Lecture 4: Tensor Product States

(slides), except the following section:

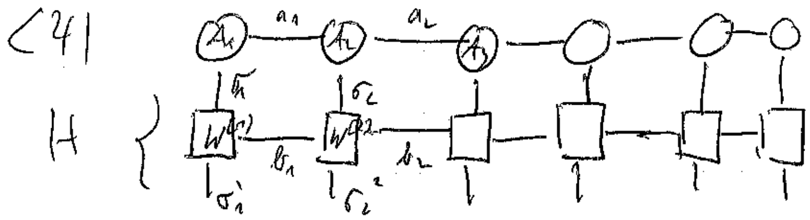
Expressing spin-Hamiltonians as Matrix-product operators

see Schollwöck Ann. Phys. 326 (2011), 96

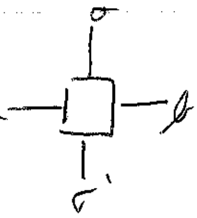
e.g. X-X-Z model in 1D

$$\mathcal{H} = \sum_i \frac{J}{2}\left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+\right) + J^z \hat{S}_i^z \hat{S}_{i+1}^z - h\sum_i S_i^z$$

Aim: write $\mathcal{H}$ in a form that is easily applied to MPS



$$H \sim \hat{W}^{[1]}\hat{W}^{[2]}\hat{W}^{[3]}\ldots$$

where $\left[W_{bb'}\right]^{\sigma\sigma'} = \sum_{\sigma\sigma'}$  $|\sigma\rangle\langle\sigma'|$

The Hamiltonian is a sum of many tensors of the form

$$\mathbb{1}\otimes\mathbb{1}\otimes\ldots\mathbb{1}\otimes\hat{S}_i^+\otimes\hat{S}_j^-\times\mathbb{1}\times\mathbb{1}$$

or $\mathbb{1}\otimes -h S_i^z\otimes\mathbb{1}\otimes\mathbb{1}\otimes\ldots\otimes\mathbb{1}\otimes\mathbb{1}$

or $\mathbb{1}\otimes\mathbb{1}\times\hat{S}_i^+\otimes J\hat{S}_{i+1}^z\otimes\mathbb{1}\ldots$

these terms have a definite syntax: we can have sequences

$$\mathbb{1}\to\mathbb{1}$$
$$\mathbb{1}\to -h\hat{S}^z\to\mathbb{1}$$
$$\mathbb{1}\to S^+\to\frac{J}{2}S^-\to\mathbb{1}$$
$$\mathbb{1}\to S^-\to\frac{J}{2}S^+\to\mathbb{1}$$

or $\mathbb{1}\to\hat{S}_i^z\to\hat{S}_{i+1}^z\to\mathbb{1}$.

These can be expressed in terms of the following 'states' of the string:

$$\{ \mathbb{1}, \hat{S}^+, \hat{S}^-, \hat{S}^z, 5 \} \in M(d \times d)$$

where we introduced '5' to signify an incomplete interaction term to the left.

Now, the construction yields the following matrix form $[W^{[i]}]_{bb'}$

$$W^{[i]} = \begin{pmatrix} \mathbb{1} & & & & \\ \hat{S}^+_i & & \mathcal{O} & & \\ \hat{S}^-_i & & & & \\ \hat{S}^z_i & & & & \\ -h S^z_i & \frac{J}{2}\hat{S}^-_i & \frac{J}{2}\hat{S}^+_i & J^z \hat{S}^z_i & \mathbb{1} \end{pmatrix} \qquad 2 \leq i \leq N-1$$

in the centre of the chain, and at the extremities:

$$W^{[1]} = \begin{pmatrix} -h S^z_1 & \frac{J}{2}\hat{S}^-_1 & \frac{J}{2}\hat{S}^+_1 & J^z \hat{S}^z_1 & \mathbb{1} \end{pmatrix}$$

$$W^{[N]} = \begin{pmatrix} \mathbb{1} \\ \hat{S}_+ \\ \hat{S}_- \\ \hat{S}_z \\ -h S^z \end{pmatrix}$$

Show by explicit multiplication of $W^{[1]} \times W^{[2]}$ that this generates all required terms of the Hamiltonian!

Longer-range interactions: Create intermediate states

e.g. $\hat{H} = J_1 \sum_i S^z_i S^z_{i+1} + J_2 \sum_i S^z_i S^z_{i+2}$

$$W^{[j]} = \begin{pmatrix} \mathbb{1} & & & \\ S^z & 0 & & \mathcal{O} \\ & \mathbb{1} & 0 & \\ & J_1 S^z_i & J_2 S^z_i & \mathbb{1} \end{pmatrix}$$

allowing sequences of states

$\mathbb{1} \to S_z \Rightarrow J_1 S_z \to \mathbb{1}$

or $\mathbb{1} \to S_z \Rightarrow \mathbb{1} \to J_2 S^z \to \mathbb{1}$

$\Rightarrow$ dimension of $W^{[j]}$ grows with range of interaction (exception: exponential decay)