ d) $W(\omega) = \dfrac{\gamma\omega}{m}\coth\dfrac{\omega}{2T}$

Statistics and dynamics are separated as Wigner transformation

$x \longleftrightarrow$ quantum dynamics

$X \longleftrightarrow$ classical evolution of wave packet

For $S_1 + S_2$ we can ~~use~~ obtain Langevin equation

(Similar calculation done in Feynman & Hibbs without Keldysh)

Crucial point – enviroment has a finite density of states

γ describes how energy of particle decreases

– mostly emission and not T dependent

$W(\omega)$ describes thermal fluctuations which agitate particles

– no vanishing even at $T = 0$ (zero-point fluctuations)

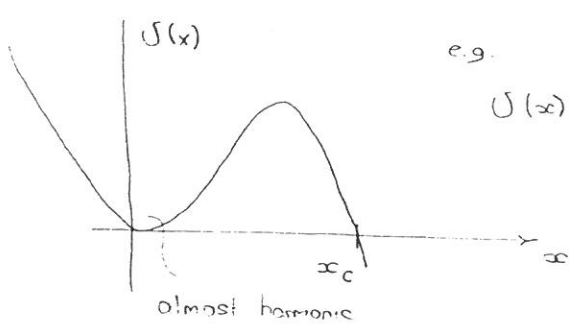
Caldeira – Legget Problem Ann. Phys. 149 (1983), 374

153 (1984), 445

Legget et al Rev. Mod. Phys 59, v1, p1

A. Shmidt Ann. Phys. 170 (1985), 333

Now assume that potential is metastable



e.g.

$$U(x) = \frac{m\omega_0^2}{2} x^2 \left(1 - \frac{x}{x_c}\right)$$

Particle has a tendency to tunnel out of this well (recall instanton calc.)

Now add coupling to some enviroment

1. Local vibration modes (I.M.Lifshits 1960)

Consider a perfectly periodic chain of atoms located at points

$$R_i = ia \qquad i = 0, \pm1, \pm2,.....$$

with a harmonic interaction between them. Let all of the atoms to have the mass m, except the one at R_0, which has a mass $m(1 + \Delta)$, so that the Hamiltonian is

$$H = (1/2)\ \Sigma_i\ \{m(1 + \Delta\delta_{i0})(du_i/dt\ )^2 + k(u_i\text{-}u_{i+1})^2\}$$

where u_i is a displacement of i-th atom.

a) Write down the equations of motion. Determine the spectrum of the phonons and scetch the density of states in the phonon band.
b) Consider excitations of the system above the top of the phonon band For what sign of Δ this exitations exist?
c) Argue that this excitations are localized near zero.
d) Estimate the size of this local excitation.

2. Solid solutions

Semiconductor solid solution $A_xB_{1\text{-}x}$ is a crystal with each site occupied by an atom A with probability x and by an atom B with a probability (1-x). Such systems are known to be pretty well described by so-called "virtual crystall" approximation where each site is occupied by x atoms A and (1-x) atoms B. The gap in this approximation is a linear function of x; $\Delta(x) = \Delta(0)x + \Delta(1)(1\text{-}x)$. Density fluctuations create a random potential.

a) calculate a variance of atom A occupancy of a site.
b) assuming that actual scales are much bigger than a lattice constant, prove that the random potential is Gaussian and calculate the correlation function. (Assume no correlations between neighbouring sites).
c)Evaluate the exponential factor in the tail of the conductance band, considering a well, which can host an electron with an energy E and estimating the probability of this fluctuation.
d) How the tail depends on the concentration x? When it is muximal?

Massachusetts Institute of Technology
Physics Department
Spring 1994

8.382: Selected Topics | Problem Set #3 | Due: **04/06/94**

Boltzman equation for electron - electron collisions

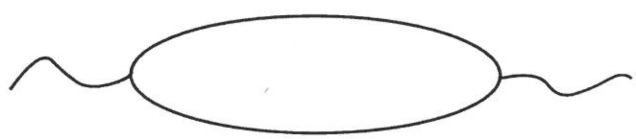
Consider a system of fermions with the pair interaction given by a potential

$$U(\mathbf{r}_1-\mathbf{r}_2).$$

a) Write the bare Greens function of the interaction $D_0(\omega,\mathbf{q})$ in Keldysh notation.

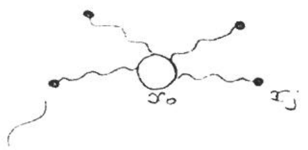
b) Draw the first order diagrams for the self energy. Argue that there is no contribution to the collision integral from these terms. Why?

c) Write down the expression for the loop $\Pi(\omega,\mathbf{q})$ in Keldysh notation. Show that $\Pi_{22} = 0$, that Π_{12} (Π_{21}) have retarded(advanced) analytical properties and that in the equilibrium $\Pi_{11}= (\Pi_{12}-\Pi_{12})(2N_\omega+1)$



d) Write down the Keldysh expression for the Greens function of the interaction.

e) Derive the expression for the collision integral.



Small masses attached to particle by springs

If particle tunnels, it must carry the masses

$$\mathcal{L} = \frac{1}{2} M \dot{x}_0^2 - U(x_0) + \frac{1}{2} \sum_{j=1}^{N} m_j \left[\dot{x}_j^2 - \omega_j^2 \left(x_j - \frac{C_j}{m_j \omega_j^2} x_0 \right)^2 \right]$$

$$I(\omega) = \frac{\pi}{2} \sum_j c_j^2 \, \delta(\omega - \omega_j) / m_j \omega_j \quad \propto \omega \quad \text{(assumption)}$$

$x_j \Rightarrow x_z \qquad T = 0 \qquad$ if string

$$\mathcal{L} = \int dz \left\{ \left[\rho \dot{x}_z^2 - \sigma x_z'^2 \right] + \delta(z) \left[\frac{m}{2} \dot{x}_z^2 - U(x_z) \right] \right\}$$

$$x' = \frac{\partial x_z}{\partial z}$$

Equation of Motion

$$\rho \ddot{x}_z - \sigma x_z'' + \delta(z) \left[m \ddot{x}_z - \frac{\partial U}{\partial x_z} \right] = 0$$

x_z is continuous, $\quad x_z'$ is not necessarily so

$$\rho \ddot{x}_z - \sigma x_z'' = 0 \qquad z \neq 0$$

$$m \ddot{x}_0 - \frac{\partial U}{\partial x_0} = \sigma \left[x'_{0+} - x'_{0-} \right]$$

Instanton: $t = i\tau$ — Fourier transform

$$\rho \omega^2 x_z(\omega) - \sigma x_z''(\omega) = 0$$

$$x_z(\omega) = x_0(\omega) \, e^{-\omega |z| / c}, \qquad c = \sqrt{\frac{\sigma}{\rho}}$$

$$m\omega^2 x_0(\omega) + \left[\frac{\partial U}{\partial x_0}\right]_\omega = \underbrace{-2\rho c|\omega| x_0(\omega)}_{\text{friction}}$$

Non-linear and non-local problem can not be solved (apparently)

but Calderra-Legget solved for $m \to 0$, $\gamma = \frac{\rho c}{m} \to \infty$

$$\int d\tau' \, W(\tau-\tau') x_0(\tau') + U'(x_0(\tau)) = 0$$

$\frac{x_c}{m} \to$ const

$$W(\omega) = 2\rho c|\omega|$$

$$x_0(\tau) = \frac{2a\tau_0^2}{\tau^2+\tau_0^2}, \quad \tau_0 = \frac{2\rho c}{m\omega_0^2} \quad a^2 = \frac{2x_c}{3m\omega_0^2}$$

$$\Gamma \propto e^{-S_0}, \quad S_0^{CL} = 2\int_{-\infty}^{0} d\tau \, m\left[\dot{x}^{CL}\right]^2 = \pi\rho c a^2$$

bigger friction

$\to$ slower decay

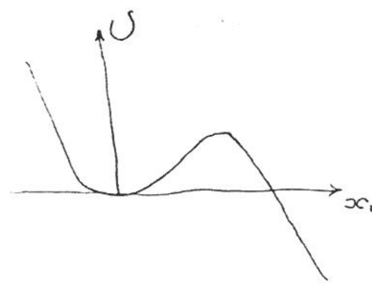
14/3/94

Recall

Calderra-Leggett problem

Macroscopic Quantum tunneling

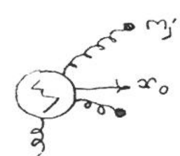
Tunneling with dissipation



$$\mathcal{L} = \frac{m}{2}\dot{x}_0^2 - U(x_0)$$

$$+ \frac{1}{2}\sum_{j=1}^{N} m_j \dot{x}_j^2 - \mathcal{L}_j$$

$$- \sum_{j=1} \frac{\omega_j^2}{2}\left(x_j - \frac{c_j}{m_j\omega_j} x_0\right)^2$$

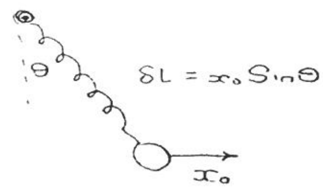


in $U(x_0)$

$$= \mathcal{L}_0 + \sum_{j=1} \mathcal{L}_j + A x_0^2 + \sum_j \frac{c_j}{m_j} x_j x_0$$

Lagrangian of quantum particle

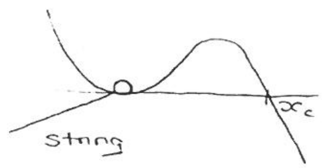
$$\frac{c_j}{m_j\omega_j^2} = \sin\theta_j$$



$$\sum_j \frac{c_j}{m_j\omega_j^2}\,\delta(\omega-\omega_j) \rightarrow \text{const} \propto \gamma$$

Enviroment has constant DOS

String



z – coordinate along string

$x(z,t)$ – displacement

$$z \neq 0 \qquad \rho\ddot{x} - \sigma x'' = 0$$

$$z = 0 \qquad m\ddot{x}_0 - \frac{\partial U}{\partial x_0} = \sigma\left[x'_{+0} - x'_{-0}\right]$$

Instanton

$$t \longrightarrow i\tau \xrightarrow{F.T.} \omega$$

$$x(z,\omega) = x_0(\omega)\, e^{-\omega|z|/c} \qquad c = \sqrt{\frac{\sigma}{\rho}}$$

$$m\omega^2 x_0(\omega) + \left[U'(x)\right]_\omega = -2\rho c|\omega| x_0$$

CL : $\quad m = 0$

$$\int d\tau'\, W(\tau-\tau')\,x_0(\tau') + U'(x_0(\tau)) = 0$$

$$x_0(\tau) = \frac{2a\tau_0^2}{\tau^2+\tau_0^2}, \qquad \tau_0 = \frac{2\rho c}{m\omega_0^2}$$

Decay rate of state $\quad \Gamma \propto e^{-S_0}, \quad S_0 = \pi\rho c a^2, \quad a^2 = \frac{2x_c}{3m\omega_0^2}$

When dissipation increased particle becomes more classical and tunneling is suppressed.

This problem leads to interesting physics ( interacting instanton gas, etc.)

Josephson junction series.

Non-Equilibrium Dynamics

Hamiltonian $\quad H = \underbrace{H_0}_{\text{free particles in external fields}} + \underbrace{H_i}_{\text{interaction}}$

$$\underbrace{H_0(t)}_{\substack{t\text{-dependent}\\ \text{fields}}} = \int d\vec{r}\, \psi_0^+(r,t)\left\{ \underbrace{\xi}_{\text{e.g. } \xi(p) = \frac{p^2}{2m}}\left(-i\nabla - \frac{e}{c}A(r,t)\right) + e\phi(r,t)\right\} \psi_0(r,t) \quad + \underbrace{H_T}_{\text{Thermal part}}$$

A – vector potential

ϕ – scalar potential

$t = -\infty$ – particles assumed to be free $(H_i = 0)$ and in equilibrium

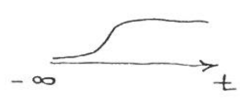
$$H = H_0$$

Density Matrix $\rho(t)$, $\quad \rho(-\infty) = \rho_0 = \exp\left[\frac{\Omega - H_0}{T}\right]$

Example of H_i

Electron – Phonon interaction

$$H_{el-ph} = g\int \psi_0^+(r,t)\,\psi_0(r,t)\left(\varphi(r,t) + \varphi^+(r,t)\right)$$



Interaction Representation

$$i\frac{\partial \psi_0}{\partial t} = [\psi_0, H_0]$$

$$i\frac{\partial \rho}{\partial t} = [H_i, \rho]$$

$$\begin{array}{ccc} & H & \\ \psi \leftarrow H_0 & & H_i \rightarrow \rho \end{array}$$

Solution for $\rho(t)$:

$$\rho(t) = S(t,-\infty)\rho_0 S^+(t,-\infty)$$

$$= S(t,-\infty)\rho_0 S(-\infty,t)$$

S-Matrix , $S = T\exp\left\{-i\int_{-\infty}^{t} H_i(t')\,dt'\right\}$ (Can be found in Mahan)

Consider an operator L_0

Average $\langle L_0(t)\rangle = \mathrm{Tr}\{\rho(t) L_0(t)\}$

or Dynamics : better in Heisenberg representation

$$i\frac{\partial\Psi}{\partial t} = [\Psi, H] \qquad *$$

Let us take ρ at certain time $t=0$,

$$\rho = \rho(0)$$

$$\rho(0) = S(0,-\infty)\overbrace{\rho_0}^{\text{at } -\infty} S^+(0,-\infty)$$

In the equilibrium , $\exp\left[\frac{\Omega - H}{T}\right]$

But we don't want equilibrium — we want a general procedure.

Let us require $\Psi(\vec{r},t)\big|_{t=0} = \Psi_0(r,t=0)$

$$\Psi(r,t) = S(0,t)\Psi_0(r,t)S^+(0,t)$$ (formal solution of $*$)

$$S(t,t') = T\exp\left\{-i\int_{t}^{t'} H_i(t')\,dt'\right\}$$

$$= S(t',-\infty)\,S^+(t,-\infty)$$

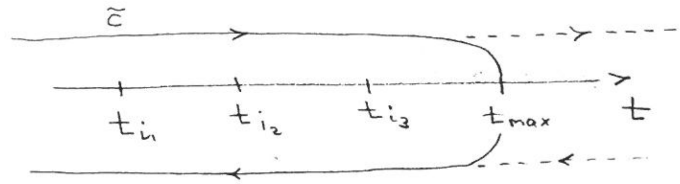
Consider

$$\langle T L_1(t_1) L_2(t_2) \ldots \rangle$$

$$= \mathrm{Tr}\left\{ \rho L_1(t_1) L_2(t_2) \ldots \right\}$$

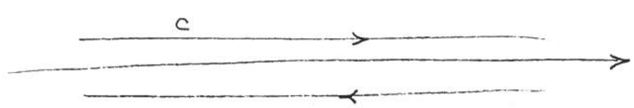
$$= \mathrm{Tr}\left\{ S(0,-\infty)\rho_0 S(-\infty,0)\, T\left[S(0,t_1) L_{1_0} S(t_1,t_2) L_{2_0} S(t_2,t_3) \ldots \right] \right\}$$

$$= \mathrm{Tr}\left\{ \rho_0 \tilde{T}_c \left[S_c L_{1_0}(t_1) L_{2_0}(t_2) \ldots \right] \right\}$$



Better to $\times$ $S(t_{max}, +\infty) S(+\infty, t_{max})$ (equal to unity)

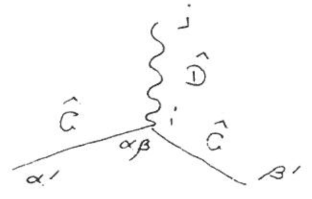
$$= \mathrm{Tr}\left\{ \rho_0 T_c \left[S_c L_{1_0}(t_1) L_{2_0}(t_2) \ldots \right] \right\}$$



Before rotation $\quad G_{\alpha'\alpha} = \begin{pmatrix} G^c & G^- \\ G^+ & \tilde{G}^c \end{pmatrix}$ electron GF

$$D_{ij} = \begin{pmatrix} D^c & D^- \\ D^+ & \tilde{D}^c \end{pmatrix}$$ phonon GF

For interaction, we need vertex



$$\gamma^i_{\alpha\beta} = \delta_{\alpha\beta} \otimes (\sigma^z)_{\alpha i}$$

Contour remains same

from T-ordering $\quad T\psi(t)\psi^\dagger(t') = \begin{cases} \psi(t)\psi^\dagger(t') \\ \psi^\dagger(t')\psi(t) \end{cases}$

Rotation

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1 - i\sigma_y}{\sqrt{2}}$$

$$F(\varepsilon) = S_0(\varepsilon)\left[G^A(\varepsilon) - G^R(\varepsilon)\right]$$
$$S_0(\varepsilon) = 2n_F(\varepsilon)-1 = \tanh \varepsilon/2T$$

$$G = \begin{pmatrix} 0 & G^A \\ G^R & F \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D^A \\ D^R & D^K \end{pmatrix}$$

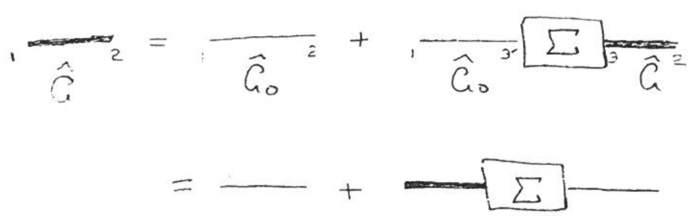
$$D^K(\varepsilon) = S_0(\varepsilon)\left[D^A(\varepsilon) - D^R(\varepsilon)\right]$$
$$S_0(\varepsilon) = 2N_\varepsilon + 1 = \coth \varepsilon/2T$$

Vertices $\quad \gamma^1_{\alpha\beta} = \frac{1}{\sqrt{2}}\,\delta_{\alpha\beta}, \quad \gamma^2_{\alpha\beta} = \frac{1}{\sqrt{2}}(\sigma_x)_{\alpha\beta} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\left[\begin{pmatrix} 0 & G^A \\ G^R & F \end{pmatrix}^{-1}\right] = \begin{pmatrix} -F/G^RG^A & (G^R)^{-1} \\ (G^A)^{-1} & 0 \end{pmatrix}$$

$$\Rightarrow \quad \Sigma_{\alpha\beta} = \begin{pmatrix} \Sigma^K & \Sigma^R \\ \Sigma^A & 0 \end{pmatrix} \qquad \text{Self-energy}$$

Dyson Equation



This correspondence is trivial for usual but works even for Keldysh GF

$$\hat{G} = \hat{G}_0 + \hat{G}\hat{\Sigma}G_0$$

$$= \begin{pmatrix} 0 & G_0^A + G^A\Sigma^A G_0^A \\ G_0^R + G^R\Sigma^R G_0^R & F_0 + G^R\Sigma^K G_0^A + F\Sigma^A G_0^A + G^R\Sigma^R F_0 \end{pmatrix}$$

Three Dyson equations —

$$G^{R(A)}_{(\varepsilon,p)} = \frac{1}{\varepsilon - \xi(p) - \Sigma^{R(A)}}$$

As for equilibrium but true even if there is external field

What is F_0 ? F contains distribution function. F_0 does depend on ρ_0 but physically it should not be relevant. Avoid by writing in differential form.

Dyson equation in the Differential form

– Boltzman Equation

$$G_0^{-1}(r,t) = i\frac{\partial}{\partial t} - \xi\left(-i\nabla - \frac{e}{c}A(r,t)\right) - e\Phi(r,t)$$

$$= i\frac{\partial}{\partial t} - H_0$$

Apply to GF before rotation

$$\begin{cases} G_0^{-1}(1)\, \underbrace{G_0^c(2,1)}_{r_2,t_2} = \delta^3(r_1-r_2)\,\delta(t_1-t_2) \\ \text{real GF} \\ G_0^{-1}(1)\, \tilde{G}_0^c(2,1) = -\delta^3(r_1-r_2)\,\delta(t_1-t_2) \\ G_0^{-1}(1)\, G^{\mp}(2,1) = 0 \end{cases}$$

$$G_0^{-1}(1)\, \hat{G}_0(2,1) = \hat{\sigma}_z\, \delta^4(2-1)$$

After rotation

$$G_0^{-1}(1)\, \hat{G}_0(2,1) = \underbrace{\hat{\sigma}_x}_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\, \delta^4(2-1)$$

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In Dyson equation

$$G_0^{-1}(1)\, G(2,1) = \sigma_x\left[\delta^4(1-2) + \int d^4(3)\, \hat{G}(2,3)\,\hat{\Sigma}(3,1)\right]$$

$$\left(G_0^{-1}(2)\right)^* G(2,1) = \sigma_x\left[\delta^4(1-2) + \int d^4(3)\, \hat{\Sigma}(2,3)\,\hat{G}(3,1)\right]$$

To proceed we need to apply a semi-classical approximation

Semi-Classics

$$G_0^{-1}(1) = i\frac{\partial}{\partial t} - \xi\left(-i\nabla - \frac{e}{c}\vec{A}(1)\right) - e\phi(1)$$

Wigner - Transformation $\quad t = \frac{t_1 + t_2}{2} \qquad \vec{r} = \frac{\vec{r}_1 + \vec{r}_2}{2}$

$$\theta = t_1 - t_2 \qquad \vec{\rho} = \vec{r}_1 - \vec{r}_2$$

In classical mechanics we consider motion in phase space. In QM it is impossible to fix both. Here we will keep $\vec{r}$ semi-classical and use FT in $\vec{\rho}$ to define momentum

Scales $\quad \theta \sim \hbar/\varepsilon \qquad \rho \sim \hbar/p \qquad (\varepsilon, p)$ are fermi if there is a fermi surface

t, r are macroscopic

Semi-classical approximation — lowest order in $\rho\frac{\partial}{\partial r}$ or $\theta\frac{\partial}{\partial t}$

Fourier Transform $\quad (\vec{\rho}, \theta) \longrightarrow (\vec{p}, \varepsilon)$

$$\underbrace{\hat{G}(\varepsilon, \vec{p}; \vec{r}, t)}_{\text{definition}} \equiv \int d\theta\, d\vec{\rho}\; \hat{G}(\vec{\rho}, \theta; \vec{r}, t)\, e^{i\theta(\varepsilon + e\phi(\vec{r},t)) - i\vec{\rho}\cdot(\vec{p} + e\vec{A}(\vec{r},t)/c)}$$

$$\frac{\partial^2}{\partial r\,\partial p} \ll 1, \qquad \frac{\partial^2}{\partial t\,\partial \varepsilon} \ll 1 \qquad \text{approximation}$$

necessary if e-m field is changing

So $\hat{G}$ is still Gauge dependent

but equations of motion must be Gauge invariant (classically)

Taking the two Dyson equations we can either add or subtract. First we will consider the difference.

$$\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} = 2\frac{\partial}{\partial t}$$

because of conjugation

$$\xi(-i\nabla_1) - \xi(-i\nabla_2) = \xi\left(p + i\frac{\partial}{\partial r}\right) - \xi\left(p - i\frac{\partial}{\partial r}\right)$$

$$= \frac{\partial \xi}{\partial \vec{p}}\frac{\partial}{\partial \vec{r}} = \vec{V}\frac{\partial}{\partial \vec{r}}$$

velocity

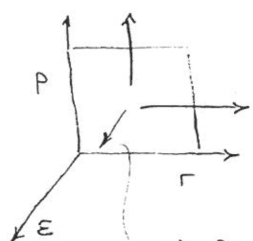
$$\text{lhs} \quad 2\left(\frac{\partial}{\partial t} + \vec{V}\frac{\partial}{\partial \vec{r}} + e\vec{V}\vec{E}\frac{\partial}{\partial \varepsilon} + \left(e\vec{E} + \frac{e}{c}\vec{V}\wedge\vec{H}\right)\frac{\partial}{\partial \vec{p}}\right)\hat{G}$$

in presence of e.m. field

$$-\frac{\partial \phi}{\partial \vec{r}} - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \nabla\wedge\vec{A}$$

This results from choice of FT — we got only fields and not potentials — manifestly Gauge invariant. If this was not so we would have a problem. The form above is very similar to Boltzman equation

Classically



particles can exit this region of phase space in either direction

$\vec{E}$ field can change energy (but not $\vec{H}$)

To determine

$G^{R,A}$ — add the two Dyson equations (do so in FT)

$$\text{lhs} \qquad 2(\varepsilon - \xi_p)G^{R,A} = 2 + \Sigma^{R,A}G^{R,A} + G^{R,A}\Sigma^{R,A} \quad \text{rhs}$$

$$2(\varepsilon - \xi_p - \Sigma^{R,A})G^{R,A} = 2$$

If order is irrelevant and we get usual expression

$$G^{R,A} = \frac{1}{\varepsilon - \xi_p - \Sigma^{R,A}}$$

Correct even out of equilibrium

Consider $h(2,1) = \int d(3)\, f(2,3)\, g(3,1)$

let us find the expression for the convolution in terms of Wigner coordinates

If $f = f(2-3)$, $g = g(3-1)$ then after FT we will get product

What if we take into account dependence on sum (slightly), $\frac{2+3}{2}$, $\frac{1+3}{2}$?

We should go up to lowest order in derivatives

$$h(r,t;p,\varepsilon) = f(r,t;p,\varepsilon)\, g(r,t;p,\varepsilon) + \frac{1}{2}\{f,g\}$$

where $\{f,g\}$ is slightly generalised Poisson bracket.

Without e-m field,

$$\{f,g\} = \frac{\partial f}{\partial \varepsilon}\frac{\partial g}{\partial t} - \frac{\partial g}{\partial \varepsilon}\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{p}}\frac{\partial g}{\partial \vec{r}} - \frac{\partial g}{\partial \vec{p}}\frac{\partial f}{\partial \vec{r}}$$

With e-m field

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \frac{e}{c}\frac{\partial \vec{A}}{\partial t}\frac{\partial}{\partial \vec{p}} - e\frac{\partial \phi}{\partial t}\frac{\partial}{\partial \varepsilon}$$

$$\frac{\partial}{\partial \vec{r}} \to \frac{\partial}{\partial \vec{r}} - \frac{e}{c}\frac{\partial \vec{A}}{\partial \vec{r}}\frac{\partial}{\partial \vec{p}} - e\frac{\partial \phi}{\partial \vec{r}}\frac{\partial}{\partial \varepsilon}$$

$$\{f,g\} = -\{g,f\}$$

$$\{f,g\} = \frac{\partial f}{\partial \varepsilon}\Big[\frac{\partial}{\partial t} + \vec{V}_F\frac{\partial}{\partial \vec{r}} + \Big(e\vec{E}\,\vec{V}_F - \frac{\partial f/\partial t}{\partial f/\partial \varepsilon}\Big)\frac{\partial}{\partial \varepsilon}$$

$$+ \Big(e\vec{E} + \frac{e}{c}\vec{V}_F \wedge \vec{H} + \frac{\partial f/\partial \vec{r}}{\partial f/\partial \varepsilon}\Big)\frac{\partial}{\partial p}\Big]\, g$$

$$\vec{V}_F = \frac{\partial f/\partial \vec{p}}{\partial f/\partial \varepsilon}$$

So $\{f,g\}$ has same construction as Boltzman equation

Above lhs can be written $2\{\varepsilon - \xi_p, \hat{G}\}$

Equation for F

$$i\{\varepsilon-\xi_p, F\} = \Sigma^K(G^A-G^R) - F(\Sigma^A-\Sigma^R)$$

$$+\ \frac{i}{2}\{\Sigma^K, G^A+G^R\} + \frac{i}{2}\{F, \Sigma^A+\Sigma^R\}$$

this term is semi-classical correction

two terms are small

At equilibrium, $\{\ ,\ \} = 0$ (No r dep, t dep., or external fields)

$$F = \frac{\Sigma^K(G^A-G^R)}{\Sigma^A-\Sigma^R}$$

should be $\tanh \varepsilon/2T$ (apparent in calculation but don't have general proof)

Classical Equation

Want distribution function $f(\vec{p},\vec{r},t)$ as opposed to $n(\varepsilon)$

in equilibrium $f = n_F(\xi_p)$ independent or $\vec{r}$ and t

Recall, in random potential we found

$$\Sigma^A - \Sigma^R = 2i\,\mathrm{Im}\,\Sigma^A = i/\tau$$

Our approximation is $\hbar/\varepsilon\tau \ll 1$ (classically $i/\tau = 0$)

Perturbation theory is therefore good and we can return to equation above

$$i\{\varepsilon-\xi_p, F\} = \Sigma^K(G^A-G^R) - F(\Sigma^A-\Sigma^R)$$

Take $G^{R,A} \rightarrow G_0^{R,A} = \dfrac{1}{\varepsilon-\xi_p \pm i0}$ (drop i/τ)

(Note that in general it is not always safe to neglect two terms above)

In classical situation $1/\tau \ll T$ units ?

$$2f(\vec{r},\vec{p},t) - 1 = \int F(\varepsilon,\vec{p};\vec{r},t)\,\frac{d\varepsilon}{2\pi i} \qquad \Big(F = \underbrace{(2f-1)}_{\varepsilon\, f^{?}}(G^A - G^R) \Big)$$

$$\left[\frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{r}} + \left(e\vec{E} + \frac{e}{c}\vec{v}\wedge\vec{H}\right)\frac{\partial}{\partial\vec{p}}\right] f = I(f)$$

Boltzman Equation

Classical equation for the distribution function.

Another possibility

$$F = S(\varepsilon,\vec{p};\vec{r},t)(G^A - G^R) + \frac{i}{2}\{S, G^A + G^R\}$$

(In space / time rep. $F = SG^A - G^R S$)

After some algebra

$$i\{\varepsilon - \xi_p, S\}(G^A - G^R) = \left[\Sigma^K - S(\Sigma^A - \Sigma^R) + \frac{i}{2}\{\Sigma^A + \Sigma^R, S\}\right](G^A - G^R)$$

$$+ \underbrace{\frac{i}{2}\{\Sigma^K - S(\Sigma^A - \Sigma^R), G^A + G^R\}}$$

this term can be neglected since it is small (0 classical, equilibrium)

$$\left[\frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial\vec{r}} \overset{+eE\vec{v}\frac{\partial}{\partial\varepsilon}}{} + \left(e\vec{E} + \frac{e}{c}\vec{v}\wedge\vec{H}\right)\frac{\partial}{\partial\vec{p}}\right] S = I(S) + \Delta(S)$$

$$I(S) = -i\left[\Sigma^K - S(\Sigma^A - \Sigma^R)\right]$$

$$\Delta(S) = \frac{i}{2}\{\Sigma^A + \Sigma^R, S\} - i\left(\delta\Sigma^K - S(\delta\Sigma^A - \delta\Sigma^R)\right)$$

$$\hat{\Sigma} \to \hat{\Sigma} + \delta\hat{\Sigma}$$

Gives route to study non-equilibrium problem in external field

KELDYSH DIAGRAMS

G.F. $$\hat{G} = \begin{pmatrix} 0 & G^A \\ G^R & F \end{pmatrix}$$

Dyson equation

$$G^{R,A}(\varepsilon,p) = \frac{1}{\varepsilon - \xi_p - \Sigma^{R/A}(\varepsilon,p)}$$

$$\xi_p = \frac{p^2}{2m}$$

$$v = \frac{\partial \xi_p}{\partial p}$$

for F, Dyson equation in differential form

$$G_0^{-1}\hat{G} = \delta(\;\;)$$

$\Rightarrow$ Result similar to Boltzman equation

for $f(\vec{p},\vec{r};t)$, # particles in an element of the phase space volume at t $d\vec{p}\,d\vec{r}$

Boltzman equation when

1/ $\quad \mathrm{Im}\,\Sigma^R = \frac{1}{2\tau} \ll \varepsilon_F \quad$ (States far from localization)

2/ $\quad \frac{1}{\tau} \ll T \quad$ (more strict condition – leads to important corrections)
– strange T dependencies

$$\left\{ \frac{\partial}{\partial t} - v\frac{\partial}{\partial r} - \left(eE + \frac{e}{c}[\vec{v}\wedge\vec{H}]\right) \right\} f(\vec{p},\vec{r};t) = I\{f\}$$

Collision integral

$$G^A(\varepsilon,p) - G^R(\varepsilon,p) = \delta(\varepsilon-\xi_p)\,2\pi i$$

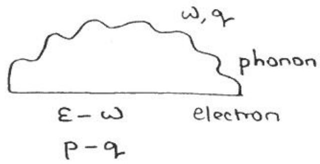
$$f(p,r;t) = \int \frac{d\varepsilon}{2\pi i}\left(F + G^A - G^R\right)\frac{1}{2}$$

$$I\{f\} = \int \frac{d\varepsilon}{2\pi i}\left\{ \Sigma^K (G^A - G^R) - F(\Sigma^A - \Sigma^R) \right\}$$

Lowest order of perturbation theory.

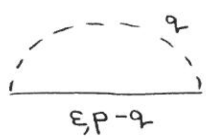
Consider collision integral in several situations

1) Electron-phonon interaction



$$\sim\sim\sim = \hat{D}(\omega, q)$$

2) Electron-impurity scattering

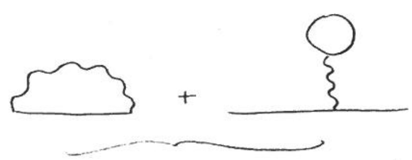


$$- - - - = W(q)$$

3) Electron-electron scattering

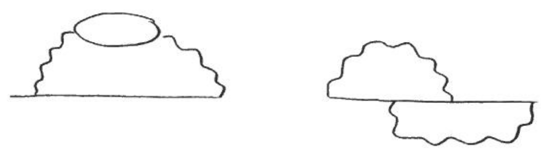
$$\sim\sim\sim = U(q)$$

has no imaginary part
and does not contribute to $\mathrm{Im}\,\Sigma$



Hartree-Fock

neither contribute to collision integral



(See problem set 3)

Calculating diagrams

$$G^A - G^R = 2\pi i\, \delta(\varepsilon - \xi_p)$$

$$D^A - D^R = 2\pi i \left[\delta(\omega - \omega_q) + \delta(\omega + \omega_q) \right]$$

can be dropped with ω integration

$$F = 2\pi i\, (2f - 1)\, \delta(\varepsilon - \xi_p)$$

$$D^K = 2\pi i \left(2N_q + 1\right) \delta(\omega - \omega_q)$$

if phonons in equilibrium have Bose - Einstein distribution

$\gamma^i_{\alpha\beta}$: $\gamma^1_{\alpha\beta} = \frac{1}{\sqrt{2}} \delta_{\alpha\beta}$ $\hat{D} = \begin{pmatrix} 0 & D^A \\ D^R & D^K \end{pmatrix}$

$\gamma^2_{\alpha\beta} = \frac{1}{\sqrt{2}} (\sigma_x)_{\alpha\beta}$

$$\hat{\Sigma}_{\alpha\beta} : \quad \gamma^i_{\alpha\alpha'} G_{\alpha'\beta'} D_{ij} \gamma^j_{\beta'\beta} = D^K \begin{pmatrix} F' & G'^R \\ G'^A & 0 \end{pmatrix} + D^R \begin{pmatrix} G'^R & F' \\ 0 & G'^A \end{pmatrix}$$

$$+ D^A \begin{pmatrix} G'^A & 0 \\ F' & G'^R \end{pmatrix}$$

prime indicates different arguments
$G = G(\varepsilon, p)$
$G' = G(\varepsilon - \omega, p - q)$

$$\Sigma_{22} \propto \int \frac{d\omega}{2\pi i} \left[D^R(\omega)\, G^A(\varepsilon - \omega) + D^A(\omega)\, G^R(\varepsilon - \omega) \right]$$

$= 0$ due to analytical properties of $D^{R,A}$ and $G^{R,A}$

(as required)

By inspection

Σ^K: $D^K F' + D^R G^{R'} + D^A G^{A'}$

Σ^R: $D^K G^{R'} + D^R F'$ $\int d\omega\, D^R(\omega)\, F'(\varepsilon - \omega)$

Σ^A: $D^K G^{A'} + D^A F'$ $= \int d\varepsilon'\, F'(\varepsilon') D^R(\varepsilon - \varepsilon')$

has retarded analytical properties

$$D^R G^{R'} + D^A G^{A'} = D^A (G^{A'} - G^{R'}) + D^R (G^{R'} - G^{A'})$$ — just adding term $= 0$

$$= (D^A - D^R)(G^{A'} - G^{R'})$$

$$\Sigma^{K}(G^{A}-G^{R}) = \frac{1}{2}\int \delta(\varepsilon-\xi_p)\,\delta(\varepsilon'-\xi_{p'})\,(2\pi i)^2 \times \left(s' D^{K} + D^{A} - D^{R}\right) \frac{dq\, d\omega}{(2\pi)^d\, 2\pi i}$$

$s = 2f - 1$

for collision integral

$$I\{f\} = \int \frac{d\varepsilon}{2\pi i}\frac{d\omega}{2\pi i}\,\delta(\varepsilon-\xi_p)\,\delta(\omega-\omega_q) \int \delta(\underset{\varepsilon-\omega}{\xi_p - \omega_q} - \xi_{p-q})\, R\{f\}\, dq\, W^2_{ij}$$

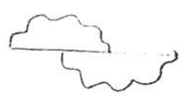
everything $\propto$ to this (F' or $G^{A'}-G^{R'}$)

$D^A - D^R$

matrix element

$$R\{f\} = f_p(1-f_{p-q})(N_q+1) - (1-f_p)f_{p-q}N_q$$

(This formula only relies on $\frac{1}{\tau} \ll$ any energy)

Can not treat the correction  consistently – does not make sense at level of Boltzman equation for non-classical behaviour.

Assumption to write Boltzman equation is that the quasi-particles are well defined.

and $1/\tau > T$ otherwise can not use δ fn.

Equilibrium $\quad I\{f\} = 0$

Elastic Scattering

Energy transfer in collision is negligible

$$\omega_q \ll T$$

e.g. $- T \gg \hbar\omega_D$

Debye frequency

$- \quad p_F \ll T/c$

– Elastic scattering by impurities (random potential)

$$\omega_q = \frac{q^2}{2m}, \quad m \to \infty$$

effective description of impurities

$$\text{if} \quad \omega_q \ll T, \quad N_q = \frac{1}{e^{\omega_q/T} - 1} \simeq \frac{T}{\omega_q} \gg 1$$

$$N_q + 1 \simeq N_q$$

$$R\{f\} = \frac{T}{\omega_q}\left(f_p - f_{p-q}\right) \qquad \text{(see before)}$$

In general

$$\int d\omega \, N_\omega \left[D^A(q,\omega) - D^R(q,\omega) \right]$$

$$\simeq P \int d\omega \, T \left[\frac{D^A - D^R}{\omega} \right]$$

$$= T\left[D^A(q, \omega = 0) - D^R(q, \omega = 0) \right]$$

for electron-impurity scattering

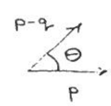
$$I\{f\} = N_{im} \int dq \left(f_p - f_{p-q} \right) W(q) \, \delta\left(\xi_p - \xi_{p-q}\right)$$

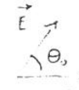
\# concn. of impurities

elastic scattering

Conductivity

$$\vec{j}_\alpha = \sigma_{\alpha\beta} \vec{E}_\beta \qquad \vec{q} \to (\theta, \phi)$$





$$-e\vec{E} \frac{\partial}{\partial \vec{p}} f = \int \frac{d\Omega}{4\pi} W(\cos\theta) \left[f_p - f_{p-q} \right]$$

$$\int \frac{d\Omega}{4\pi} W = \frac{1}{\tau}$$

Solution $\quad f = f_0 + g\cos\theta_0 \qquad \cos(p-q, E) = \cos\theta\cos\theta_0$

$$-e\vec{E}\vec{v}\,\frac{\partial n_F(\xi)}{\partial\xi} = \int_{-1}^{1} d\cos\theta \;\; W(\cos\theta)\,(1-\cos\theta)\,g\cos\theta_0$$

$$= \frac{g\cos\theta_0}{\tau_{Tr}}$$

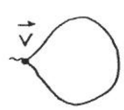
$$\frac{1}{2}\int_{-1}^{1} d\cos\theta\; W(\cos\theta)\,(1-\cos\theta) = \frac{1}{\tau_{Tr}} \sim \text{transport time}$$

$$\vec{j} = e\int (dp)\;\vec{v}f(p) = \int (dp)\,\tau_{Tr}\,e^2 v\,\frac{\partial n}{\partial\xi}$$

$$= \underbrace{\frac{e^2\tau N_{el}}{m}}_{\sigma}\,\vec{E}$$

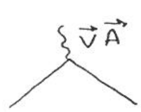
Without transport equation

$$\vec{j} = \frac{1}{2}\int v\,[F + G^A - G^R]\,(dp)\,\frac{d\varepsilon}{2\pi i}$$



with external field, use as perturbation

$$\vec{A}(t) = \vec{A}_\omega\, e^{i\omega t} \quad \text{- vector potential}$$



$\sim$ addition term in H

$j =$  allows to determine σ without transport equation.

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LOCALISATION

Random Potential $V(r)$

$$\langle V(r) V(r') \rangle = W(r-r') = \gamma\,\delta(r-r')$$

Extended States

ℓ - mean free path $\qquad p_F \ell \gg \hbar$

τ - time of mean free path $\qquad E\tau \gg \hbar$

$\Big\}$ far from localisation

(for $p_F \ell \sim \hbar$, ℓ is not well defined)

Parameters $\dfrac{1}{p\ell}$ or $\left(\dfrac{W}{I}\right)^2$ for Anderson Model

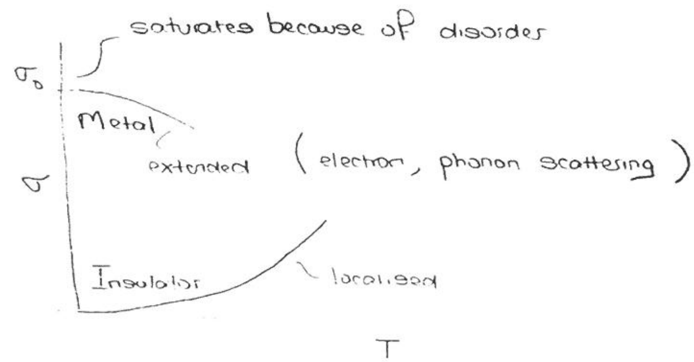
Goal is to understand small corrections to this picture of good metal as a first step towards understanding real localisation transition

Good metal $\quad p_F \ell \gg \hbar$

Quantum corrections?

Conductivity $\quad \sigma_{\alpha\beta}$

$$j_\alpha = \sigma_{\alpha\beta} E_\beta$$



Classical Conductivity

Two approaches 1) Boltzman eqn.

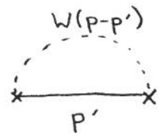
2) Diagrams

1) Boltzman eqn.

$$\left(\frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{r}} + e\vec{E}\frac{\partial}{\partial \vec{p}} \right) f(\vec{p},\vec{r},t) = \frac{\bar{f} - f}{\tau}$$

Approximation (δ-correlated disorder)

$$\bar{f}(\vec{r},|\vec{p}|,t) = \int f(\vec{r},\vec{p},t)\,\frac{d\Omega_p}{4\pi}$$

(Recall $\Sigma(p) =$ 

$= \int (dp')\, f(p')\, \delta(\xi_p - \xi_{p'})$ gives angle averaging)

Can use to find
- Thermopower
- Non-linear conductivity

etc. from Boltzman equation

We want linear response.

Conductivity
$$f = f_0 + \delta f(p) = f_0 - \tau e \vec{E}\frac{\partial f_0}{\partial p}$$

f_0: equilibrium distribution f^n (isotropic); $\delta f(p)$: correction

$$j(\vec{r},t) = \int f(p,r,t)\,\vec{v}\,(dp) \qquad v = \frac{\partial \xi}{\partial p} \text{ velocity}$$

$$\sigma = \frac{Ne^2}{m}\tau$$

As before (In this approach all temperature dependence is in τ)

2) Diagrams

$$\mathcal{H} = \frac{(\vec{p} - e\vec{A}/c)^2}{2m} + V(r)$$

$$-\frac{e\vec{A}\vec{p}}{cm} = -\frac{e}{c}\vec{A}\vec{v} \quad \text{perturbation} \quad (\text{time dependent})$$

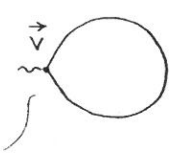
Electric field $\quad E(t) = -\frac{1}{c}\frac{dA}{dt}$

$$A(t) = A_\omega e^{i\omega t}$$

$$\vec{J} = e\int (dp)\frac{d\varepsilon}{2\pi i}\,\vec{v}\,G^{+}(p,\varepsilon) \quad \text{— Keldysh}$$

$$\langle \psi^{+}\psi \rangle$$

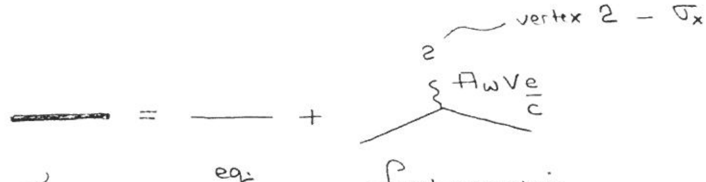
$$= e\int (dp)\frac{d\varepsilon}{2\pi i}\,\vec{v}\,\frac{1}{2}\left[F + G^{A} - G^{R}\right]$$



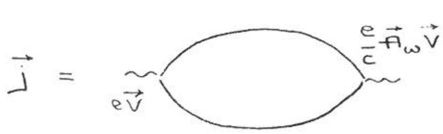
two points r, t equivalent

In equilibrium, angular integration over $\vec{v}$ vanishes ; current is zero

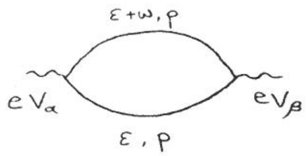
Perturbation



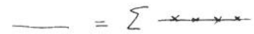
G.F in field.



Now angular integral will not vanish

$\sigma_{\alpha\beta}(\omega) =$ 

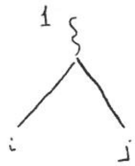
$\frac{1}{i\omega}$ — from $A \leftrightarrow E$ relation

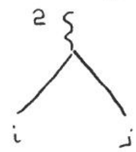


Note that in usual diagrammatic approach it is necessary to determine diamagnetic-term

— with Keldysh it is dealt with automatically

for external field

Recall 

$= \frac{1}{\sqrt{2}}\delta_{ij}$ 

$= \frac{1}{\sqrt{2}}(\sigma_x)_{ij}$

$$\hat{G}\hat{\sigma}_x\hat{G} = \begin{pmatrix} 0 & G^A G^A \\ G^R G^R & G^R F + F G^A \end{pmatrix} \qquad \hat{G} = \begin{pmatrix} 0 & G^A \\ G^R & F \end{pmatrix}$$

$$\int d\varepsilon\, G^R G^R = 0 = \int d\varepsilon\, G^A G^A \qquad (\text{using analytical properties})$$

$$\sigma_{\alpha\beta} = \frac{1}{2}\int (dp)\frac{d\varepsilon}{2\pi i} V_\alpha V_\beta \left\{ G^R(\varepsilon)F(\varepsilon+\omega) + F(\varepsilon)G^A(\varepsilon+\omega)\right\}\frac{1}{i\omega}$$

Linear response — equilibrium

$$F = (2n-1)(G^A - G^R)$$

$$\int (dp) = \int \frac{d\Omega}{4\pi}\underbrace{d\xi_p\, \nu(\xi_p)}_{dp}$$

DOS

change of variables

$$\int d\xi\, G^R G^R = 0 \qquad , \qquad G^R = \frac{1}{\varepsilon - \xi - i/2\tau}$$

$\Rightarrow$ neglect $G^R G^R$ and $G^A G^A$ above

ε, ξ enter in a similar form

Can sometimes cause problems as we will see later

$$\sigma_{\alpha\beta}(\omega) = \int \frac{d\varepsilon}{2\pi i} \left(n_F(\varepsilon+\omega) - n_F(\varepsilon) \right) \int \frac{d\Omega_p}{4\pi} V_\alpha V_\beta$$

$$\times \int \nu \, d\xi \; G^R(\varepsilon) G^A(\varepsilon+\omega) \; \frac{1}{i\omega}$$

after substitution of equilibrium expression for F

$$\int \frac{d\Omega_p}{4\pi} V_\alpha V_\beta = \frac{V^2}{d} \delta_{\alpha\beta} \qquad \left(\text{e.g. } d=3 \quad \frac{V^2}{3} \right)$$

$$\nu \int d\xi \, \frac{1}{\epsilon - \xi + i/2\tau} \cdot \frac{1}{\varepsilon + \omega - \xi - i/2\tau} = -2\pi i \frac{\nu}{\omega - i/\tau} \xrightarrow{\omega \to 0} 2\pi\tau$$

$$\sigma_{\alpha\beta}(\omega) = e^2 \int d\varepsilon \left(\frac{n_F(\varepsilon+\omega) - n_F(\varepsilon)}{\omega - i/\tau} \right) \frac{V^2}{d} \nu \frac{1}{i\omega} \quad \delta_{\alpha\beta}$$

For $T \ll \varepsilon_F$

$$= e^2 \nu \frac{V^2}{d} \tau \int d\varepsilon \, \frac{n_\varepsilon - n_{\varepsilon+\omega}}{\omega(1+i\omega\tau)} \quad \delta_{\alpha\beta}$$

a) $\omega \gg T$, $n_\varepsilon = \Theta(-\varepsilon + \mu)$ $\qquad \int_{\mu-\omega}^{\mu} d\varepsilon = \omega$

b) $\omega \ll T$ $\quad n_\varepsilon - n_{\varepsilon+\omega} = -\omega \frac{dn}{d\varepsilon}$ $\qquad \int \frac{dn}{d\varepsilon} d\varepsilon = -1$

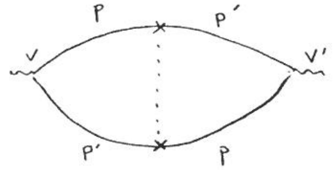
$$\sigma_{\alpha\beta}(\omega) = e^2 \nu \frac{V^2}{d} \tau \frac{1}{1+i\omega\tau} \delta_{\alpha\beta} \qquad \text{Drude formula}$$

in either case a) and b)

σ is retarded function of frequency (as it should be)

In this calculation we have included the reducible part of diagram, but not irreducible part.

e.g



x- - - - x $W(q) = \text{const}$

$$\int \bar{v} \, d\Omega_p = 0$$

But when W is no constant, there is no vanishing and the loop is renormalised – effect is to replace τ by transport time τ_{Tr}. Can be found in AGD book.

Situation without external field – fluctuations

Hydrodynamic Limit

$\rho(\vec{r},t)$ density
$j(\vec{r},t)$ current
$\Big\}$ averaged over a certain volume

$\Delta v \, \Delta t$

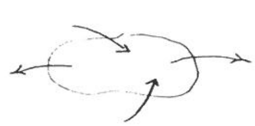
$$j_D(\vec{r},t) = -D \nabla \rho(\vec{r},t)$$

local equation for average quantities

(assume fluctuations are small)

Higher derivatives become irrelevant at $r, t \to \infty$

Continuity Relation



$$\nabla . \vec{j} = \frac{\partial \rho}{\partial t}$$

Number of particles is conserved

Together, these equations imply

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho \qquad \text{diffusion equation}$$

diffusion constant

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \vec{r}} = \frac{\bar{f} - f}{\tau}$$

$$f = \bar{f} + f_a \qquad f_a \ll \bar{f}$$

isotropic

i) $$\frac{\partial \bar{f}}{\partial t} + \overline{v\frac{\partial f_a}{\partial \vec{r}}} = 0 \qquad \text{average over } \Omega_p$$

ii) $$\frac{\partial \overline{v_\alpha f_a}}{\partial t} + \overline{v_\alpha v_\beta}\frac{\partial \bar{f}}{\partial r_\beta} = -\frac{\overline{v_\alpha f_a}}{\tau} \qquad \times \vec{v} \text{ and average}$$

$$\overline{v f_a} \simeq -\tau\, \overline{v_\alpha v_\beta}\frac{\partial \bar{f}}{\partial r_\beta}$$

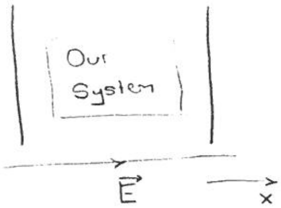
$$\frac{\partial \bar{f}}{\partial t} - \tau\, \overline{v_\alpha v_\beta}\, \nabla^2 \bar{f} = 0$$

$$\rho(r,t) = \int (dp)\, f(p,r;t)$$

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho, \qquad D = \tau\, \overline{v_\alpha v_\beta} = \tau \frac{v^2}{d}\delta_{\alpha\beta}$$

So $$\sigma_{\alpha\beta} = e^2 \nu D_{\alpha\beta} \qquad \text{Einstein relation}$$

This connection between diffusion constant and conductivity is very generic



$$\vec{j} = 0 = \vec{j}_E + \vec{j}_D$$

$$\vec{j}_D = -D\nabla\rho \qquad \text{hydrodynamics}$$

$$\vec{j}_E = \sigma E \qquad \text{Ohm's law (definition)}$$

potential constant over sample

$$\underbrace{\mu(x)}_{\text{chemical potential}} + \underbrace{e\Phi(x)}_{\text{electric potential}} = \text{const.} \qquad \text{Thermodynamics}$$

$$\rho(x) = \rho_0 + \delta\mu(x)\frac{\partial\rho}{\partial\mu}$$

$$= \rho_0 - e\Phi\nu$$

$$j_D = D\nu e^2\nabla\Phi = -De^2\nu E$$

$$\text{So} \quad \sigma = e^2\nu D$$

It is a type of Gauge invariance

So we can examine diffusion constant or conductivity – both are related

Repeat of diffusion

$\rho(r,t)$ – density $\qquad \dfrac{\partial\rho}{\partial t} = D\nabla^2\rho$

or $\quad D_{\alpha\beta}\dfrac{\partial}{\partial r_\alpha}\dfrac{\partial}{\partial r_\beta} \qquad$ D – is diffusion constant

From Boltzman Eq

$$D_{\alpha\beta} = \overline{V_\alpha V_\beta}\,\tau = \frac{v^2\tau}{d}\,\delta_{\alpha\beta} \qquad \text{in a classical limit}$$

$$= \frac{v\,\underbrace{l}_{\text{mean free path}}}{d}\,\delta_{\alpha\beta}$$

Einstein Relation

$$\sigma_{\alpha\beta} = e^2 \nu D_{\alpha\beta} \qquad \nu = \frac{\partial\rho}{\partial\mu} \text{ is density of states}$$

derived from general assumptions even without Boltzman eq.

In metal σ, D constants

Insulator $\sigma, D \to 0$ as $T \to 0$

How does quantum mechanics drive these constants to zero?

Density - density correlation function

Wavefunction $\Psi(1) \longrightarrow$ Green function $G(1,2)$ $\qquad 1 = t_1, r_1$

$$(H - E)\Psi = 0$$

$$(H_1 - E)\,G(1,2) = \delta(1-2)$$

$$\rho(r,t) \longrightarrow P(r_2 t_2; r_1, t_1)$$

$$\left(\frac{\partial}{\partial t_1} - D\nabla^2\right)P(2,1) = \delta(r_1 - r_2)\,\delta(t_1 - t_2)$$

P - Green function of the diffusion equation

$P(2,1) = 0$ for $t_2 < t_1$ get retarded Green function P^R

Physical meaning of P is probability to get from 1 to 2

$$\rho(r,t_1) = \delta(r-r_1)$$

$\Rightarrow$ $P(r_2,t_2;r_1,t_1)$ is density distribution at $t=t_2$

$$P^R(r_2,t_2;r_1,t_1) = |G^R(2,1)|^2 = G^R(2,1)G^A(1,2)$$

$$P^R(2,1) = \frac{1}{(2\pi D t_{12})^{d/2}} \cdot \exp\left[-\frac{r_{12}^2}{2Dt_{12}}\right]\Theta(t_{12})$$

$t_{12} = t_2 - t_1$

$r_{12} = r_2 - r_1$

$\downarrow$ Fourier transform

$$P^R(q,\omega) = \int P^R(2,1)\, e^{i\omega t_{12} + iqr_{12}}\, dt_{12}\, dr_{12}$$

r_{12} $\quad$ t_{12}

$$= \frac{1}{-i\omega + Dq^2}$$

singularity $\omega = -iDq^2$

$\rightarrow$ retarded

Another meaning is that P is density - density correlation function

- this would be fine if just single particle problem

For many - particle system, we do not want to add particles.

$$\int \rho(r,t)\,dr = \text{constant} = \rho_0 V$$

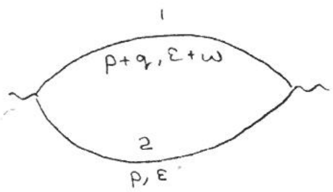
So better not to start with δ function.

$$\rho(r,0) = \delta(r-r_1) - \frac{1}{V}$$ - fluctuation of density

Let us try to derive this from Keldysh diagram technique

$$\langle \delta\rho(0,0)\,\delta\rho(r,t)\rangle^R_{q,\omega} = ?$$

$\delta\rho = \rho - \rho_0$ — from now let us call $\delta\rho \rightarrow \rho$ and understand that it is the fluctuation



Some field proportional to density fluctuation

$$\hat{G} = \begin{pmatrix} 0 & G^A \\ G^R & F \end{pmatrix}$$

Vertices $\quad \gamma^1_{\alpha\beta} = \frac{1}{\sqrt{2}}\,\delta_{\alpha\beta} \quad , \quad \gamma^2_{\alpha\beta} = \frac{1}{\sqrt{2}}\,\sigma^x_{\alpha\beta}$

$$\hat{\Pi} = \Pi_{ij}(q,\omega) = \int (dp)\,\frac{d\varepsilon}{2\pi i}\,\mathrm{Tr}\,\hat{\gamma}^j\,\hat{G}(\varepsilon+\omega)\,\hat{\gamma}^i\,\hat{G}(\varepsilon)$$

$$\begin{pmatrix} 0 & \Pi^A \\ \Pi^R & \Pi^K \end{pmatrix}$$

$$\Pi^R = \Pi_{21} = \int (dp)\,\frac{d\varepsilon}{2\pi i}\,\frac{1}{2}\mathrm{Tr}\begin{pmatrix} 0 & G^A G^A \\ G^R G^R & F G^A + G^R F \end{pmatrix}$$

$$= \frac{1}{2}\int (dp)\,\frac{d\varepsilon}{2\pi i}\left[G^R(\varepsilon+\omega)F(\varepsilon) + F(\varepsilon+\omega)G^A(\varepsilon)\right]$$

$\varepsilon' = \varepsilon+\omega$, $\varepsilon = \varepsilon'-\omega$ } structure already retarded

$$F(p,\varepsilon) = \left[2n_F(\varepsilon)-1\right]\left(G^A(p,\varepsilon) - G^R(p,\varepsilon)\right)$$

$$\Pi^R = \int (dp)\,\frac{d\varepsilon}{2\pi i}\Big\{G^R(\varepsilon+\omega)G^A(\varepsilon)\left[n_F(\varepsilon) - n_F(\varepsilon+\omega)\right]$$
$$- n_F(\varepsilon)G^R G^R + n_F(\varepsilon+\omega)G^A G^A\Big\}$$

$$= \Pi_1 + \Pi_2 + \Pi_3$$

What about $\Pi_2 + \Pi_3 = ?$

Hydrodynamics : $\quad q,\omega \longrightarrow 0$

$$G^R(p,\varepsilon)^2 = \frac{\partial G^R}{\partial \varepsilon}$$

$$G^R = (\varepsilon - \xi_p + i/2\tau)^{-1} \quad \text{for } \tau \text{ indep. of } \varepsilon \quad \left(\text{true to } O\left(\frac{1}{\varepsilon_F \tau}\right)\right)$$

$$G^A(p,\varepsilon)^2 = \frac{\partial G^A}{\partial \varepsilon}$$

after $\int$ by parts and neglecting ω

$$\Pi_2 + \Pi_3 = \int (dp)\, \frac{d\varepsilon}{2\pi i}\, \frac{\partial n_\varepsilon}{\partial \varepsilon}\,(G^A - G^R)$$

$$\nu = \frac{1}{2\pi i}\int (dp)\left(G^A(\varepsilon) - G^R(\varepsilon)\right)$$

$$= \int d\varepsilon\, \underbrace{\frac{\partial n_F(\varepsilon)}{\partial \varepsilon}}_{\text{quick}}\, \underbrace{\nu(\varepsilon)}_{\text{smooth}} \simeq \underbrace{\nu(\varepsilon_F)}_{\text{This constant is very important}}$$

$$\Pi_1 = \int \frac{d\varepsilon}{2\pi i}\left[n_F(\varepsilon) - n_F(\varepsilon+\omega)\right]\int (dp)\, G^R(\varepsilon+\omega)G^A(\varepsilon)$$

$$G^A - G^R = (-i\tau)^{-1}\, G^R G^A$$

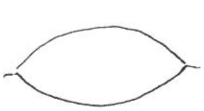
$$\Pi_1 = -i\tau \int \frac{d\varepsilon}{2\pi i}\left[n_F(\varepsilon) - n_F(\varepsilon+\omega)\right]\int (dp)\,(G^A - G^R)$$

$$= i\tau\, \nu(\varepsilon_F)\int d\varepsilon\left[n_F(\varepsilon) - n_F(\varepsilon+\omega)\right]$$

$$= i\omega\tau\, \nu(\varepsilon_F)$$

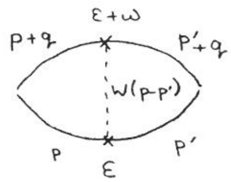
This result is badly wrong – it is nothing like diffusion equation

– Something very important has been missed – renormalisation of vertices

 we took into account  and neglected 

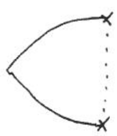
For conductivity, we had velocity at vertex and corrections were zero

Here, in first order we have



$W(k) = \text{constant} = \gamma$

$\frac{1}{\tau} = 2\pi\gamma\nu$



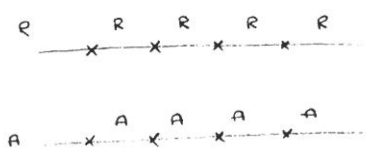
$$= \gamma \int (dp)\, G^R(p+q) G^A(p) = \zeta(q,\omega)$$

first order, so vertex correction

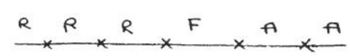
first term

(integrating over p only)

$$\Gamma(q,\omega) = 1 + \zeta(q,\omega) + \ldots$$



in Keldysh we have



1) $q = 0,\ \omega = 0$

$$\zeta(q=0,\omega=0) = \gamma\, 2\pi\tau\nu = 1 \; ! \quad (\text{see above})$$

using identity between product and difference

2) q, ω finite but small (can do in general and get Lindhard function in presence of disorder)

and expand

$$G^R(p+q, \varepsilon+\omega) = G^R(p,\varepsilon) - \omega G^{R^2} + \vec{q}\,\vec{v}\, G^{R^2}$$

$\xi_{p+q} = \xi_p + \vec{q}\,\vec{v}$

angular integration vanishes

$\omega \ll 1/\tau, \quad q \ll 1/l$

$$- \frac{(\vec{q}\,\vec{v})^2}{2} 2 G^{R^3} + O(\omega\tau)^2 + O(ql)^3$$

$$\zeta(q,\omega) = 1 + i\omega\tau - \tau^2 \int (\vec{q}\vec{v})^2 \frac{d\Omega}{4\pi} + \cdots$$

$$= 1 + i\omega\tau - \underbrace{D}_{v^2\tau/d} q^2\tau + \cdots \qquad \omega\tau \ll 1, \quad Dq^2\tau \sim (ql)^2 \ll 1$$

$$\int (dp)\, G^{R^2} G^A = \int \nu(\xi)\, d\xi\, G^{R^2} G^A = \int \frac{d\xi\ \nu(\xi)}{(\varepsilon - \xi + i/2\tau)^2 (\varepsilon - \xi - i/2\tau)}$$

$$\varepsilon_F \tau \gg 1$$

$$= \frac{\nu(\varepsilon)\, 2\pi i}{(i/\tau)^2} = -2\pi i \nu \tau^2$$

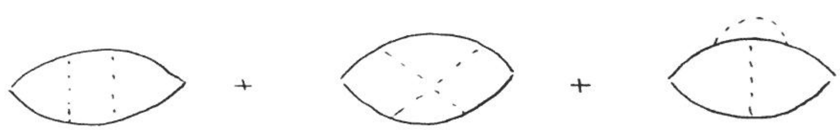
When typical time longer than time of m.f.p

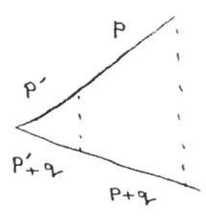
or when typical length is longer than m.f.p. corrections can not be ignored

(Note we have used already many times $\varepsilon_F \tau \gg \hbar$)

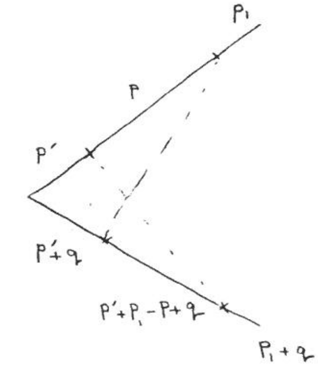
This is a problem because each correction is of the same order, However we can get classical result from ladder approximation.

Second order





$$= \zeta(q,\omega)^2 \sim 1$$



$$= \gamma^2 \int (dp)(dp')\, G^R(p') G^A(p') G^R(p) G^A(p'+p_1-p)$$

is small $\sim \hbar/\varepsilon_F\tau$

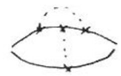
for this to big, must have poles close

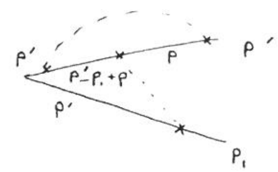
$$\xi_p - \xi_{p'+p_1-p} \lesssim 1/\tau$$

$$\xi \simeq \frac{p^2}{2m} \quad \Rightarrow \quad \frac{1}{2m}\left[(p'+p_1)^2 - p(p'+p_1)\cos\theta\right] \lesssim 1/\tau \ll \varepsilon_F$$

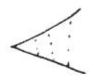
$$\cos\theta = \frac{p'+p_1}{\ \ } + O(1/\ )$$

Integral over θ gives $\sim \frac{\hbar}{\varepsilon_F \tau}$ and it is consistent to neglect it

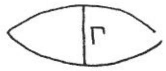
A similar argument applies to 



Lowest order in $\hbar/\varepsilon_F \tau$ – neglecting all crossings

$$\Gamma = \sum$$  $$= \sum_{n=0}^{\infty} \zeta(q,\omega)^n$$

$$= \frac{1}{-\zeta(q,\omega)+1} = \frac{1}{(-i\omega + Dq^2)\tau}$$

$$\Pi =$$ 

$$= \left(1 + \frac{i\omega\tau}{(-i\omega + Dq^2)\tau}\right)\nu$$ (other terms in π unaffected)

$$= \frac{Dq^2 \nu}{-i\omega + Dq^2}$$

$$\Pi^R(q,\omega) = \frac{Dq^2\nu}{-i\omega + Dq^2}$$

1) It is retarded

2) It obeys diffusion equation

3) $\Pi(q=0) = 0$

$$\int \delta\rho(r,t)\,dr = 0 \quad \left(\begin{array}{c}\text{Conservation}\\ \text{Law}\end{array}\right)$$

4) Correct form for any value of $\hbar/\varepsilon_F\tau$

– Crossings renormalise D

5) $\Pi^A = (\Pi^R)^*$

Thermal density fluctuations

$$\Pi^K = (2N_\omega + 1)(\Pi^A - \Pi^R)$$

$$= \mathrm{Coth}\left(\frac{\omega}{2T}\right) \cdot \frac{2i\omega Dq^2}{\omega^2 + (Dq^2)^2}$$

(it is a challenge to derive it !)

Remarks about classical limit.

- We have discussed $\langle \rho(0,0)\rho(r,t)\rangle = \Pi(r,t) \to \Pi(q,\omega)$

$$\Pi^R = \nu \frac{Dq^2}{-i\omega + Dq^2} = \nu\left(1 + \frac{i\omega}{-i\omega + Dq^2}\right) = (\Pi^A)^*$$

This is exact formula for small enough ω and q

$$\Pi^K(q,\omega) = (2N_\omega + 1)(\Pi^A - \Pi^R)$$

$$\to \langle \rho\rho\rangle_{q,\omega} = N_\omega \nu \frac{\omega D q^2}{\omega^2 + (Dq^2)^2} \xrightarrow[\omega \ll T]{} T\nu \frac{Dq^2}{\omega^2 + (Dq^2)^2}$$

$N_\omega = \dfrac{1}{e^{\omega/T} - 1}$ get from

pair of fermions act as boson - bose statistics

$$\int n_\varepsilon (1 - n_{\varepsilon+\omega}) \to \omega N_\omega$$

This form for Keldysh G.F. is exact in equilibrium

Only the value we have used for D is quasi-classical. It becomes renormalised by higher order terms.

Current - Current Correlator

$\vec{j}(r,t)$ - current density

$$\langle j_\alpha(0,0)\, j_\beta(r,t)\rangle = ?$$

Can make use of continuity relation $\quad \dfrac{\partial \rho}{\partial t} = -\nabla . \vec{j}$

$$\nabla \to i\vec{q}$$

$$\frac{\partial}{\partial t} \to -i\omega$$

$$\langle j_\alpha j_\beta \rangle_{q,\omega} q_\alpha q_\beta = \omega^2 \langle \rho\rho \rangle_{\omega,q}$$

$$\langle j_\alpha j_\beta \rangle_{q,\omega} = \underbrace{\langle j_\alpha j_\beta \rangle^{Sym}_{q,\omega}}_{= \langle j_\beta j_\alpha \rangle^{Sym}_{q,\omega}} + \underbrace{\langle j_\alpha j_\beta \rangle^{Asym}_{q,\omega}}_{= -\langle j_\beta j_\alpha \rangle^{Asym}_{q,\omega}}$$

Can be obtained.

This is complex part of calculation – Hall...

In isotropic situation

$$\langle j_\alpha j_\beta \rangle^{Sym} = \langle j\, j \rangle \delta_{\alpha\beta}$$

why not the tensor ?

longitudinal – transverse

$$\langle j\, j \rangle^{Sym}_{q,\omega} = \frac{\omega^2}{q^2} \langle \rho\rho \rangle_{q,\omega}$$

$$\langle \rho\rho \rangle_{q,\omega} = \frac{D_{\alpha\beta}\, q_\alpha q_\beta}{-i\omega + D_{\alpha\beta}\, q_\alpha q_\beta}\ \nu$$

$$\Rightarrow \quad \langle j_\alpha j_\beta \rangle^{Sym} = \nu \frac{D_{\alpha\beta}\, \omega^2}{-i\omega + D_{\alpha\beta}\, q_\alpha q_\beta}$$

e^2 if electric current

Einstein relation $\quad \sigma_{\alpha\beta} = e^2 \nu D_{\alpha\beta}$

$$\langle j_\alpha j_\beta \rangle^{Sym}_{q,\omega} = \sigma^{Sym}_{\alpha\beta} \frac{\omega^2}{-i\omega + Dq^2}$$

$$\xrightarrow[q \to 0]{} \sigma^{Sym}_{\alpha\beta}\, i\omega \qquad \text{Kubo Formula}$$

$$\langle j_\alpha j_\beta \rangle^{K \sim \text{Keldysh}}_{q,\omega} = \operatorname{Coth}\frac{\omega}{2T} \cdot 2i \operatorname{Im} \langle j_\alpha j_\beta \rangle^R$$

$$\xrightarrow[q \to 0]{} 2i \operatorname{Coth}\frac{\omega}{2T} \operatorname{Re}\sigma \cdot \omega$$

$$\xrightarrow[\omega \ll T]{} 2iT \operatorname{Re}\sigma$$

Fluctuation - Dissipation Theory

This completes semi-classical theory of metallic state

Conclusions

1) Boltzman Equation $\longleftrightarrow$ Diagrams (ladder approximation)

2) Based on parameter $\hbar/p_F l = \hbar/E_F\tau$

3) Quantum corrections - crossing over impurity lines

4) Inconvenient - whole series of diagrams - delayed understanding of theory of localisation by 25 years!

Diffusion Modes

For only elastic processes, many more objects have diffusive form

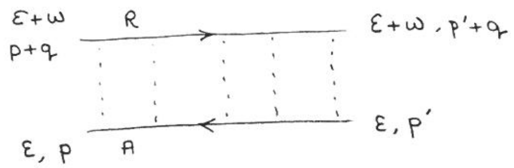
Have conservation law of particle number, but there are many more symmetries.

Elastic scattering - number of particles with given energy is also conserved

Laddars were renormalisation of vertex $\sum$  $= \Gamma(\varepsilon, q, \omega)$

$$\Gamma(\varepsilon, q, \omega) = \sum_{n=0}^{\infty} \zeta^n = \frac{1}{1-\zeta(q,\omega)} = \frac{1}{\tau(-i\omega + Dq^2)}$$

Can instead two particle Green function



$$\cdots\cdots = \gamma = \frac{1}{2\pi\nu\tau}$$

$$= \frac{1}{2\pi\nu\tau^2} \cdot \frac{1}{(-i\omega + Dq^2)} = \mathcal{D}^R(q,\omega) = (\mathcal{D}^A)^*$$

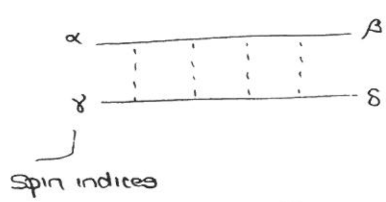
(Keldysh one doesn't have any physical sense)

$\mathcal{D}^R$ looks like Goldstone Mode of some particle

More symmetries

Disorder violates spatial symmetry

a) if no magnetic scattering => spin rotation symmetry



$$= \mathcal{D}^{\alpha\gamma}(q,\omega)\, \delta_{\alpha\beta}\, \delta_{\gamma\delta} \quad \propto \frac{1}{i\omega + Dq^2}$$

Same diffusion pole even if we consider certain components of spin

If there is weak inelastic scattering 'particles' have a finite lifetime τ_ε – inelastic time

$$\frac{1}{-i\omega + Dq^2} \longrightarrow \frac{1}{-i\omega + Dq^2 + 1/\tau_\varepsilon}$$

Only approximate conservation of particle number

In principle there will also be a spin relaxation time $1/\tau_s$

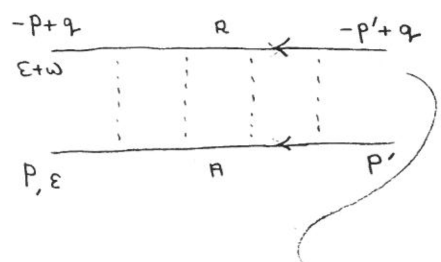
b) T – invariance

(easily broken by Magnetic field)

$t \longrightarrow -t$

$p \xrightarrow{T} -p$ momenta change sign

$\psi \longleftrightarrow \psi^*$

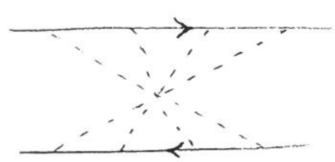


T transform to upper line

$$G(p) = G(-p)$$

$$\gamma \int G^R_{\varepsilon+\omega}(-p+q)\, G^A_{\varepsilon}(p)\, (dp) = \zeta(q,\omega)$$

$$\frac{1}{2\pi\nu\tau^2} \quad \frac{1}{(-i\omega + Dq^2)}$$



maximally crossed diagram

Correct when $ql \ll 1$

$\omega\tau \ll 1$

This mode, in contrast to first, can be destroyed by breaking T-invariance as well as # of particles with a given energy invariance.

τ_ϕ – phase relaxation time

The simple ladder is diffusion pole, the crossed ladder is Cooper pole

<u>D</u>iffuson $\qquad\qquad$ <u>Coop</u>eron

Conductivity

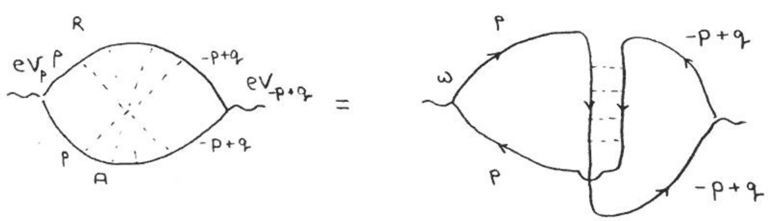
$\sigma =$  $\frac{1}{i\omega}$

Calculation of  is difficult and gives small correction

$$\sigma_0 = \frac{Ne^2\tau}{m} \quad \text{- Drude conductivity} = \text{constant}$$

Does quantum Mechanics give interesting dependence

$$\sigma(\omega, T, H, L, \dots)\ ?$$



V_p and V_{-p+q} are still well correlated and differ by a sign

$$\delta\sigma_{\alpha\beta} = e^2 \int (dq)\, C(q,\omega) \int (dp) \frac{d\varepsilon}{2\pi\omega} (n_\varepsilon - n_{\varepsilon+\omega})$$

Cooper pole

$$\times\, G^A(p,\varepsilon)\, G^R(p,\varepsilon+\omega)\, G^A(-p+q,\varepsilon)\, G^R(-p+q,\varepsilon+\omega)$$

$$\times\, (V_p)_\alpha (V_{-p+q})_\beta$$

$G(p) = G(-p)$ and neglect q, ω on r.h.s

$$\int (dp)\, G^R(p,\varepsilon)^2\, G^A(p,\varepsilon)^2 = \int \nu\, d\xi\, G^{R^2} G^{A^2} = 4\pi\nu\tau^3$$

$$\int (V_p)_\alpha (V_{-p})_\beta \frac{d\Omega}{4\pi} = -\frac{v^2}{3}\delta_{\alpha\beta}$$

$$\delta\sigma_{\alpha\beta} = -\delta_{\alpha\beta}\frac{De^2}{\pi} 2\pi\nu\tau^2 \int (dq)\, C(q,\omega)$$

$$\delta\sigma = -\frac{De^2}{\pi} 2\pi\nu\tau^2\, C(\omega, \vec{r}_i = \vec{r}_f)$$

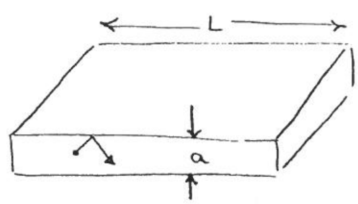
$$= -\frac{De^2}{\pi}\int^{\ell^{-1}} \frac{d\vec{q}}{(2\pi)^d}\frac{1}{Dq^2 - i\omega}$$

$$\delta\sigma = \begin{cases} \text{const} + \dfrac{e^2}{2\pi^2\hbar}\sqrt{\dfrac{-i\omega}{D}} & d=3 \\[2ex] -\dfrac{e^2}{2\pi^2\hbar}\ln\left(\dfrac{1}{\omega\tau}\right) & d=2 \\[2ex] -\dfrac{e^2}{\pi\hbar}\sqrt{\dfrac{D}{\omega}} & d=1 \end{cases}$$

($d=2$ and $d=1$:) diverge when $\omega \to 0$

What is the meaning of $d \neq 3$?

$d = 2$



Boundary conditions – particles do not leave system but are reflected at surface

$$\left.\frac{\partial G(r,r';\omega)}{\partial \vec{n}}\right|_{\vec{r}\in S} = 0$$

$$\int (dq) \to \frac{1}{a}\sum_{q_n} \qquad q_n = \frac{\pi n}{a}, \quad n = 0, 1$$

a) $\frac{D}{a^2} \ll \omega$, $n \gg 1$

$$\frac{1}{a}\sum \rightarrow \int (dq) \qquad d = 3$$

b) $\frac{D}{a^2} \gg \omega$, $n = 0$ dominates

$$\int (dq) \longrightarrow \frac{1}{a}\int (dq_{\parallel}) \qquad d = 2$$

$$\begin{cases} a \gg L_\omega = \sqrt{\frac{D}{\omega}} & 3d \\ a \ll L_\omega & 2d \end{cases}$$

Physically, in time $\hbar/\omega$ particle has reflected many times and its density is uniform in perpendicular direction

– Can have wire and dot $(d = 0)$

Conductance

Not local, but global property

g – dimensionless conductance in units of e^2/h

$$e^2/2\pi^2\hbar = 1.26 \times 10^{-5}\ \Omega^{-1} \qquad g = \frac{\sigma\hbar}{e^2} L^{d-2}$$

For $L \sim L_\omega$, $\delta g \sim 1$

$$g_0 \propto (k_F \ell)(k_F L)^{d-2}(k_F a)^{3-d} \qquad \text{In} \quad \begin{matrix} d : L \\ 3-d : a \end{matrix}$$

$$\frac{\delta g}{g_0} = \frac{\delta\sigma}{\sigma} \sim \frac{1}{k_F \ell}\cdot\left(\frac{L}{a}\right)^{2-d} \begin{cases} \xrightarrow{L\to\infty} 0 & d = 3 \\ \xrightarrow{L\to\infty} \infty & d \lesssim 2 \end{cases}$$

In low d, quantum corrections are not small by $1/k_F l$ but can become large due to IR divergency

Let G = conductance

g = dimensionless conductance unit $= e^2/h$

$$\left[\frac{e^2}{2\pi^2\hbar} = 1.26 \times 10^{-5}\ 1/\text{ohm} \right]$$

$$g = \frac{\sigma\hbar}{e^2} L^{d-2}$$

Suppose sample size $L \simeq L_\omega$, then we get $\delta g \sim 1$ to logarithmic accuracy.

effective dimension as discussed

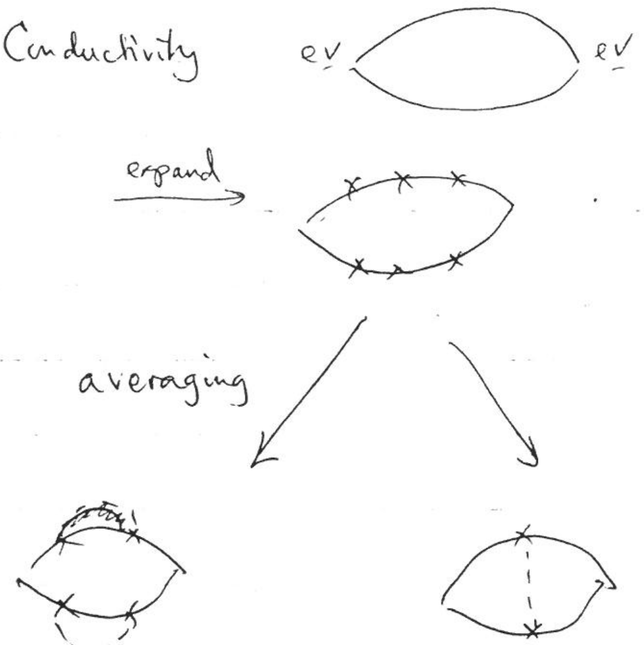
Drude: $g_0 \propto (k_F l)(k_F L)^{d-2}(k_F a)^{3-d}$ ~~Sample~~ $d=2$ for $a \ll L_\omega$; $d=3$ .. $a \gg L_\omega$

$$\frac{\delta g_0}{g_0} = \frac{\delta\sigma}{\sigma} \sim \frac{1}{k_F l}\left(\frac{L}{a}\right)^{2-d} \to \begin{cases} 0 & d=3 \\ \infty & d=2 \quad (\ln L/a) \\ \infty & d=1 \end{cases}$$

as $L \to \infty$

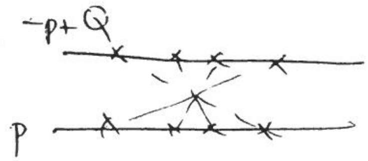
- Summary

Conductivity



Drude $\sigma = \frac{ne\tau^2}{m}$. quantum corrections

Cooperon correction = (small Q)



$$= \frac{1}{2\pi^2\nu\tau^2}\;\frac{1}{-i\omega + DQ^2}$$

Symmetries : # particles at given energy $\rightarrow$ diffuson gapless

$\downarrow$ time reversal

Cooperon gapless .

Dimensions : $L_x \times L_y \times L_z$

Let $L_x \geq L_y \geq L_z$

Diffusion : $L_\omega = \sqrt{D/\omega}$

(1) $L_x \gg L_\omega \gg L_y, L_z$: $d = 1$

(2) $L_x, L_y \gg L_\omega \gg L_z$: $d = 2$

(3) $L_x, L_y, L_z \gg L_\omega$: $d = 3$

For simplicity, consider : d dimensions of size $L \gg L_\omega$

: $3-d$ " " " $a \ll L_\omega$.

Conductivity : $\sigma = \frac{1}{R} a^{3-d} L^{d-2}$

$\Downarrow$

$\sigma_d = \frac{L^{d-2}}{R} = \sigma a^{d-3}$ is d-dimensional conductivity.

$$\delta\sigma_d = \begin{cases} \text{Const} + \frac{e^2}{2\pi^2\hbar}\sqrt{\frac{-i\omega}{D}} & d=3 \\ -\frac{e^2}{2\pi\hbar}\ln \frac{1}{\omega\tau} & d=2 \\ -\frac{e^2}{\pi\hbar}\sqrt{D/\omega} & d=1 \end{cases}$$

Let g = dimensionless conductance = Conductance / $e^2/\hbar$

Effective parameter : $\frac{\delta g}{g} \propto \frac{1}{k_F \ell}\left(\frac{L}{\ell}\right)^{2-d} (k_F a)^{d-3}$ $(k_F\ell,\ k_F a \gg 1$

Assume that, when $\delta g \sim g$, $L \sim \xi$ localisation length

$\longrightarrow$ All states localised in $d \leq 2$!!

$d=2$: $\xi \sim \ell\, e^{k_F^2 \ell a}$

$d=1$: $\xi \sim \ell\, (k_F a)^2$

Anderson PR (1958)

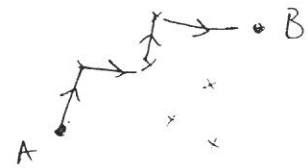
Ioffe-Regel (1960)

Mott & Twose (1961) Adv. Physics : Localisation in pure 1d case

Thouless PRL (1977) " " Wire with finite cross-section

Abrahams, Anderson, Liciardello & Ramakrishnan . PRL (1979)

Physical Picture



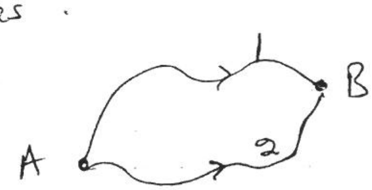
$P(A \xrightarrow{t} B)$ Probability from A to B in time t

$$P(A \xrightarrow{t} B) = P(r_{AB}, t) = \frac{1}{(2\pi D t)^{d/2}} e^{-r_{AB}^2/2Dt}$$

if we have simple diffusion equation.

Quantum mechanics : need sum over trajectories $\rightarrow$ interference

Consider two trajectories.



($r_{AB} \gg$ wavelength)

$$P = |\underbrace{A_1}_{\text{amplitude}} + A_2|^2 = p_1 + p_2 + 2\,\mathrm{Re}(A_1 A_2^*)$$

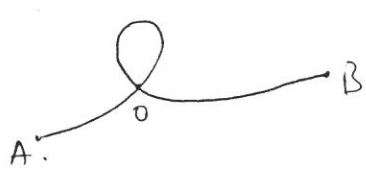
Let $A_{1,2} = |A_{1,2}| e^{i\phi_{1,2}}$ where $\phi_i = \frac{1}{\hbar}\int_i \underline{p}\cdot d\underline{r} = \frac{1}{\hbar}\int \sqrt{2m(E-V)}\, d\underline{r}$

$$\text{i.e. } P = P_1 + P_2 + 2\sqrt{P_1 P_2}\cos(\phi_1 - \phi_2)$$

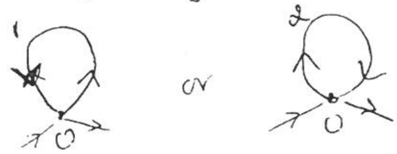
$\downarrow$ average

kills interference term.
with <u>exceptions</u>.

<u>Exceptions</u> :



Self-crossing trajectory : 2 directions to go around loop



e.g. impurity at O.

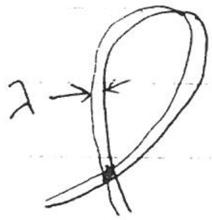
$$\phi_1 = \int \underline{p}(r)\, d\underline{r}$$
$$\phi_2 = \int [-\underline{p}(r)][-d\underline{r}] = \phi_1 \quad \Longrightarrow \quad \cos(\phi_1 - \phi_2) = 1$$

<u>not</u> cancelled by averaging

$\Rightarrow \quad P = |A_1 + A_2|^2 = 2 \times P_{classical}$ for this ~~case~~ loop .

Coherent backscattering

We want trajectory to return back with accuracy of wavelength λ.



δP = correction to probability P due to these paths .

$$\frac{\delta P}{P} \propto \frac{\delta D}{D} \sim \frac{\delta \sigma}{\sigma}$$

since P should depend on diffusion constant D only .

(1) $d=3$: $dt \times \delta P = \frac{1}{(Dt)^{3/2}} \cdot \lambda^2 \cdot v\, dt$

$\frac{1}{(Dt)^{3/2}}$: prob to return to unit volume

λ^2 : cross-section

$$\Rightarrow \quad \delta P = \int_{\tau}^{t^*} \frac{\lambda^2 v}{(Dt)^{3/2}}\, dt \qquad t^* = \min\left(\frac{1}{\omega}, \tau_\phi, \frac{D}{L^2}\right)$$

τ : elastic scattering

cutoff : $\frac{1}{\omega}$ finite freq. , τ_ϕ inelastic , $\frac{D}{L^2}$ finite size

$\frac{\delta P}{P} \sim \delta P$ give

$$\frac{\delta D}{D} = -\int_{\tau}^{t^*} \frac{\lambda^2 v\, dt}{(Dt)^{3/2}}$$

$$= O\left(\frac{1}{k_F l}\right) - \frac{\lambda^2 v}{D} \frac{1}{L^*} \qquad \left(L^* = \sqrt{Dt^*}\right)$$

$O\left(\frac{1}{k_F l}\right)$: from lower limit

$\left(\lambda^2 v \propto 1/\nu\right)$ So, $\delta D \propto -\frac{1}{\nu}\frac{1}{L^*}$

$$\Rightarrow \quad \delta\sigma = e^2 \nu\, \delta D \sim -\frac{e^2}{\hbar}\frac{1}{L^*}$$

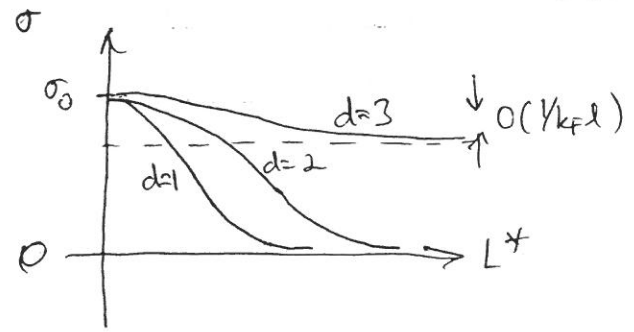
[cf. Diagrammatics $\delta\sigma = \frac{e^2}{2\pi^2\hbar}\sqrt{\frac{-i\omega}{D}}$

$\underline{d<3}$: $(a < L^*)$

After $t_\perp = a^2/D < t^*$, distribution in $\perp$ direction is uniform.

$$\frac{\delta D}{D} \propto -\left(\frac{\lambda}{a}\right)^{3-d} \int_{t_\perp}^{t^*} \frac{\lambda^{d-1} v \, dt}{(Dt)^{d/2}} = \frac{\lambda^2 v}{a^{3-d}} \underbrace{\frac{2}{2-d}\left(\frac{t^*}{D}\right)^{(2-d)/2}}_{\log \text{ as } d \to 2}$$

$$\Rightarrow \delta\sigma \propto -\frac{e^2}{\hbar} \frac{L^{*(2-d)}}{2-d} \frac{1}{a^{3-d}}$$



- Magnetic field — violates T-invariance

$$p \to p - \frac{e}{c}A \quad \text{kinematic momentum}$$

$$\left.\begin{array}{l} p \xrightarrow{T} -p \\ A \xrightarrow{T} +A \end{array}\right\}$$

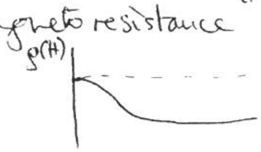
So, phases: $\left.\begin{array}{l} \varphi_1 \to \varphi_1 - \frac{e}{c}\oint_1 \underline{A}\cdot d\underline{r} \\ \varphi_2 \to \varphi_2 + \frac{e}{c}\oint_1 \underline{A}\cdot d\underline{r} \end{array}\right\}$

$$\Rightarrow \varphi_2 - \varphi_1 = 2\,\Phi/\Phi_0 \cdot 2\pi \qquad \left\{\begin{array}{l} \Phi = \text{flux enclosed} \\ \Phi_0 = \text{flux quantum} = hc/e. \end{array}\right.$$

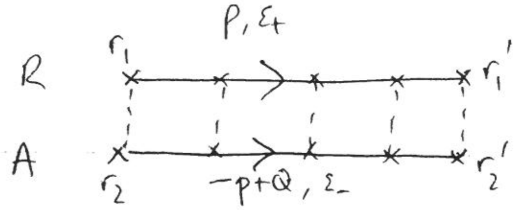
Aharonov – Bohm

→ destruction of coherent backscattering

→ Conductivity increases with H "negative magnetoresistance".



Cooperon as a particle



$r_1 = r_2$, $r_1' = r_2'$ for delta-function correlated disorder.

$$= \frac{1}{2\pi^2 \nu \tau^2} \frac{1}{-i\omega + DQ^2 + 1/\tau_\varphi} \longleftarrow \text{phase breaking} \qquad \left(\varepsilon_\pm = \epsilon \pm \frac{\omega}{2} \right)$$

$= \xi$

Fourier transform

$C_\omega(r, r')$ obeys :

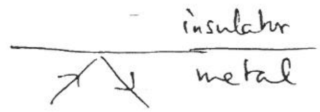
$$\left(-i\omega - D\nabla_r^2 + \frac{1}{\tau_\varphi} \right) C_\omega(r, r') = \frac{1}{2\pi\nu\tau^2} \delta(r - r')$$

"Schrödinger equation" for Cooperon

$$\left.\begin{array}{llll} \text{Mass} = & \dfrac{1}{2D} \sim \dfrac{\hbar}{p_F l} m_e & & \underline{\text{small}} \\ \text{Energy} = & -i\omega + \dfrac{1}{\tau_\varphi} & & \text{complex!} \end{array}\right\}$$

Boundary condition :

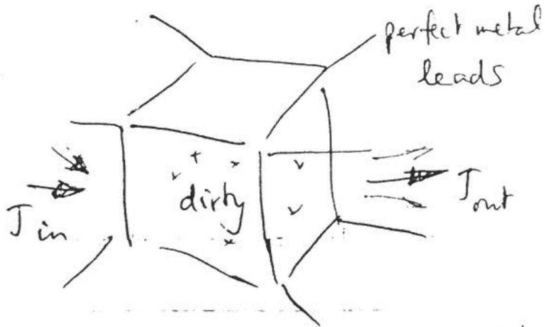
(a) Metal - insulator interface :



$$j_n = 0 , \quad \underline{n} \cdot \underline{\nabla} \rho \Big|_{\text{Surface } S} = 0$$

$$\Rightarrow \quad \underline{n} \cdot \underline{\nabla}_r C_\omega(r, r') \Big|_{r \in S} = 0$$

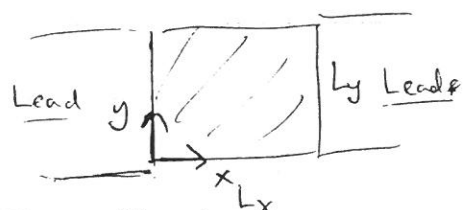
(b) Metal-metal interface



$$\delta\varphi\Big|_{\text{interfaces}} = 0 \quad \Rightarrow \quad C(r,r')\Big|_{r\in S} = 0$$

for perfect metal leads
or for leads in which electron loses phase memory completely

Therefore,



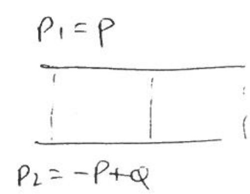
~~use eigenfunctions~~ Eigenfunctions are:

$$\begin{aligned} \psi_n(y) &\propto \cos Q_{yn} y \quad ; \quad Q_{yn} = \frac{\pi n}{L_y} \quad n = 0,1,2,\dots \\ \phi_m(x) &\propto \sin Q_{xm} x \quad ; \quad Q_{xm} = \frac{\pi m}{L_x} \quad m = 1,2,\dots \end{aligned}$$

i.e. $\quad D Q_x^2 \geq \dfrac{\pi^2 D}{L_x^2}$ $\qquad$ RHS = Thouless energy.

$$\sigma(\omega) \propto \int (dQ)\, C_\omega(Q) \longrightarrow \sum_{nm} C(\underline{Q}, \omega).$$

Effect of magnetic field,



$$p_{1,2} \to p_{1,2} - \frac{e}{c} A$$

$$\therefore \quad Q = p_1 + p_2 \longrightarrow \underline{Q} - 2\frac{e}{c}\underline{A}$$

Cooperon charge $= 2e$
[diffuson has no charge]

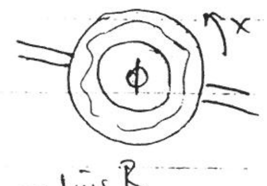
$$\text{i.e.} \quad \left[-i\omega + \frac{1}{\tau_\phi} - D\left(-i\underline{\nabla}_r - \frac{2e}{c}\underline{A}\right)^2 \right] C_\omega(r,r') = \frac{\delta(r-r')}{2\pi\nu\tau^2}$$

Conductivity:

$$\delta\sigma(\omega, \tau_\phi, L) = -2D\nu\tau^2 \int C(Q)(dQ)$$
$$= 2D\nu\tau^2 \; C(r,r)$$

"Density of states" for Cooperon

e.g. Aharonov-Bohm effect



radius R

$$\phi = \frac{e}{c} \cdot A \; 2\pi R \quad ; \quad A = \frac{1}{2\pi R} \frac{\phi}{\phi_0}$$

$$C_\omega(x,x') = \sum_m \frac{\psi_m(x)\psi_m^*(x')}{-i\omega + E_m + 1/\tau_\phi} \qquad \begin{cases} \psi_m \sim e^{iQ_m x} \\ E_m = D(Q_m - 2eA)^2 \end{cases}$$

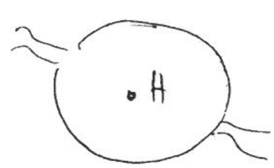
$$Q_m = \pi m / 2\pi R$$

$$\Rightarrow \delta\sigma \propto \frac{e^2}{\hbar} \sum_{m=-\infty}^{\infty} \left[\left(-i\omega + \frac{1}{\tau_\phi}\right) + \frac{D}{R^2}\left(m - \frac{2\phi}{\phi_0}\right)^2 \right]^{-1}$$

NB: periodic in ϕ

e.g.

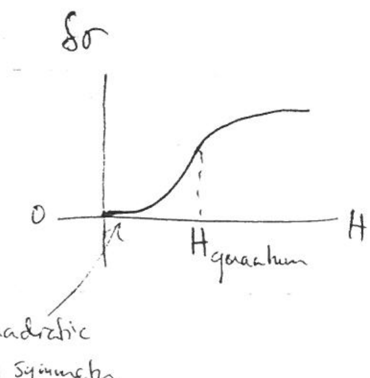
Magnetoresistance



Eigenfns: Landau levels $\psi_m(\underline{r})$ with $E_m = \Omega_H (m + 1/2)$

$$\Omega_H = \text{"}\frac{eH}{mc}\text{"} = \frac{(2e)H}{(2D)^{-1} c} \quad \text{for Cooperon}$$
$$= 4eDH/c$$

$$\rightarrow \delta\sigma(H) \propto \frac{e^2}{\hbar} f(\Omega_H \tau_\phi)$$
$$\propto \frac{e^2}{\hbar} f\left(\frac{4eD\tau_\phi}{\hbar c} H\right)$$



What is the scale $H_{quantum}$ for the magnetic-field crossover?

classical	quantum
$\omega_c \tau \sim 1$	$\Omega_H \tau_\varphi \sim 1$

but $\Omega_H \sim (k_F l)\,\omega_c$

$$\Rightarrow \Omega_H \tau_\varphi \sim \omega_c \cdot \varepsilon_F \tau \cdot \frac{\tau_\varphi}{\tau}$$

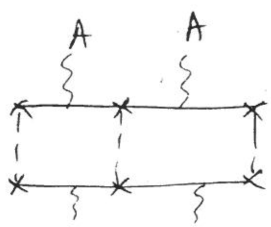
$$\propto \omega_c (\varepsilon_F \tau_\varphi)$$

$\frac{1}{\tau_\varphi} \simeq 0.1\,K$, $\varepsilon_F \sim 10^5 K$

$\Rightarrow H_{quantum} \sim H_{classical} \times (10^{-5} - 10^{-6})$

Derivation of Cooperon Schrodinger equation

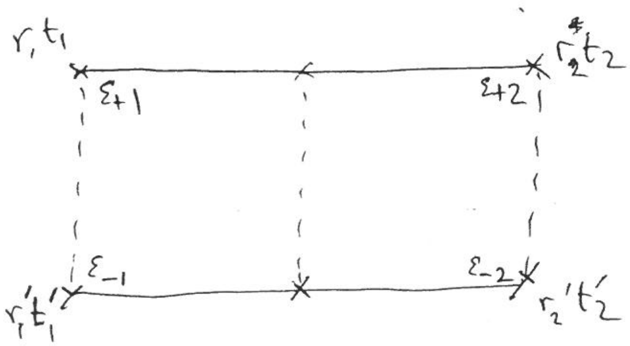
Let $\underline{A}(\underline{r},t) = A_{k\Omega}\, e^{i(\underline{k}\underline{r} - \Omega t)}$



For $\Omega = 0$:

$\frac{1}{2m}\left(p^2 - \frac{e}{c}A\right)^2 \rightarrow -\frac{e}{c}\,\underline{v}\cdot\underline{A}$ vertices.

General case with time dependence



$\varepsilon_\pm = \varepsilon \pm \frac{\omega}{2}$ for 1 and 2

$\varepsilon_1 \neq \varepsilon_2$ because of $\underline{A}(r,t)$

Variables :

$$t = \tfrac{1}{2}(t_1 + t_1') \;;\quad t' = \tfrac{1}{2}(t_2 + t_2')$$
$$\eta = t_1' - t_1 \;;\quad \eta' = t_2' - t_2$$

Set: $r_1 = r_1' = r$, $r_2 = r_2' = r'$

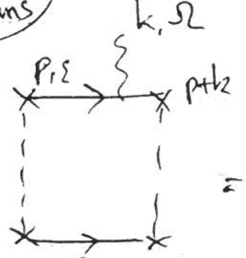
Want to calculate $C^{tt'}_{\eta\eta'}(r, r')$

Dyson equation :

$$C(Q, \varepsilon^+, \varepsilon^-) = C^{(0)}(Q, \varepsilon^+ \varepsilon^-)\left[1 + \hat{\zeta}_Q(\varepsilon^+, \varepsilon^-)\, C(Q, \varepsilon^+, \varepsilon^-)\right]$$

where $C^{(0)} = \dfrac{1}{2\pi\nu\tau^2}\,\dfrac{1}{-i\omega + DQ^2}$

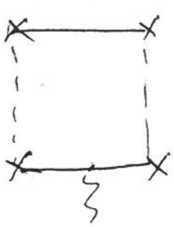
1st order diagrams



so, $Q \to Q + k$ and $\varepsilon^+ \to \varepsilon^+ + \Omega$

$= \dfrac{e}{c} A \dfrac{\partial}{\partial Q} \zeta_0$ where $\zeta_0 = 1 + i\omega - DQ^2$ $\quad(\omega = \varepsilon^+ - \varepsilon^-)$

and

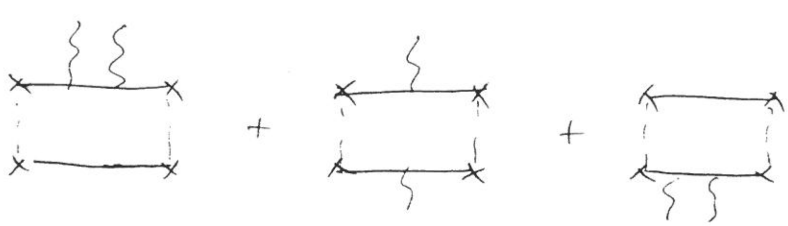


take $Q \to Q + k$ and $\varepsilon^- \to \varepsilon^- + \Omega$.

Use Shift operator : $f(\varepsilon) \longrightarrow f(\varepsilon + \Omega) = e^{\Omega \frac{\partial}{\partial \varepsilon}} f(\varepsilon)$

2nd order diagrams

$\propto \dfrac{\partial^2 \zeta_0}{\partial Q_\alpha \partial Q_\beta} A_\alpha A_\beta$



So, combining 1st & 2nd order diagrams,

$$\left[C^{o(0)^{-1}}(Q, \varepsilon^+ - \varepsilon^-) - \hat{\varsigma} \right] C = 1$$

$$\Rightarrow \quad -i(\varepsilon^+ - \varepsilon^-) + D\left[\underline{Q} - \frac{e}{c}\left(A_{k\Omega}\left(e^{\Omega \partial/\partial\varepsilon^+} + e^{\Omega \partial/\partial\varepsilon^-} \right) e^{k\frac{\partial}{\partial Q}} \right)^2 \right] C = 1$$

Fourier transform $\longrightarrow$

$$\frac{\partial}{\partial\eta} + D\left[-i\underline{\nabla}_r - \frac{e}{c}\left(\underline{A}(\underline{r}, t - \eta/2) + \underline{A}(r, t + \eta/2) \right)^2 \right]^2 C^{tt'}_{\eta\eta'}(\underline{r}, \underline{r}')$$

$$= \delta(t - t')\delta(\eta - \eta')\delta(\underline{r} - $$

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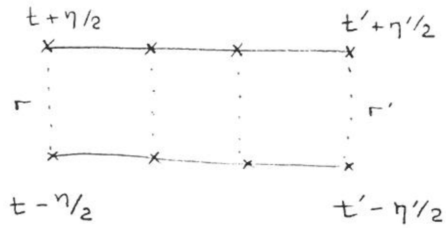
Cooperon



$$= C^{tt'}_{\eta\eta'}(r,r')$$

$$\left\{ \frac{\partial}{\partial\eta} + D\left[-i\nabla_r - \frac{e}{c}\mathcal{A}\right]^2 \right\} C^{tt'}_{\eta\eta'}(r,r') = \frac{1}{2\pi\nu\tau^2}\,\delta(\eta-\eta')\,\delta(t-t')\,\delta(\underline{r}-\underline{r}')$$

$$\mathcal{A} = A(r, t+\eta/2) + A(r, t-\eta/2)$$

derived from diagrammatics

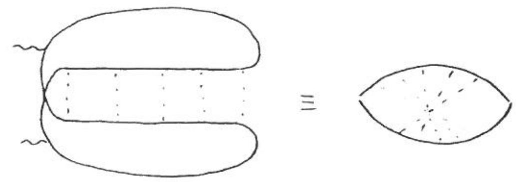
effective vector potential

t is a parameter (there are no $\partial/\partial t$'s)

So $\quad C^{tt'}_{\eta\eta'}(r,r') = C^{t}_{\eta\eta'}(r,r')\,\delta(t-t')$

$$\delta\sigma = -8\sigma_0\,\tau^2\int d\eta\, C^{t-\eta/2}_{\eta,-\eta}(\vec{r},\vec{r})$$

classical



Gives AC conductivity, etc.

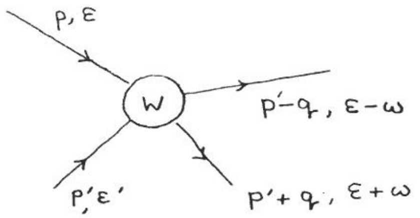
Infrared Dressing

Insert $1/\tau^*_\varphi$ into Cooperon equation $\qquad \frac{\partial}{\partial\eta} \to \frac{\partial}{\partial\eta} + 1/\tau^*_\varphi$

Electron-phonon scattering

$$\hbar/\tau_{e\text{-}ph} \propto T^3/\Theta_D^2$$

Electron-electron scattering



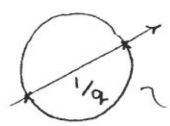
Within Landau Fermi-Liquid Theory

$$\hbar/\tau_{e\text{-}e} \propto T^2/E_F = \int^{f(\varepsilon/T)} d\varepsilon \int_0^{\varepsilon} d\omega \int_{-\omega}^{0} d\varepsilon' \int (dq)\, \frac{W(q)}{(q.v_F)^2}$$

Clean case, $l \to \infty$

$\varepsilon, \varepsilon'$ are counted from the Fermi level

$$1/qv_F = t_{int}$$



In dirty case $\quad l$ is finite $\qquad t_{int} \sim \dfrac{1}{|-i\omega + Dq^2|}$

$$ql \ll 1$$

$$\hbar/\tau_{ee} \propto \int_0^{\varepsilon} d\omega \int_{-\omega}^{0} d\varepsilon' \; \frac{(dq)\, W(q)}{|-i\omega + Dq^2|^2} \propto \int_0^{\varepsilon} \frac{\omega\, d\omega}{(i\omega)^{2-d/2}}$$

$$\propto \begin{cases} \varepsilon^{3/2} & d=3 \\ \varepsilon & d=2 \\ \varepsilon^{1/2} & d=1 \end{cases}$$

But we assume $\varepsilon\tau_{ee} \gg 1$ for Fermi-liquid theory $\left(\begin{array}{c} \text{satisfied if } L < \xi \\ \text{i.e. metallic} \end{array} \right)$

What happens in dirty case, $T \neq 0$

$$1/\tau_{ee} \propto \int \frac{\omega\, d\omega}{\omega^{2-d/2}} \coth \omega/2T$$

Bose distribution function

$$\to \int_0 \frac{d\omega}{\omega^{2-d/2}} T$$ diverges at $d = 1, 2$ /

What is happening?

$\omega \ll T$, $\Rightarrow N_\omega \gg 1$

Coulomb interaction generates e-m field - do we have to quantise? No

Classical electromagnetic field

How does electron-electron interaction look formally?

Fluctuation dissipation theorem

σ is finite $\longrightarrow$ fluctuations of j in the eq

Correlation function

$$\langle A_\alpha(r,t) A_\beta(r',t') \rangle_{k,\omega}$$

$\nearrow$ longitudinal $\propto k_\alpha k_\beta$

$\searrow$ transverse $\propto \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right)$

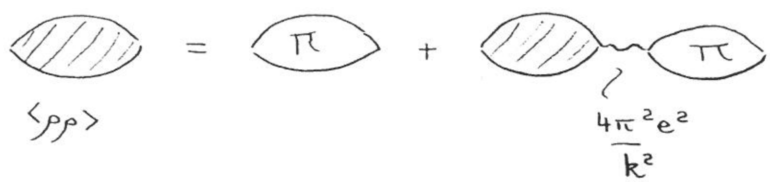
$$\langle A_\alpha A_\beta \rangle = \frac{c^2}{\omega^2} \langle E_\alpha E_\beta \rangle \Rightarrow \frac{(4\pi)^2 c^2}{\omega^2} \langle \rho\rho \rangle = k_\alpha k_\beta \langle A_\alpha A_\beta \rangle$$

Poissons Equation

$$i k_\alpha E_\alpha = 4\pi \tilde{\rho}$$ excess density

transverse part disappears

$$\langle \rho_0 \rho_0 \rangle = e^2 \nu \frac{Dk^2}{(-i\omega + Dk^2)} = \Pi(k,\omega)$$ without Coulomb interaction



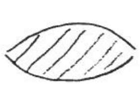
$$= \frac{e^2 \Pi}{1 + \frac{4\pi e^2}{k^2}\Pi} = \frac{e^2 \nu D k^2}{(-i\omega + Dk^2)} \frac{1}{\left(1 + \frac{4\pi e^2}{k^2} \frac{Dk^2}{(-i\omega + Dk^2)} \nu \right)}$$

$$= \frac{\sigma k^2}{-i\omega + Dk^2 + 4\pi\sigma}$$

peculiar to coulomb – pole disappears

$4\pi\sigma = D\kappa^2$

$\frac{1}{\kappa} = r_D$ – screening length



K ~ Keldysh

$$= \coth\frac{\omega}{2T} \frac{\omega\sigma k^2}{(4\pi\sigma)^2}$$

dropping $(-i\omega + Dq^2)$ because its small

real density-density correlation in Coulomb gas

$$k_\alpha k_\beta \langle A_\alpha A_\beta \rangle = \frac{c^2}{\sigma} \frac{k^2}{\omega^2} \omega \coth \omega/2T$$

$$\langle A_\alpha A_\beta \rangle^L = c^2 \frac{k_\alpha k_\beta}{k^2} \frac{\coth \omega/2T}{\omega\sigma} \xrightarrow[\omega \ll T]{} \frac{2Tc^2}{\sigma\omega^2} \frac{k_\alpha k_\beta}{k^2}$$

transverse part

$$\langle A_\alpha A_\beta \rangle^t = \frac{2T}{\sigma(\omega^2)} c^2 \frac{1}{1+(k^2\delta_\omega^2/2)^2}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)$$

Skin depth $\delta_\omega = \dfrac{c}{\sqrt{2\pi\omega\sigma}}$

(Note the hope (FLT) assumes that internal lines just renormalise D and do not change form of expression)

Path Integral

$$C^t_{\eta\eta'} = \frac{1}{2\pi\nu\tau^2}\int_{r(\eta)=r}^{r(\eta')=r} \mathcal{D}r(t_1)\, \exp\left\{\int\left[\frac{\dot r^2}{4D} + i\dot r A + \frac{1}{\tau_\varphi^*}\right]dt_1\right\}$$

mass $\frac{1}{2D}$

for a certain realisation of A

Average Gaussian fluctuation in A $\quad \langle e^{i\phi}\rangle = e^{-\langle\phi^2\rangle/2}$

$$C^t_{\eta-\eta}(r,r) = \frac{1}{2\pi\nu\tau^2}\int_{r(-\eta)=r}^{r(\eta)=r}\mathcal{D}r(t_1)\,\exp\left[-\int_{-\eta}^{\eta}dt_1\left[\frac{\dot r^2}{4D}\right] - \frac{2\eta}{\tau_\varphi^*} - S_N\right]$$

$$S_N = \frac{e^2 T}{2\sigma}\iint_{-\eta}^{\eta} dt_1\, dt_2\, \dot r_\alpha(t_1)\,\dot r_\beta(t_2)\,\langle A_\alpha(1) A_\beta(2)\rangle$$

$$\langle A_\alpha(1) A_\beta(2)\rangle = 2\int_{-\infty}^{\infty}(dk)\frac{d\omega}{2\pi}\left(\cos\frac{\omega}{2}(t_1+t_2) + \cos\frac{\omega}{2}(t_1-t_2)\right)\underbrace{e^{ik(r_1-r_2)}}_{}$$

$r(t_1)$

$\langle A_\alpha A_\beta\rangle_{k,\omega}$

$$\int(dk)\, k_\alpha k_\beta\, \dot r_\alpha(t_1)\,\dot r_\beta(t_2)\, e^{ik(r_1-r_2)} = \frac{\partial}{\partial t_1\,\partial t_2}\int(dk)\, e^{ik(r_1-r_2)}$$

$$S_N = \frac{e^2}{\sigma} T \int \frac{(dk)}{k^2} \left\{ 1 - \cos k \left[\vec{r}(t_1) - \vec{r}(t_2) \right] \right\}$$

Since

$$= \frac{e^2}{\sigma} T \int dt_1 dt_2 \int (dk) \frac{e^{ik(r_1 - r_2)}}{k^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\cos \frac{\omega}{2} t_+ + \cos \frac{\omega}{2} t_- \right]$$

$$= \frac{e^2}{\sigma} T \int \frac{(dk)}{k^2} \int_{-\eta}^{\eta} dt_1 \left(1 - \cos k \left(r(t) - r(-t) \right) \right)$$

New variables

$$R(t) = \frac{r(t) + r(-t)}{\sqrt{2}}, \quad \rho(t) = \frac{r(t) - r(-t)}{\sqrt{2}}$$

$$\dot{r}^2 = \dot{R}^2 + \dot{\rho}^2$$

$$C_{\eta,-\eta} = \frac{1}{2\sqrt{2}\pi\nu\tau^2} \int dR \int_{R(0)=R}^{R(\eta)=r\sqrt{2}} \mathcal{D}R(t) \exp\left[-\int dt \frac{\dot{R}^2}{4D} \right]$$

$$\times \int \mathcal{D}\rho(t) \exp\left\{ -\int_0^{\eta} dt_1 \left[\frac{\dot{\rho}^2}{4D} + \frac{2}{\tau_\varphi^*} + \frac{2e^2 T\sqrt{2}}{\sigma} V(\rho) \right] \right\}$$

$$V(\rho) = \int \frac{(dk)}{k^2} \left[1 - \cos k\rho\sqrt{2} \right]$$

P.I. of particle in $V(\rho)$

$d = 1 \qquad V(\rho) = \dfrac{|\rho|}{\sqrt{2}}$

$$\left[\frac{\partial}{\partial\eta} - D\frac{\partial^2}{\partial\rho^2} + \frac{2}{\tau_\varphi^*} + \sqrt{2}\frac{e^2 T}{\sigma}|\rho| \right] C_{\eta,-\eta}(\rho,\rho')$$

$$= \frac{\delta(\eta)\,\delta(\rho-\rho')}{2\sqrt{2}\pi\nu\tau^2}$$

Rescaling $\quad \rho = L_\varphi x, \quad C = \dfrac{\tau_\varphi}{2\sqrt{2}\pi\nu\tau^2 L_\varphi} \bar{C}(x, x')$

$$L_\varphi = \sqrt{D\tau_\varphi} \quad , \quad \tau_\varphi = \left(\frac{\sigma}{e^2 T\sqrt{2}\, D}\right)^{2/3}$$

$$\left[-\frac{\partial^2}{\partial x^2} + 2\left(\frac{L_\varphi}{L_\varphi^*}\right) + |x|\right]\bar{C} = \delta(x-x')$$

$$\delta\sigma = \frac{2\sqrt{2}}{\pi}\frac{e^2}{\hbar}L_\varphi\,\bar{C}(0,0)$$

$$\bar{C}(0,0) = \frac{1}{2}\frac{1}{\left[\ln A_i \frac{2L_\varphi}{L_\varphi^*}\right]'}$$

$$\frac{1}{\tau_\varphi} \propto T^{2/3}$$ — has a physical interpretation (see later)

Now there is no divergence!

d = 2

$$\left[\frac{\partial}{\partial\eta} - D\nabla^2 + \frac{2}{\tau_\varphi^*} + V(\rho)\right]C_{\eta,-\eta} = \frac{\delta(\eta)\,\delta(\rho-\rho')}{4\sqrt{2}\,\pi\nu\tau^2}$$

$$V(\rho) = -\frac{e^2 T}{2\sqrt{2}\,\pi\sigma_2}\ln\frac{\rho}{L_T} \quad , \quad L_T = \sqrt{\frac{D}{T}}$$

$$\delta\sigma_2 = -\frac{e^2}{\pi^2\hbar}\ln\frac{\tau_\varphi}{\tau}$$

where $\frac{1}{\tau_\varphi} = \frac{T}{2\pi D\nu\hbar^2}\ln\left(\pi \underbrace{D\nu}_{\text{conductivity}}\hbar\right)$

linear dependence on T, but not $\frac{T}{k_F l}$ but $\propto \frac{T}{k_F l}\ln k_F l$!

p.t. in $\frac{1}{k_F l}$ leads to divergence but just clearing ln

Both forms checked experimentally

Now, we started with

$$\frac{1}{\tau_\varphi} \propto T \int \frac{d\omega}{\omega^{2-d/2}} \qquad \left(T \int \frac{d\omega}{\omega^2 \nu L_\omega^d} \ , \ L_\omega = \sqrt{\frac{D}{\omega}} \right)$$

and it looked divergent

All problems come from low frequencies – but it looks unphysical for $\omega < 1/\tau_\varphi$.

Suppose its true and we should cut integration at $1/\tau_\varphi$.

$$\int_{1/\tau_\varphi} d\omega \ \frac{1}{\omega^{2-d/2}} \sim -\left(d/2 - 1\right)\left(\frac{1}{\tau_\varphi}\right)^{d/2-1}$$

Leads to equation and

$$\frac{1}{\tau_\varphi} \propto \left[(2-d)\,T\right]^{2/4-d} \qquad \text{for } d \leq 2$$

$$d = 1 \qquad \frac{1}{\tau_\varphi} \propto T^{2/3}$$

$$d = 2 \qquad \frac{1}{\tau_\varphi} \propto \frac{T}{k_F \ell} \ln T\tau_\varphi$$

$$\propto \frac{T}{k_F \ell} \ln k_F \ell \quad \text{after 1st iteration} \ /$$

All orders of p.t. are divergent and P.I allows calculation in correct manner

Let us recall, we have considered Quantum Corrections to Conductivity σ_d

$$\delta\sigma_1 \propto -\frac{e^2}{\hbar} L_\varphi \qquad \sigma_d = \sigma a^{3-d} = \frac{1}{R} L^{d-2}$$

$$\delta\sigma_2 \propto -\frac{e^2}{\hbar} \ln \frac{L_\varphi}{L} \qquad L_\varphi = \sqrt{D\tau_\varphi}$$

$$\delta\sigma_3 \propto \frac{e^2}{\hbar} \frac{1}{L_\varphi}$$

T - dependence

$$\frac{1}{\tau_\varphi} \propto T^\alpha \Rightarrow \delta\sigma_d \propto T^{\alpha(d/2-1)}$$

dominates when T small enough — T-dependence arises from quantum effects

Electron - electron Scattering — phase breaking

$\Rightarrow$ Johnson Noise — fluctuations of e-m field (dephasing has no free parameters but determined by σ)

d = 1 $$\frac{1}{\tau_\varphi} = \left[\frac{e^2}{\sigma_1}\sqrt{2D}\right]^{2/3} T^{2/3}$$

d = 2 $$\frac{1}{\tau_\varphi} = \frac{e^2}{2\pi\sigma_2} T \ln \frac{\pi\sigma_2}{e^2}$$ (No coupling constant, no free parameter)

d = 3 $$\frac{1}{\tau_\varphi} \propto T^{3/2}$$ (much bigger than classical)

All these results can be obtained from perturbation theory + trick

Perturbation Theory

$$\frac{1}{\tau_{e-e}} \propto \int \frac{d\omega}{\omega \nu L_\omega^d} \frac{T}{\omega} \propto \frac{1}{\nu D^{d/2}} \int_{T_0}^{T} \frac{d\omega}{\omega^{2-d/2}} \quad \text{diverges} \quad d \leq 2$$

$$T_0 = \frac{1}{\tau_\varphi} \longrightarrow \text{equation whose solution is correct answer}$$

Why is this cut-off correct ?

Meaning of τ_φ ?

How to measure energy of a quantum particle ?

Quasi-elastic collision, characterised by energy transfer ω

$\omega \tau_{in} \ll \hbar$ — Inelastic time between scattering τ_{in}

each single collision does not break phase — need collective

After $t > \tau_{in}$, inelastic broadening

$$\delta_{in} \varepsilon(t) = \omega \sqrt{\frac{t}{\tau_{in}}}$$

each collision can give or remove ω

If we are to measure, we should do it quickley but not too quick so as not to impart too much energy

Quantum uncertainty $\qquad \delta_q \varepsilon(t) \sim \hbar / t$

$$\delta \varepsilon(t) = \delta_{in} \varepsilon(t) + \delta_q \varepsilon(t)$$

$$[\delta\varepsilon(t)]_{min} = ? \qquad t_{min} : \delta\varepsilon(t_{min}) = \delta\varepsilon_{min}$$

$$\delta_{in}\varepsilon(t_{min}) \sim \delta_q\varepsilon(t_{min})$$

$$t_{min} = \left(\frac{\tau_{in}}{\omega^2}\right)^{1/3}$$

$$\delta\varepsilon_{min} \sim \hbar / t_{min}$$

What is phase breaking time τ_φ ?

$$\delta_{in}\varphi(t) \sim \delta_{in}\varepsilon(t) . \, t / \hbar$$

$$\tau_\varphi : \quad \delta_{in}\varphi(\tau_\varphi) \sim 1$$

$$\sqrt{\frac{\omega^2\tau_\varphi^3}{\tau_{in}}} \sim 1$$

$$\tau_\varphi \sim \left(\frac{\tau_{in}}{\omega^2}\right)^{1/3} = t_{min}$$

$\hbar/\tau_\varphi$ is maximum accuracy at which energy can be determined

– it is the minimal uncertainty.

This justifies the neglect of energy scales smaller than $\hbar/\tau_\varphi$ – they can not affect the phase shift

Scaling Theory of Localisation

If we want localisation in pure case, study $\sigma(L)$, L - system size

Classically σ is L-independent

$$R = \frac{L}{S\sigma} \propto L^{2-d} \quad \text{resistance}$$

$$g = \frac{\hbar}{e^2}\frac{\pi^2}{R} \quad \text{dimensionless conductance}$$

$$g \propto L^{d-2}$$

Quantum corrections - L dependence

$L_\varphi \longrightarrow L$ in previous formulae

i.e. time taken to leave sample

$$\delta\sigma_1 \propto -\frac{e^2}{\hbar} L$$

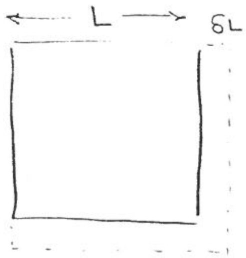
$$\delta\sigma_2 \propto -\frac{e^2}{\hbar} \ln\frac{L}{l}$$

$$\delta\sigma_3 \propto \frac{e^2}{\hbar}\frac{1}{L}$$

$$\delta g \sim \text{const (or ln)}$$

For $d \leq 2$, sooner later $\delta g \sim g_0 \longrightarrow$ localisation

Suppose we make a sample of size $L + dL$ if we know $g(L)$



$g(L)$, $g(L + \delta L)$?

RG Equation

$$\frac{d \ln g}{d \ln L} = \beta(g)$$

Importantly, we believe β depends only on g and not on g_0, l, impurity content, etc.

Let us first study what follows from this assumption

(1) Zero Order

$$g \propto L^{d-2} \quad , \quad \ln g = (d-2) \ln L$$

$$\beta_0 = d - 2$$

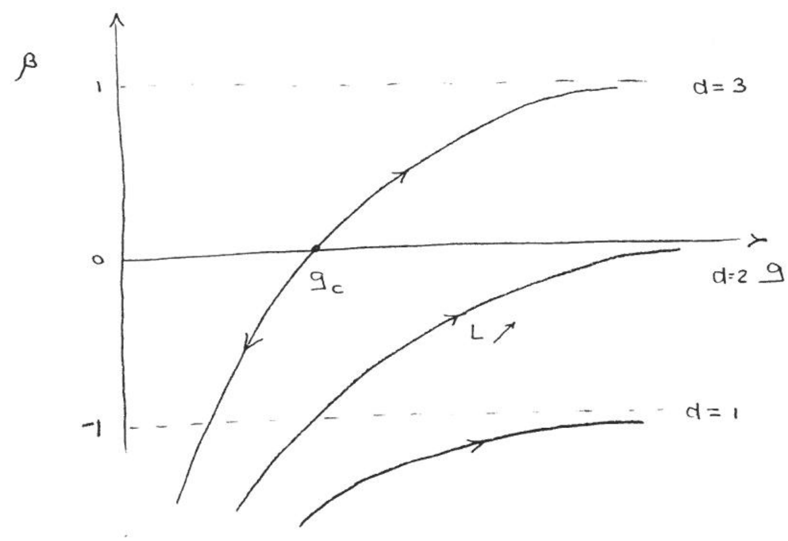
(2) First Order – first quantum correction

$$\beta = d - 2 - \frac{1}{g} \qquad (\text{correct when } g \gg 1)$$

In principle $\beta = \sum_n a_n \frac{1}{g^n}$, actually next is $\frac{1}{g^5}$ (Wegner, Hikami.)

(3) $g \ll 1 \longrightarrow$ Localisation $\quad g \propto e^{-L/\xi} \qquad \frac{d \ln g}{d \ln L} = \ln g$

β or Gell-Mann Low function allows us to make g arbitrarily small (unlike in perturbation theory)



I) In $d=1,2$, β always negative

g decreases monotonically with L

$$g \xrightarrow[L \to \infty]{} 0 \qquad \text{Insulator}$$

II) $d > 2$ $\quad \beta(g_c) = 0$

$$\beta \begin{cases} > 0 & g > g_c \\ < 0 & g < g_c \end{cases}$$

If we start with $g > g_c$, g increases with L

$g < g_c$, g decreases with L

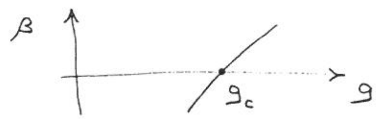
g_c is unstable fixed-point — phase transition

$d = 3$ $\quad g \xrightarrow[L \to \infty]{} L \qquad$ Metal

$$g \xrightarrow[L \to \infty]{} 0 \qquad \text{Insulator}$$

For $d=3$, $g_c \sim 1$ where calculation is difficult (no mean-field-theory)

But for $d = 2+\varepsilon$, $g_c \sim 1/\varepsilon \gg 1$ and we can apply perturbation theory to determine β function.



When $g = g_c$, $\beta = 0$ and $\dfrac{dg}{dL} = 0$

$$g = g_c \text{ for any } L$$

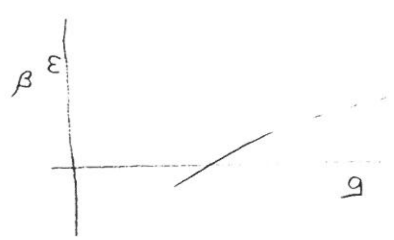
$$\sigma \propto \frac{g_c}{L^\varepsilon} \xrightarrow[L\to\infty]{} 0$$

Conductivity can be arbitrarily small

Close to g_c, $g \geq g_c$

$$\sigma(L\to\infty) = \frac{e^2}{\hbar}\xi^\varepsilon$$

metallic counterpart of localisation length



$$\beta(g) = a(g-g_c)$$

$$\frac{d\ln g}{\beta(g)} = d\ln L$$

$$\int \frac{dg}{ag(g-g_c)} = \ln\frac{L_2}{L_1}$$

Suppose

$$g_1 - g_c \ll 1 \qquad L_2 \sim \xi$$

$$g_2 \gg 1 \quad \text{(always true because } g_c \text{ large)}$$

$$\frac{1}{a g_c} \ln \frac{(g_2 - g_c)\, g_1}{(g_1 - g_c)\, g_2} = \ln \frac{\xi}{L}$$

if $g_2 \gg g_c$

$$\xi = L_1 \left(\frac{g_1 - g_c}{g_1} \right)^{-1/a g_c}$$

(Take semi conductor with certain impurity concn. $n \sim n_c$

$$\xi(n) \propto (n - n_c)^{-1/a g_c}$$

since $(n - n_c) \propto (g - g_c)$)

exponent $\mu = \dfrac{1}{a g_c}$

for $\varepsilon \ll 1$, $\quad g_c = \dfrac{1}{\varepsilon}$, $\quad a = \varepsilon^2$

$$\mu = 1/\varepsilon$$

$$\xi(n) \propto (n - n_c)^{-\mu}$$

For $d = 3$, $\quad \mu \sim 1$ is close to numerics

(Note, we have assumed β not singular at g_c — no reason why it should be) for $g \lesssim g_c$ we get similar result for ξ

How can we justify the scaling theory ?

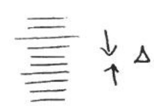
What is the Physical picture ? Thouless 1978

Recall Einstein relation

$$\sigma = e^2 \nu D$$

$$g = \hbar \nu D L^{d-2} = (\nu L^d) \frac{D\hbar}{L^2} = \frac{E_c}{\Delta}$$

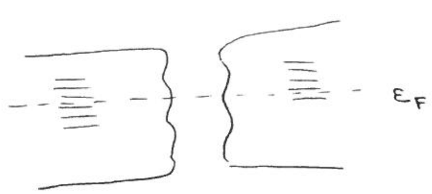
$\Delta = (\nu L^d)^{-1}$ mean spacing between energy levels



$E_c = \frac{D\hbar}{L^2} = \frac{\hbar}{t_0}$, t_0 time for particle to tunnel

from one side of system to other

(or $1/t_0$ is escape rate for open system)



join two bits of wire with same Δ

Close to ε_F

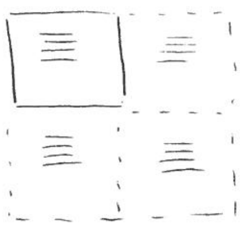
if $E_c > \Delta$, levels strongly mixed , otherwise no mixture

and states belong to one or other part of lead

So in 1d wire, with finite crossection

there is localisation

for higher dimension



$$L \longrightarrow L' = 2L$$

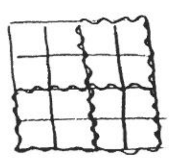
$$\Delta \longrightarrow \Delta' = \Delta / 2^d$$

$$E_c \longrightarrow E_c' = E_c\, f\left(\frac{E_c}{\Delta}\right)$$

$$g' = 2^d g\, f(g)$$

$$\frac{d \ln g}{d \ln L} = \beta(g) \qquad \text{in differential form}$$

Anderson Model : $I, W \longrightarrow E_c,\ \Delta$



In between Thouless argument and Gang of Four scaling theory came Wegners field Theory. — what we are missing for phase transition is an order parameter.

We require a Ginzburg – Landau equation to transfer to normal critical phenomena.

27/4/94

Localization Literature

1) B. Kramer and A. McKinnon Rep. in Prog of Phys 1994

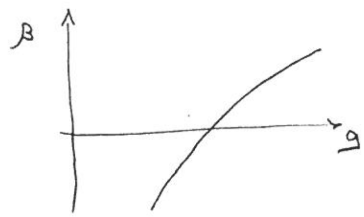
2) P.A. Lee and T.V. Ramakrishnan Rev. Mod Phys. 57, 287 (1985)

3) B.L.A and A.G. Aronov 1985 in Efros and Pollack "Electron-electron Interaction in Disorder Systems"

4) B.L.A, Aronov, Gershenson and Sharvin, Sov Sci Rev A Phys 9 223, 1987

5) Chakravarty and A Schmid, Phys. Rep. 140 193, 1986 (Path Integrals)

Scaling Theory of Localisation



$$\frac{d\ln g}{d\ln L} = \beta(g)$$

We have a phase transition

"low T" phase (metal)

"high T" phase (insulator)

Goldstone Mode – Cooperon / Diffuson $\frac{1}{-i\omega + Dq^2}$

We do not have an order parameter yet. } These will be discussed presently
nor a free energy }

We require an effective field theory – this will require a replica trick

Replica Trick

Random Potential

Interaction ?

$$\underset{r}{\times}\cdots\cdots\underset{r'}{\times} \;=\; \langle V(r)\,V(r')\rangle \;=\; \gamma\,\delta(r-r') \quad \leftarrow \quad \delta(\omega)$$

There is interaction between effective particles.

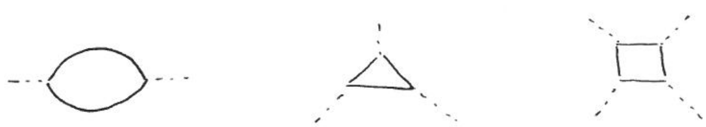
Problem

Not all diagrams are present

e.g.

since they are disconnected before averaging

We require a trick to eliminate any kind of loops.



deGennes, later Edwards and Anderson

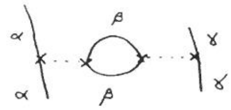
a) Replicate our system

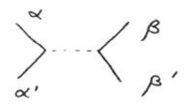
$$\psi \longrightarrow \psi_\alpha \qquad \alpha = 1, \dots N$$

b) Calculate everything as a function of N

c) $N \longrightarrow 0$

So



 $\propto \delta_{\alpha\alpha'}\, \delta_{\beta\beta'}$

$$\sum_{\beta} \to N \to 0$$

Step 1 – Functional Integral (see Popov or Berezin "Method of Second Quantisation")

Integrate over fields $\psi(r)$

$$\mathcal{H} = -\frac{\nabla^2}{2} + V(r)$$

$$S = \int \bar{\psi}(r)\,(\mathcal{H} - E)\,\psi(r)\,d\vec{r} \qquad \text{Action}$$

$$G(\vec{r}, \vec{r}\,') = -\langle \bar{\psi}(r)\psi(r) \rangle$$

$$= \frac{1}{Z} \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\, e^{iS}\, \bar{\psi}(r)\psi(r)$$

$$Z = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\, e^{iS}$$

Grassman Variables

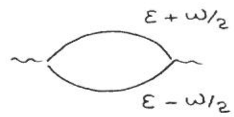
$$\psi = \begin{pmatrix} \chi \\ \kappa \end{pmatrix}, \quad \bar{\psi} = (-\kappa, \chi)$$

$$\chi\kappa + \kappa\chi \equiv \{\chi, \kappa\} = 0 = \{\chi, \chi\} = \{\kappa, \kappa\}$$

$$\int d\kappa = \int d\chi = 0$$

$$\int \kappa\, d\kappa = \int \chi\, d\chi = 1$$

Conductivity



$\varepsilon = 0$ Fermi level

$$\sigma_{\mu\nu}(\omega) = -\frac{e^2}{\pi L^d} \int \hat{V}_\mu \hat{V}_\nu \, G^R_{\omega/2}(r, r')\, G^A_{-\omega/2}(r'; r)$$

$$G^{R,A}(r,r') = \frac{i}{Z} \int \kappa(r)\, \chi(r') \exp\left[-i \int dr_1\, \chi(r_1) \left(\varepsilon + \varepsilon_F - H \pm i0 \right) \kappa(r_1) \right] D\chi\, D\kappa$$

$$\psi = \psi_{p,\alpha,s} \qquad p = \pm \quad \begin{matrix} + \text{ retarded} \\ - \text{ advanced} \end{matrix}$$

$\alpha = 1, \dots, N$ replica

$S = 1, 2 \qquad \psi_1 = \chi, \quad \psi_2 = \kappa$

Action

$$S = S_0 + S_{imp}$$

free

$$S_0 = \int \bar{\psi} \left[-\frac{\nabla^2}{2} - \varepsilon_F - \left(\frac{\omega}{2} + i0 \right) \Lambda \right] \psi \, dr$$

$(\sigma_z)_{pp'}$

$$S_{imp} = \int \bar{\psi}(r)\, V(r)\, \psi(r)\, d\vec{r}$$

$$\sigma_{\mu\nu}(\omega) = \frac{4}{\pi N^2} \frac{1}{Z} \int D\kappa\, D\chi\, \mathrm{Tr}\left[J^R_\mu(r)\, J^A_\nu(r') \right] e^{iS}$$

$$\left(\vec{J}\right)^{R,A}_{\alpha\alpha' pp' ss'} = \delta_{\alpha\alpha'} \frac{1}{2} \left(\delta_{pp'} \pm (\sigma_z)_{pp'} \right)$$

$$\times \frac{1}{2} \left(\tau_0 + i\tau_3 \right)_{ss'} \frac{i}{2} \left\{ \psi \otimes \nabla \bar{\psi} - \nabla \psi \otimes \bar{\psi} \right\}$$

We can write

$$\sigma = \frac{A}{Z}$$

$$= \frac{A\,Z^{N-1}}{Z^{N}} \quad \text{for } N \text{ replicas}$$

Field Theory allows separate averaging only $\frac{A}{Z} \longrightarrow \frac{\langle A \rangle}{\langle Z \rangle}$

We require $\left\langle \frac{A}{Z} \right\rangle$

But field theory at $N \to 0$ gives $\left\langle \frac{A\,Z^{N-1}}{Z^{N}} \right\rangle_{N\to 0} = \left\langle \frac{A}{Z} \right\rangle$

Step 2 Impurity Averaging

$$\left\langle e^{iS_{imp}} \right\rangle = e^{-\frac{1}{2}\langle S_{imp}^{2} \rangle}$$

$$= \exp\left\{ -\frac{1}{2} \int dr\,dr'\, \bar{\Psi}(r)\Psi(r)\bar{\Psi}(r')\Psi(r')\, \gamma\, \delta(r-r') \right\}$$

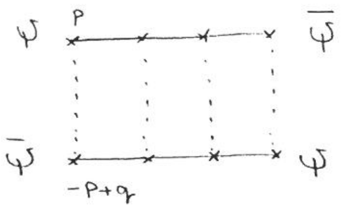
$$= \exp\left\{ -\frac{\gamma}{2} \int dr\, \bar{\Psi}(r)\Psi(r)\bar{\Psi}(r)\Psi(r) \right\}$$

interaction with coupling constant γ

Step 3 Hubbard-Stratonovitch Transformation

Something similar to derivation of Landau-Ginzburg Equation

Short-range interaction



$= Q \sim\!\sim\!\sim \bar{Q}$

long-range (q small) describes essentials of problem

a) $\int \mathcal{D}Q \exp\left(-\frac{\pi\nu}{8\tau}\int \mathrm{Tr}\, Q^2\, dr\right)$

matrix with double indices

b) $Q \rightarrow Q - \frac{2}{\pi\nu}\, \psi \otimes \bar{\psi}$

transformation cancels fourth order term.

Only quadratic terms in ψ remain

c) Integrate over ψ – Gaussian integral

Q is the order parameter — has a sense at finite N

d) Effective free energy $F[Q]$

$$F[Q] = -\mathrm{Tr}\left[\ln\left(-iH_0 - \frac{i\omega}{2}\Lambda - \frac{Q}{2\tau}\right) + \frac{\pi\nu}{4\tau} Q^2\right]$$

Step 4

Saddle-Point – suppose Q = const: indep of r

$$Q = \frac{I}{\pi\nu} \int \frac{d\vec{p}}{iE - ip^2/2 + i\omega\Lambda/2 + Q/2\tau}$$

has solution $Q = \Lambda$

$Q^2 = I$ $\qquad$ $Q = U^{-1}\Lambda U$ allowed for $\omega = 0$

Consider $Q = \Lambda + \delta Q \longrightarrow \delta Q_k$

and expand to second order

longitudinal fluctuations

$$F = F_0 + \frac{1}{4\tau^2} \mathrm{Tr} \int (dk) \left\{ G_p G_{p+k}\, \delta Q_k \delta Q_{-k} + \frac{\pi\nu}{4\tau} \delta Q_k \delta Q_{-k} \right\}$$

$$+ \frac{i\omega\pi\nu}{2} \int \mathrm{Tr}\, \Lambda \delta Q \, dr$$

works like magnetic field

Longitudinal $\delta(Q^2) \neq 0$

Transverse $Q\delta Q + \delta Q . Q = 0$

Exclude costly longitudinal modes

$$F = F_0 + \frac{\pi\nu}{4} \int dr \left\{ D\, \mathrm{Tr}\, (\nabla Q)^2 + 2i\omega\, \mathrm{Tr}\, \Lambda Q \right\}$$

with $Q^2 = I$ constraint

Corrections from Longitudinal part

— small as $\frac{1}{k_F \ell}$

transverse part

— $\frac{\delta\sigma}{\sigma} \propto \frac{1}{k_F \ell} \frac{(k_F \ell)^{-\varepsilon}}{\varepsilon}$ $\qquad \varepsilon = d - 2$

give big corrections for $d \leq 2$

When $\varepsilon = 1$, $d = 3$ both contributions are significant

For ε small we can still keep only transverse modes

Step 5

Renormalisation Group

Integrate over fast variables, k large

Monitor $D = \frac{1}{t}$

$$\frac{dt}{d\ln k} = -\varepsilon t + \frac{2N+1}{8} t^2 + O(t^3)$$

Normal dimension

of resistivity

$$\frac{d\omega}{d\ln k} = \frac{Nt}{4} \xrightarrow[N \to 0]{} 0$$

So ω is not renormalised

(number of particles is conserved)

Localisation

Degree of disorder $\frac{1}{k_F\, l}$

Mobility Edge E_g

High T phase – insulator

(strong transverse fluctuations $\langle Q\rangle \neq 0$)

Low T phase – metal

Order Parameter $\langle Q \rangle$

Diffusion Modes

Diffusion Constant

Frequency ω

Magnetic Field

Phase Transition (ferromagnet)

Temperature T

Curie temperature T_c

Paramagnetic

Feromagnetic

Magnetisation

Spin Waves

Spin Wave Stiffness

Magnetic field H

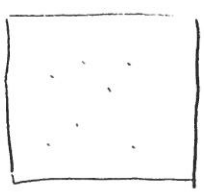
Anisotropy

(Application of Field Theory to Disordered Metals — "Mesoscopic Phenomena"

We heard last week of an application of a field theory of disordered metals to the question of localisation. Today I will speak about

Level Statistics in Metallic Grains, or "Quantum Dots"

where a non-perturbative analysis is crucial)



$$H = \frac{p^2}{2m} + V(r) \qquad \text{(Spinless)}$$

(δ-correlated impurity potential)

$$\langle V(r) V(r') \rangle = \frac{1}{2\pi\nu\tau} \delta(r-r')$$

$$D = v_F^2 \tau / d$$

(Today we will not be interested in localised states)

$$L \ll \xi$$

(and we will assume no dephasing so resonances are well defined)

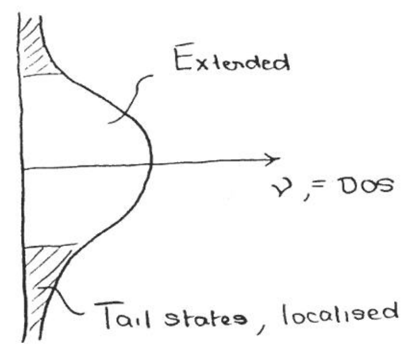
$$\ell_\varphi \rightarrow \infty$$

Length Scales

($d \lesssim 2$ or strong disorder)

Semi-classical parameter

$$\frac{1}{k_F \ell} = \frac{1}{\varepsilon_F \tau} \ll 1$$



Diffusive

$$\frac{\ell}{L} \ll 1$$

Energy levels – (discrete, random – all grains differ
– calls for statistical description)

($L \sim 100\,\mathring{A}$
$N \sim 10^5$
$\Delta / k_B \sim 1K$)

Level spacing

$$\Delta = \langle E_{i+1} - E_i \rangle = \frac{1}{V\nu}$$

(meaning of average ?)

Averages

$$\langle \ldots \rangle_{\text{Ensemble}} \equiv \langle \ldots \rangle_{\text{Spectrum}}$$

"Ergodicity"

Spectral Statistics

DoS $\quad \nu(E) = \frac{1}{V} \sum_i \delta(E - E_i)$

$$= \frac{1}{\pi V} \int \mathrm{Im}\, G^R(r,r)\, dr \equiv \frac{1}{\pi} \mathrm{Tr}\, \mathrm{Im}\, G^R$$

tadpole

$$\langle \nu \rangle = \nu = \frac{1}{V\Delta}$$

(Independent of disorder and parameters)

Fluctuation

$$k(\Omega) = \langle \nu(E)\, \nu(E+\Omega) \rangle - \langle \nu \rangle^2$$

(connected)

$$= \frac{1}{2\pi^2} \langle F(\Omega) \rangle - \frac{\nu^2}{2}$$

$$F(\Omega) = \mathrm{Tr}\left(G^R_{E+\Omega/2}\right) \mathrm{Tr}\left(G^A_{E-\Omega/2}\right)$$

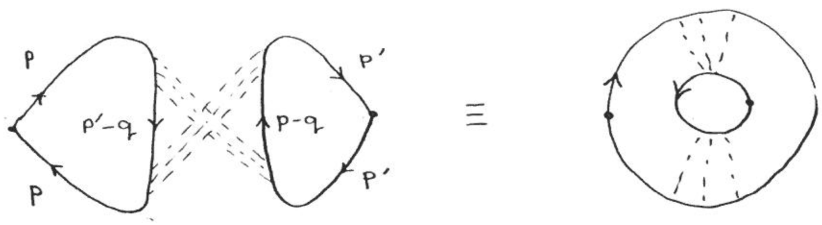
(other terms disconnected and disorder, parameter independent)

Perturbation Theory

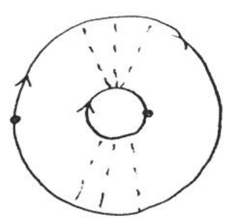
(Dominant contribution to connected correlator comes from the exchange

of two Cooperon or Diffuson lines between two tadpoles)

Diffuson



Cooperon



(If this is not clear

please ask Boris !)

$$k(\Omega) = \frac{1}{\pi^2 V^2} \; \mathrm{Re} \sum \frac{1}{(-i\Omega + Dq^2)^2}$$

Zero-Mode

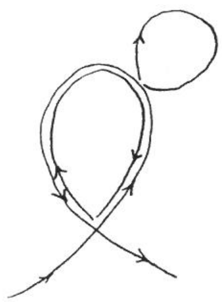
For $\quad \Omega \ll E_c = \frac{\hbar D}{L^2}$ $\qquad$ (Thouless energy)

$q = 0$ dominant

$$k(\Omega) = -\frac{\nu^2}{\pi^2} \quad \frac{\Delta^2}{\Omega^2}$$

Divergent as $\frac{\Omega}{\Delta} \to 0$

Failure of p.t. (How can we understand this divergence?)



Energy $^{\Omega}$/ Time $^{\hbar/\Omega}$ Scales

(t_c - transport time across grain)

time

$\tau = \ell / v_F$ $\quad$ $t_c = \hbar / E_c = L^2 / D$ $\quad$ $t_H = \hbar / \Delta$ $\quad$ $\tau_\varphi \to \infty$

Perturbative $\quad$ Non-perturbative

(Short dephasing time can save perturbation theory)

(Long-time scales call for a different approach)

Supersymmetry Method — K.B Efetov Adv. in Phys 32 53 (1983); J. Verbaarschot et al Phys Rep 129 367 (1985) (Nobuhiko and Anton)

(Similar to replica approach which Boris presented in previous lecture

– First few steps to the σ-model will look almost identical – but it is good to see this done again.

I will not "remind" you of Grassman algebra – its best not to think about it unless its necessary. We will begin with a place where it is crucial)

Gaussian integration (lets think about matrices rather fields)

Commuting Variables

$$\int dS_1^* dS_1 \ldots dS_n^* dS_n \, e^{-\frac{i}{2} S^+ A S} = \left[\det(A/2\pi i) \right]^{-1}$$

Anticommuting

$$\int d\chi_1^* d\chi_1 \ldots d\chi_n^* d\chi_n \, e^{+\frac{i}{2}\chi^+ A \chi} = \det(A/2\pi i)$$

Trick: Combine

$$\vec{\psi}^T = (\vec{\chi} \;\; \vec{S} \;\; \vec{\chi}^* \;\; \vec{S}^*)$$

$$\int d[S]\, d[\chi] \, \exp\left[\pm \frac{1}{2} i \psi^+ A \psi \right] = +1$$

(a remarkably useful cancellation that Avoids weight denominator)

Saddle - point

(Conventional mean-field approximation )

$$\frac{\delta}{\delta Q_{sp}} F[Q]\Big|_{Q = Q_{sp}} = 0$$

As with replicas

$$Q_{sp} = T^{-1} \Lambda T \qquad Q_{sp}^2 = 1$$

(T - unitary matrices which obey symmetry for Q)

(This invariance is a consequence of the supersymmetry and applies only to $\langle G^R G^A \rangle$ - otherwise $\mathbb{1}$ not Λ and degeneracy lifted

Q - plays role of order parameter with anticommuting elements which do not enter physical quantities)

(Allowing for) small longitudinal fluctuations and expanding in ℓ/L and $\omega\tau$

$$\langle \dots \rangle_Q = \int \mathcal{D}Q(r)\, e^{-F[Q]} (\dots)$$

$$\langle F(\Omega) \rangle_v = -\frac{\pi^2\nu^2}{4} \int \mathcal{D}Q(r)\, e^{-F[Q]}\, \mathrm{Tr}\left(Q\, P(A,b) \right) \mathrm{Tr}\left(Q\, P(R,b') \right)$$

$$F[Q] = \frac{\pi\nu}{8} \int dr\, \mathrm{STr}\left[D (\nabla Q)^2 + 2i(\Omega + i0) \Lambda Q(r) \right]$$

(Which we have seen before ! Goldstone Modes at $\Omega = 0$ (c.f. ferromagnet)

But what can be done with functional non-linear σ-model)

What about replicas ?

(Perturbation Theory

$$Q = W + \Lambda(1-W^2)^{1/2} \quad , \quad W = \begin{pmatrix} 0 & Q^{RA} \\ Q^{AR} & 0 \end{pmatrix} \quad \text{unconstrained}$$

$$F[W] \simeq \frac{\pi\nu}{8} \int \mathrm{STr}\left[D(\nabla W)^2 - i\Omega W^2 \right]$$

Gaussian – propagator

$$\frac{1}{Dq^2 - i\omega} \qquad \text{Diffuson / Cooperon}$$)

(Magnetic field

$$F = \frac{\pi\nu}{8} \int \mathrm{STr}\left[D\left(\nabla Q - \frac{ie}{c}\vec{A}\,[Q,\tau_3]\right)^2 + 2i\Omega\Lambda Q \right] dr$$

Covariant derivative

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ 'Conjugate elements'

σ-model has massive modes $[Q_c, \tau_3] \neq 0$ – Cooperons

Goldstone Modes $[Q_0, \tau_3] = 0$ – Diffusons

– topological terms etc.)

$$\int dr \, \frac{\theta}{16\pi} \, \varepsilon^{\mu\nu} \, \mathrm{STr}\left[Q \partial_\mu Q \partial_\nu Q \right]$$

Zero-Mode

$t > t_c \quad , \quad \Omega < E_c$ (Long-time, Short energy scales – non-perturbative regime)

$Q(r) \longrightarrow Q$ (independent of $\vec{r}$ – lowest spatial mode)

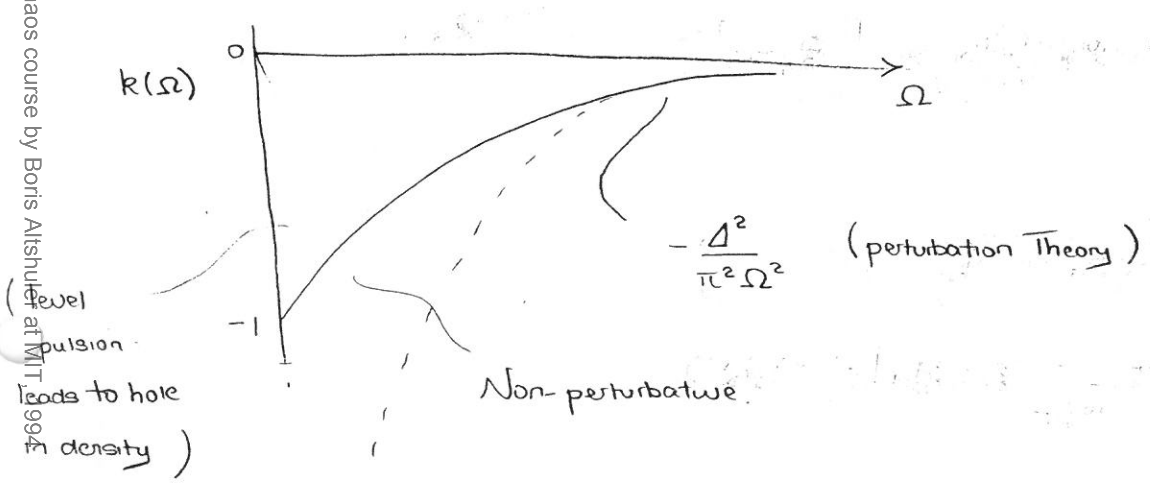
$$\langle F(\Omega) \rangle = \left(\frac{\pi}{\Delta}\right)^2 \int d[Q] \; e^{-F[Q]} \; \mathrm{Tr}\left(Q P(A,b)\right) \mathrm{Tr}\left(Q P(R,b)\right)$$

$$F[Q] = \frac{i\pi(\Omega + i0)}{4\Delta} \; \mathrm{STr}\, \Lambda Q$$ (Zero-dimensional σ-model)

Definite integral !

(Unfortunately there is no time to speak about parametrisation – but I simply state result)

$$k(\Omega) = \frac{\sin^2(\pi\Omega/\Delta)}{(\pi\Omega/\Delta)^2} + \left[\int_{\Omega/\Delta}^{\infty} \left(\frac{\sin \pi t}{\pi t}\right) dt\right] \frac{d}{d(\Omega/\Delta)} \left(\frac{\sin(\pi\Omega/\Delta)}{(\pi\Omega/\Delta)}\right)$$



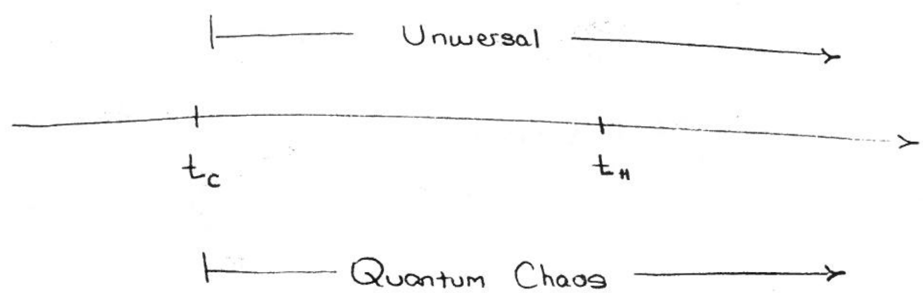
Random Matrix Theory

$$P(H)\,dH = e^{-\frac{\lambda^2}{N}\mathrm{Tr}\,H^2}\,dH$$

Repeat calculation — no fields

Get same result with $\Delta = \pi\lambda/N$ /

For $\Omega < E_c$, $t > t_c$ grain behaves as complex zero-dimensional atom (phase space is connected)



Correlators apply to Quantum Billiards

Atomic Nuclei

H in B field

Complex lattices

. . . .