# Lecture 3: Tensor Product Ansatz

TCM Graduate Lectures

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slide credits: Philippe Corboz (ETH / Amsterdam)

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Trinity Hall Cambridge

The Royal Society

Cavendish Laboratory

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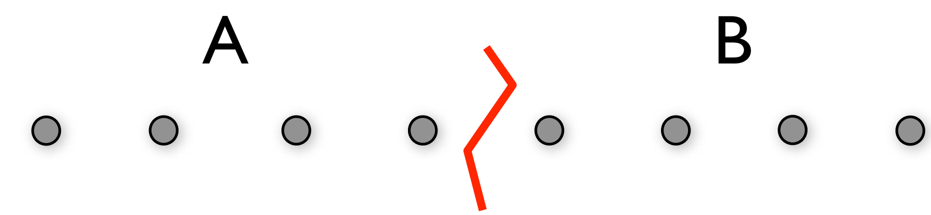
# Part I: Introduction

- on the blackboard...

# Reduced density matrix

$$|\Psi\rangle = \sum_k^M s_{kk} |u_k\rangle |v_k\rangle$$




★ Reduced density matrix of left side: *describes system on the left side*

$$\rho_A = \mathrm{tr}_B[\rho] = \mathrm{tr}_B[|\Psi\rangle\langle\Psi|] = \sum_k \lambda_k |u_k\rangle\langle u_k| \qquad \lambda_k = s_{kk}^2$$ *probability*

★ **DMRG**: Keeping the basis states on the left (right) side with largest probabilities gives the best approximation to the exact wave function

★ **Entanglement entropy:** $S(A) = -\mathrm{tr}[\rho_A \log \rho_A] = -\sum_k \lambda_k \log \lambda_k$

- Product state: $S(A) = -1 \log 1 = 0$

  ⋮

- Maximally entangled state: $S(A) = -\sum_k \frac{1}{M} \log \frac{1}{M} = log M$

# relevant states $\chi \sim \exp(S)$

*How large is S in a physical state? How does it **scale** with system size?*

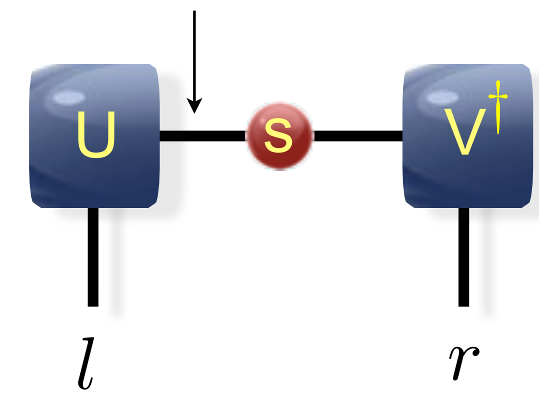
# How many relevant singular values?

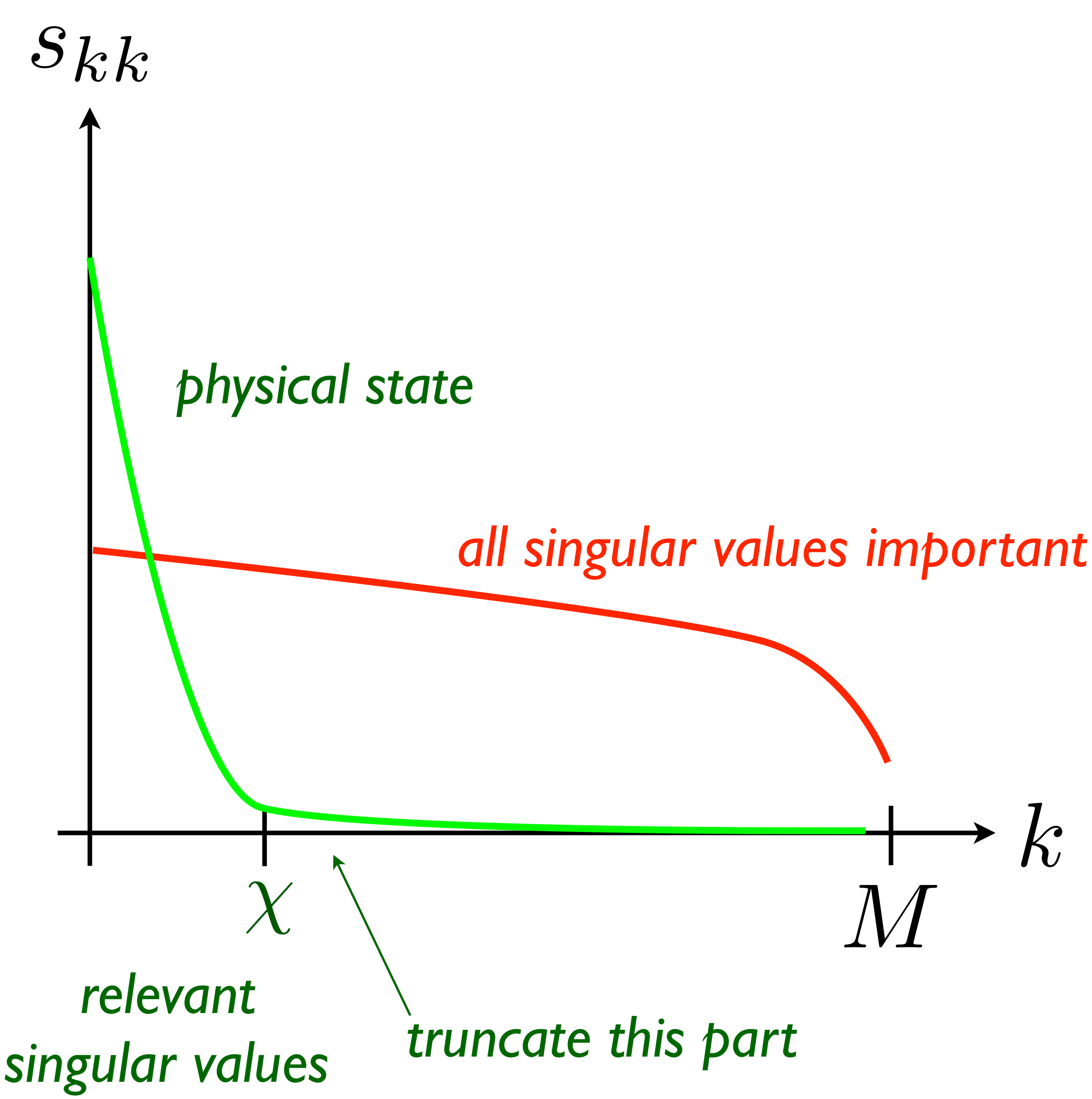
$$|\Psi\rangle = \sum_{k}^{M} s_{kk} |u_k\rangle |v_k\rangle$$

*how many **relevant** singular values?*

*bond dimension*

$D = \chi$




$$|\Psi\rangle \approx |\tilde{\Psi}\rangle = \sum_{k}^{\chi} s_{kk} |u_k\rangle |v_k\rangle$$

*keeping the $\chi$ largest singular values minimizes the error*

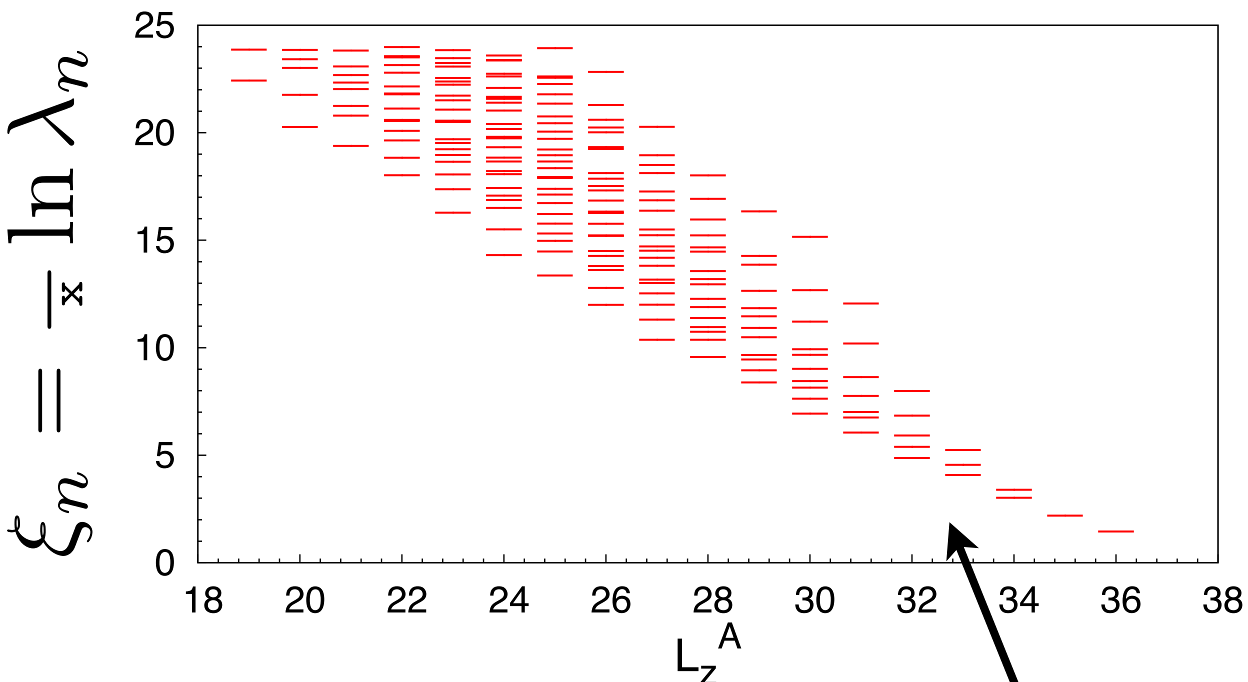
$$|||\Psi\rangle - |\tilde{\Psi}\rangle||$$

**KEY IDEA OF DMRG!**

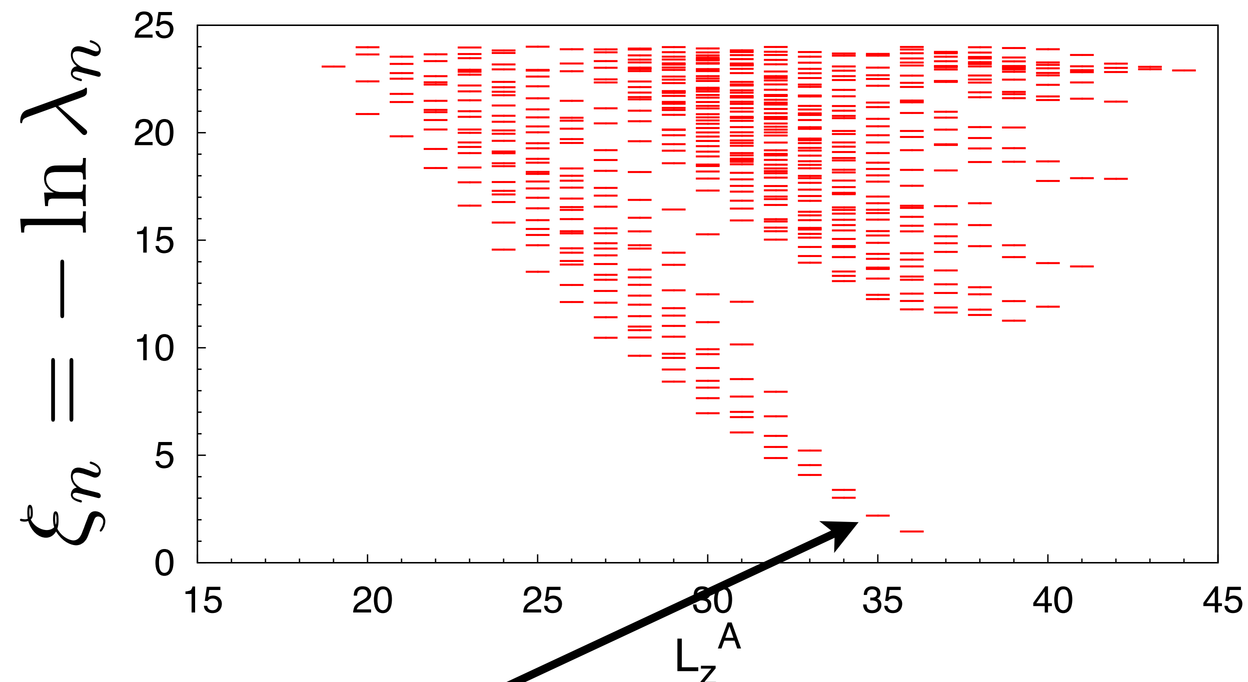
# Examples for Entanglement spectra

e.g. fractional quantum Hall states (on the sphere)

Laughlin state (bosons, N=12)




ν=1/2 Coulomb, bosons, N=12

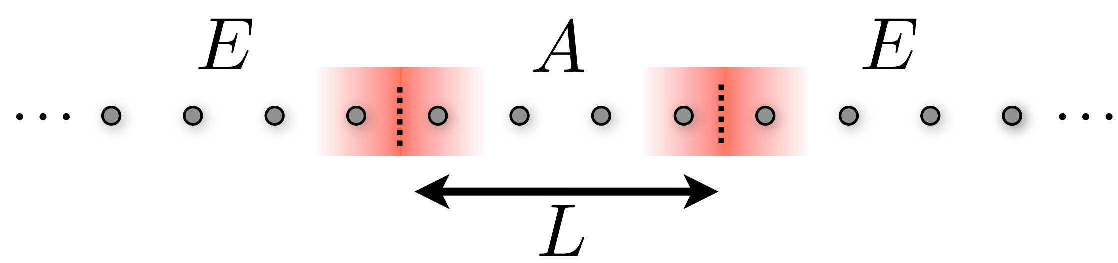



Universal part provides signature of chiral boson edge theory

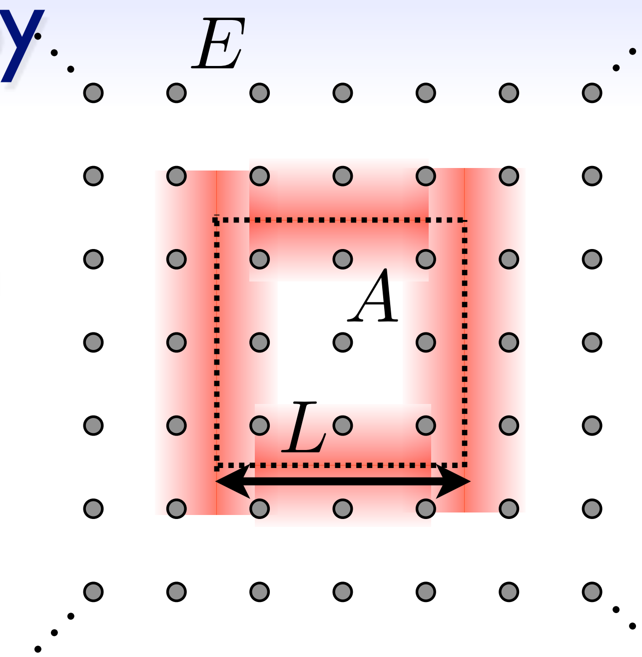
State is already well-approximated by eigenvalues of low entanglement energy

# Area law of the entanglement entropy

1D




2D




Entanglement entropy $S(A) = -\mathrm{tr}[\rho_A \log \rho_A] = -\sum_i \lambda_i \log \lambda_i$

# relevant states
$\chi \sim \exp(S)$

**General (random) state**

$S(L) \sim L^d$ **(volume)**

**Physical state** (local Hamiltonian)

$S(L) \sim L^{d-1}$ **(area law)**

| | | |
|---|---|---|
| **1D** | $S(L) = const$ | $\chi = const$ |
| **2D** | $S(L) \sim \alpha L$ | $\chi \sim \exp(\alpha L)$ |

Note: Some (critical) ground states have a **logarithmic correction** to the area law

# Entanglement entropy & area law: *Proofs* *(incomplete list)*

- Gapped 1D systems have an area law! — Hastings 2007

  $$S(L) < S_{\max} = \text{const}(\xi)$$

- 1D critical system: $S(L) = \frac{c}{3}\log(L)$ — Vidal, Latorre, Rico, Kitaev 2003; Calabrese & Cardy 2004

- Area law for (quadratic) gapped bosonic systems — Plenio, Eisert, Dreißig, Cramer 2005; Cramer & Eisert 2006

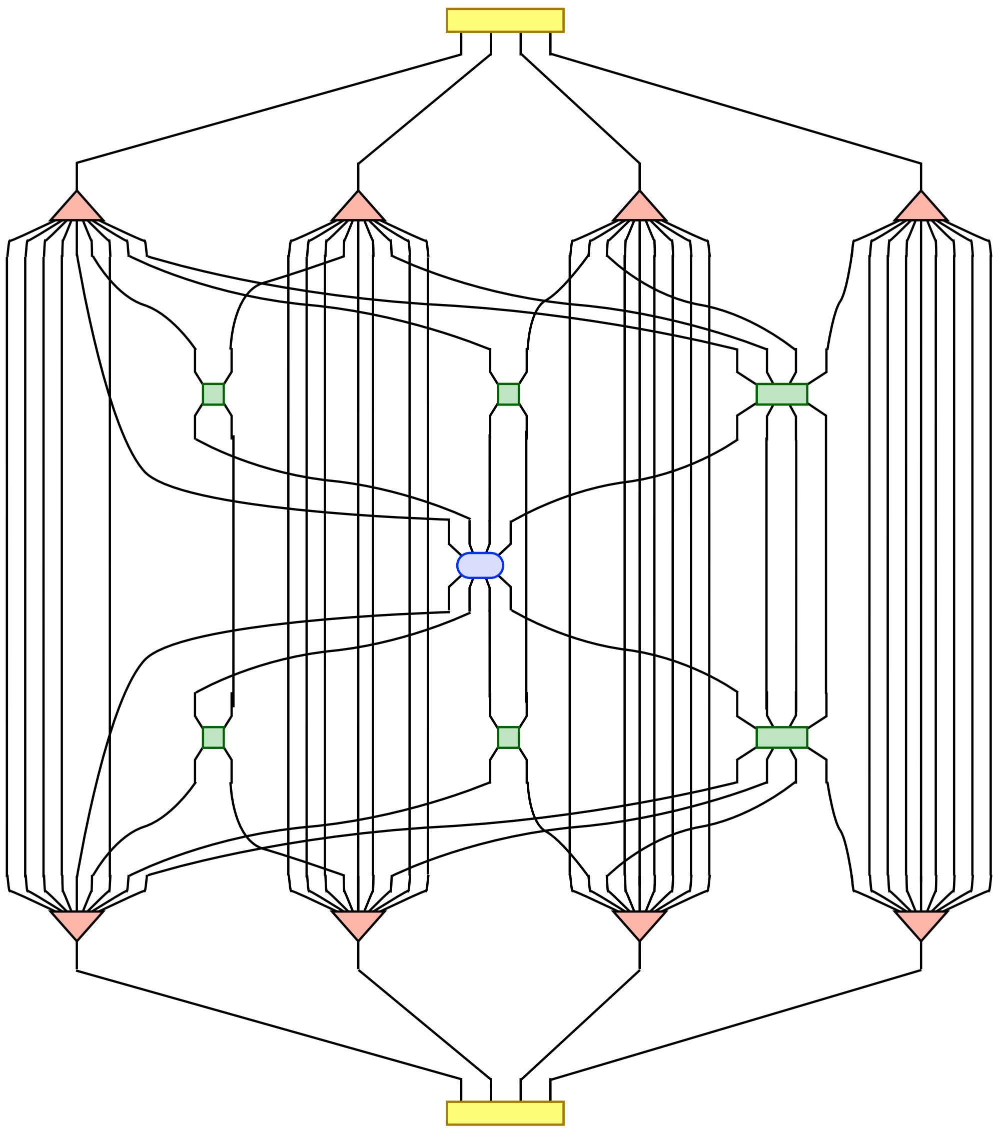
- there are gapless systems in 2D with area law (without logarithmic correction) — Verstraete, Wolf, Perez-Garcia, Cirac 2006

- 2D free fermion system: $S(L) \sim L\log(L)$ — Wolf 2006, Gioev & Klich 2006; Barthel, Chung, Schollwoeck 2006; Cramer, Eisert, Plenio 2007

- Area law holds at finite temperature (mutual information) for all local Hamiltonians — Wolf, Verstraete, Hastings, Cirac, 2008

$\vdots$ $\vdots$

# Examples of tensor networks



= some number

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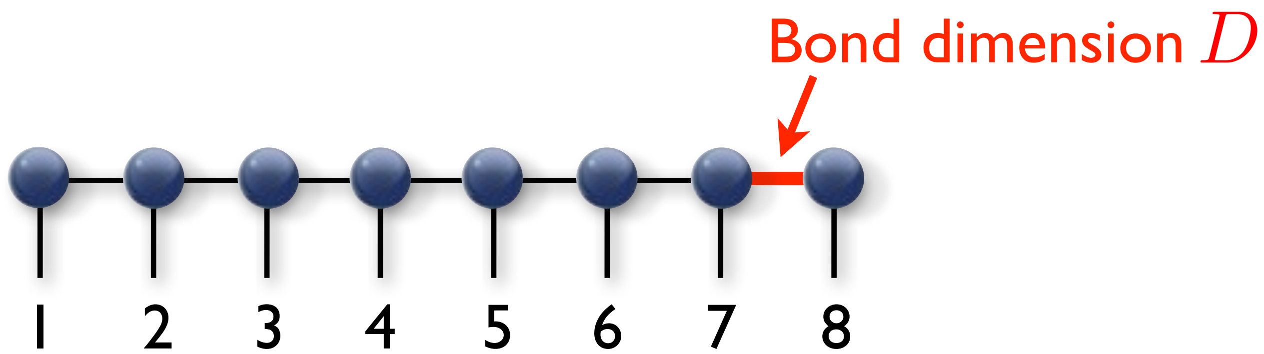
# MPS & PEPS

## 1D

**MPS**

Matrix-product state




Physical indices (lattices sites)

S. R. White, PRL 69, 2863 (1992)

Fannes et al., CMP 144, 443 (1992)

Östlund, Rommer, PRL 75, 3537 (1995)

✓ Reproduces area-law in 1D

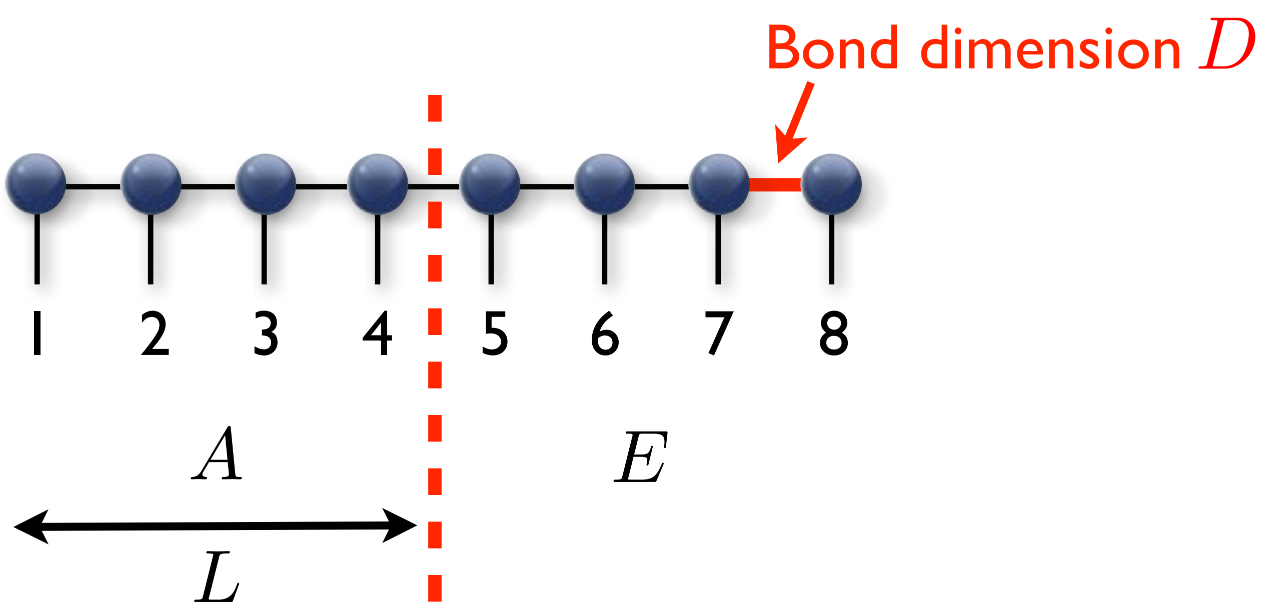
$$S(L) = const$$

# MPS & PEPS

## 1D

**MPS**

Matrix-product state




$$rank(\rho_A) \leq D \longrightarrow S(A) \leq log(D) = const$$

✓ Reproduces area-law in 1D

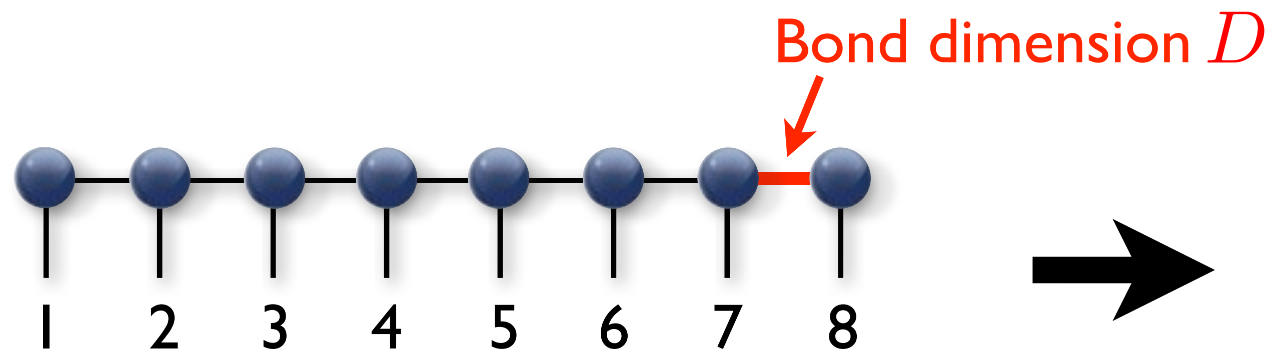
$$S(L) = const$$

# MPS & PEPS

## 1D

**MPS**

Matrix-product state

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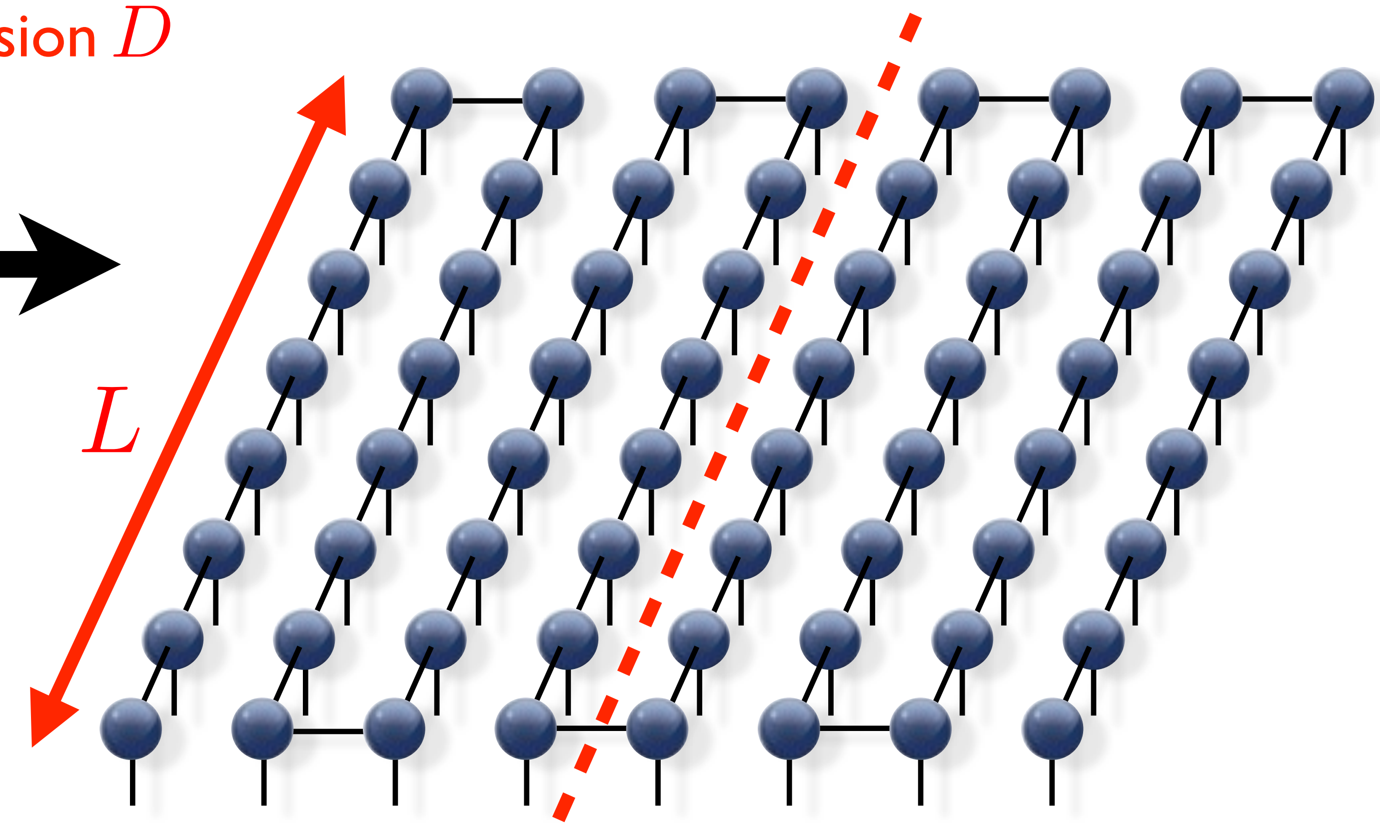
✓ Reproduces area-law in 1D

$$S(L) = const$$

## 2D

**can we use an MPS?**




**!!! Area-law in 2D !!!**

$$S(L) \sim L$$

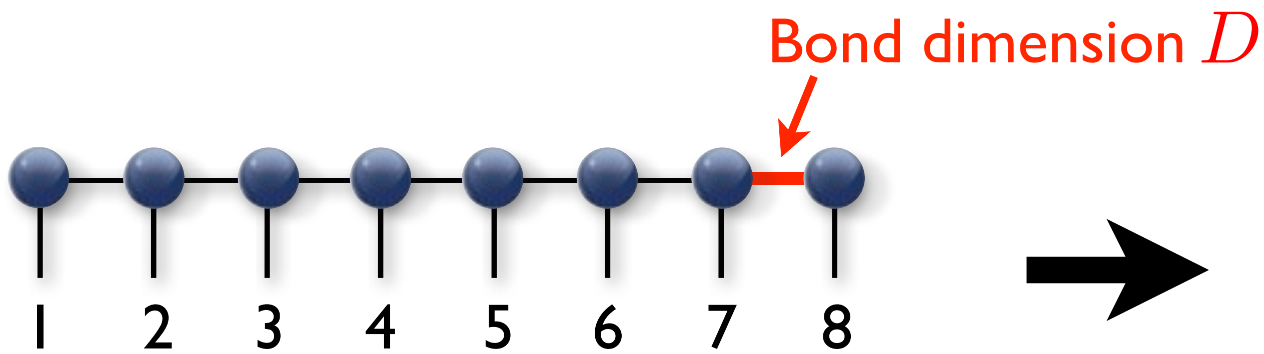
→ $D \sim exp(L)$

# MPS & PEPS

## 1D

### MPS

Matrix-product state

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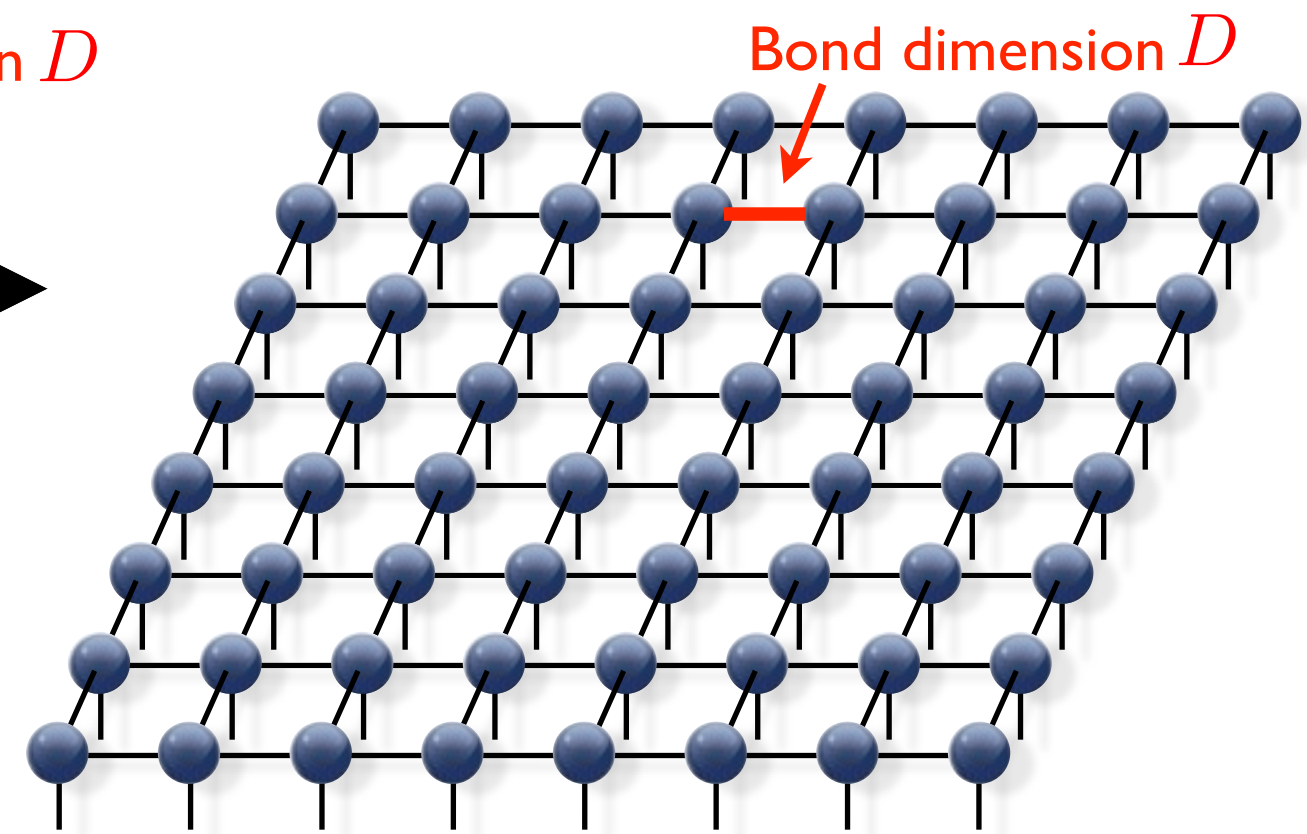
✓ Reproduces area-law in 1D

$$S(L) = const$$

## 2D

### PEPS (TPS)

projected entangled-pair state
(tensor product state)

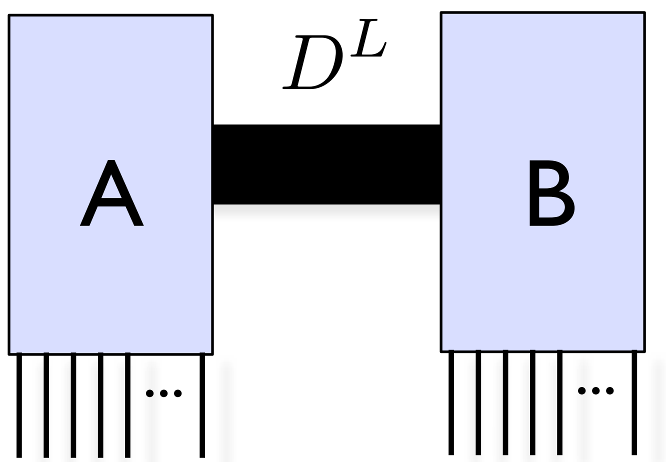



F. Verstraete, J. I. Cirac, cond-mat/0407066

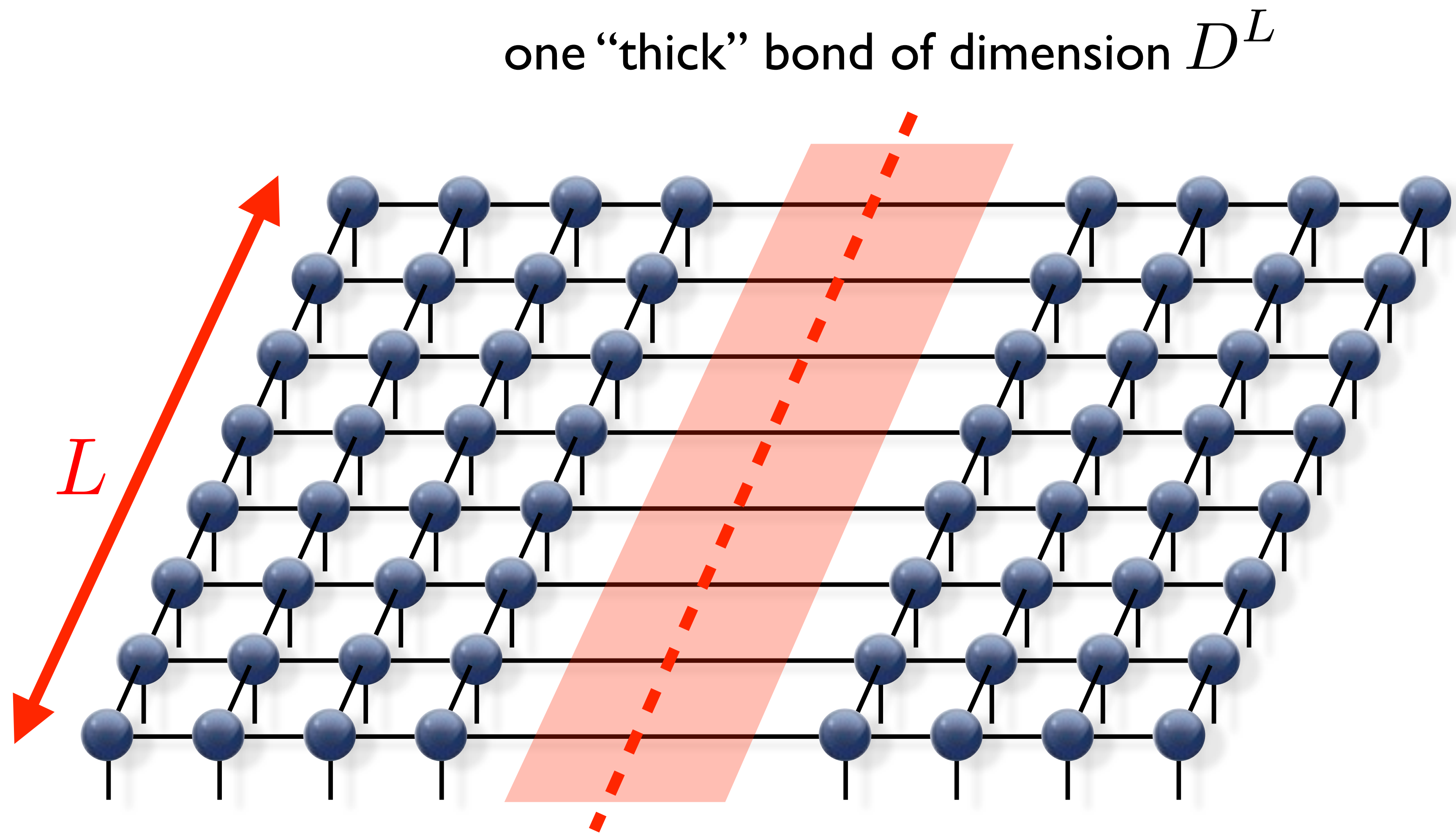
✓ Reproduces area-law in 2D

$$S(L) \sim L$$

# PEPS: Area law




$$S(A) \leq L \log D \sim L$$




each cut auxiliary bond can contribute (at most) log D to the entanglement entropy

The number of cuts scales with the cut length

✓ Reproduces area-law in 2D

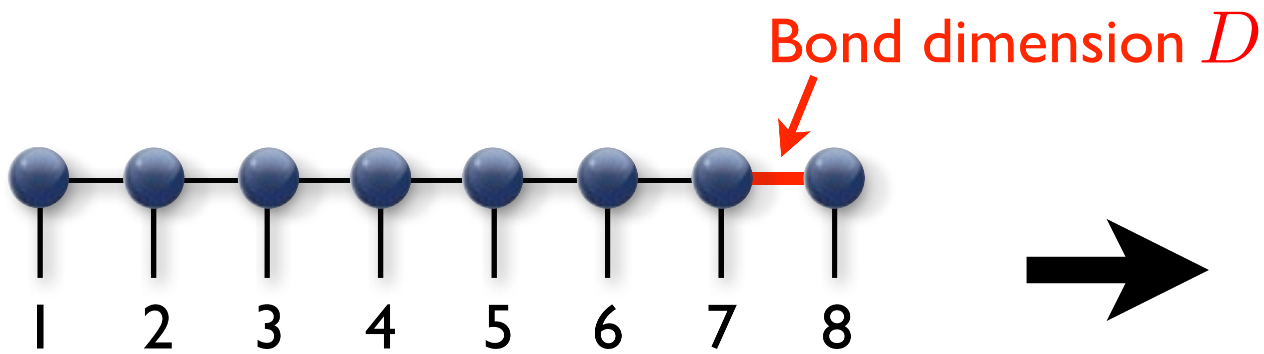
$$S(L) \sim L$$

# MPS & PEPS

## 1D

### MPS

Matrix-product state

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## 2D

### PEPS (TPS)

projected entangled-pair state
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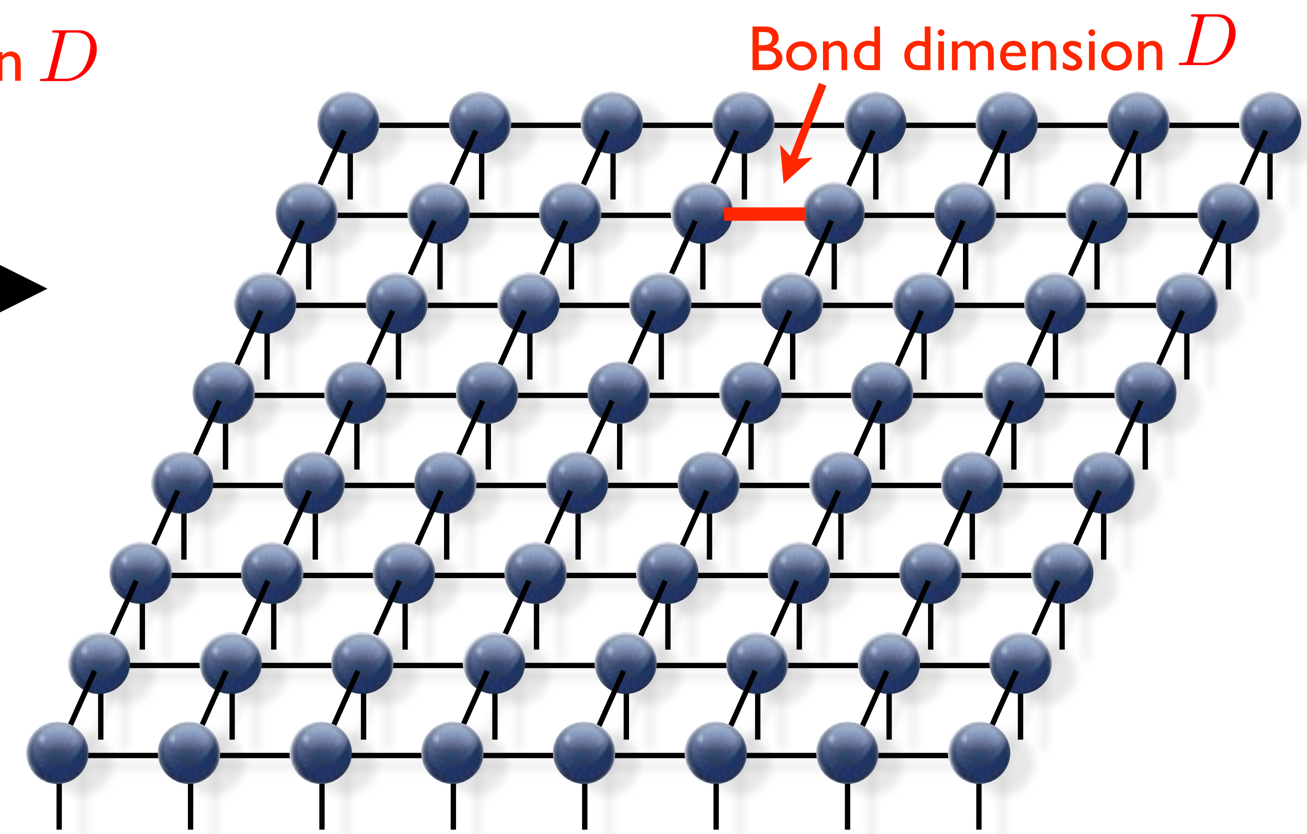


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✓ Reproduces area-law in 1D

$$S(L) = const$$

✓ Reproduces area-law in 2D

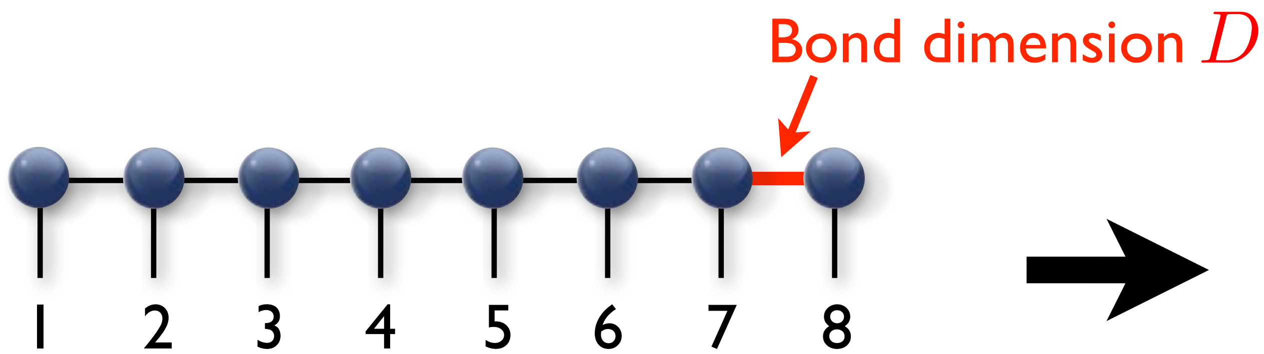
$$S(L) \sim L$$

# iPEPS

**1D**

## MPS

Matrix-product state

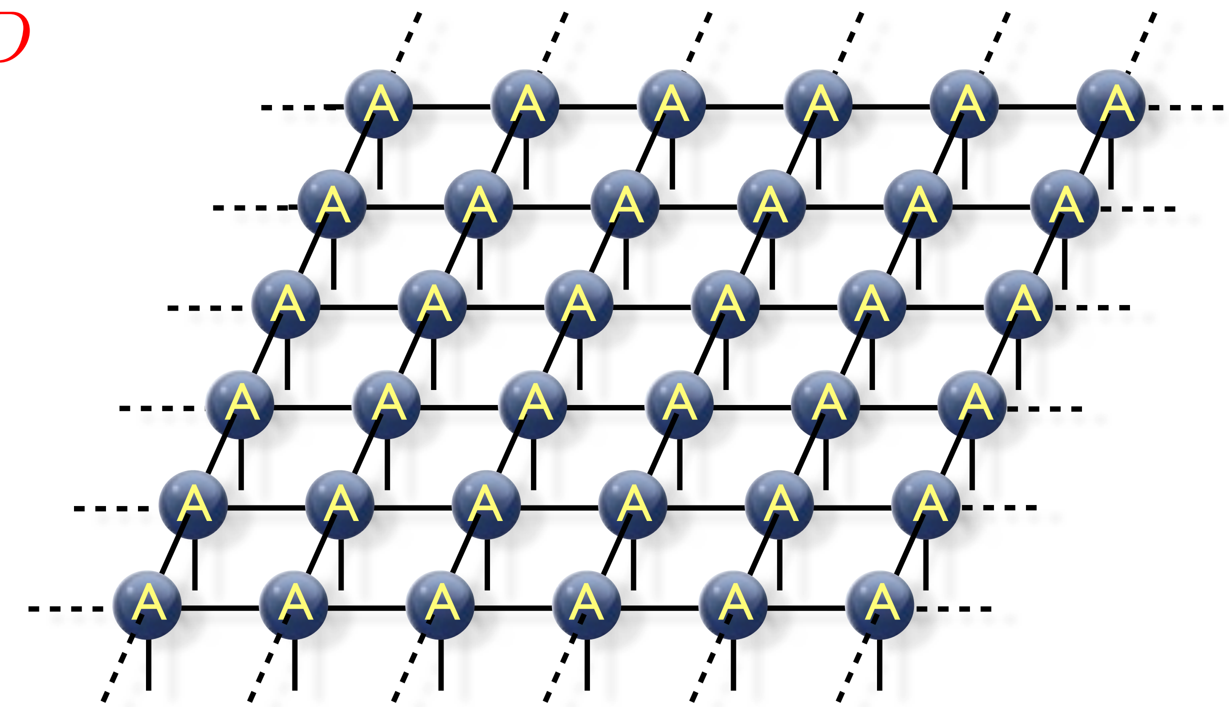



Physical indices (lattices sites)

**2D**

## iPEPS

infinite projected entangled-pair state



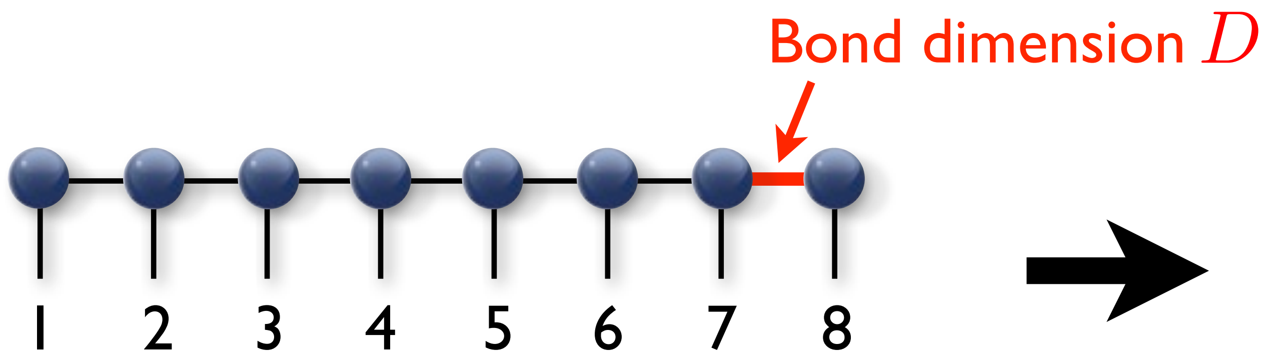

Jordan, Orus, Vidal, Verstraete, Cirac, PRL (2008)

# iPEPS with arbitrary unit cells

## 1D

### MPS

Matrix-product state

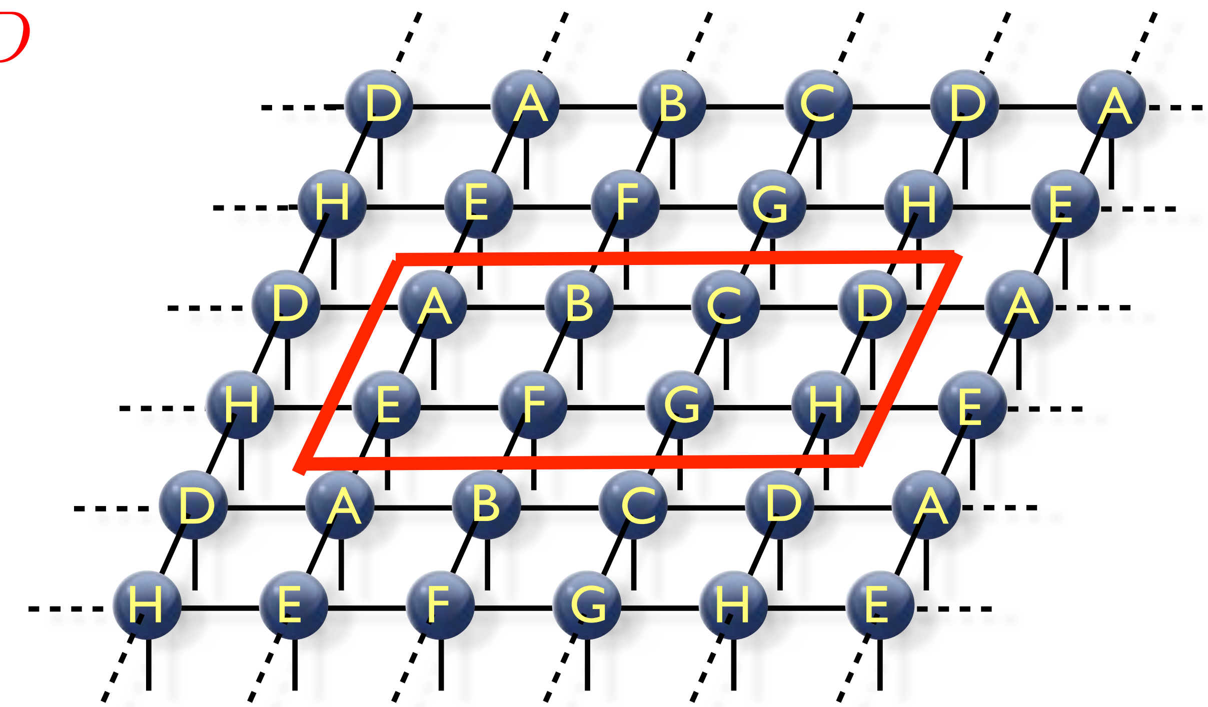



Physical indices (lattices sites)

## 2D

### iPEPS

with arbitrary unit cell of tensors



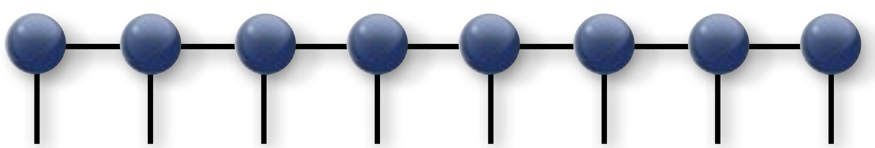

**here: 4x2 unit cell**

Corboz, White, Vidal, Troyer, PRB **84** (2011)

★ Run simulations with different unit cell sizes and compare variational energies

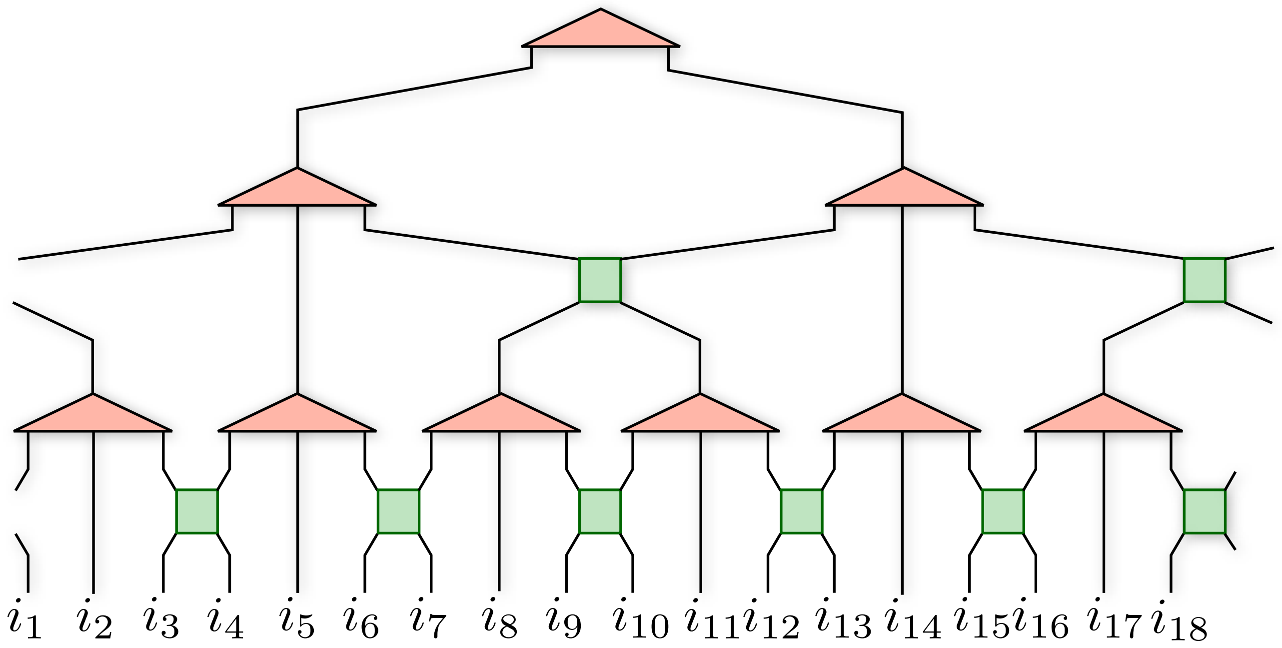
# Hierarchical tensor networks (TTN/MERA)

**MPS**



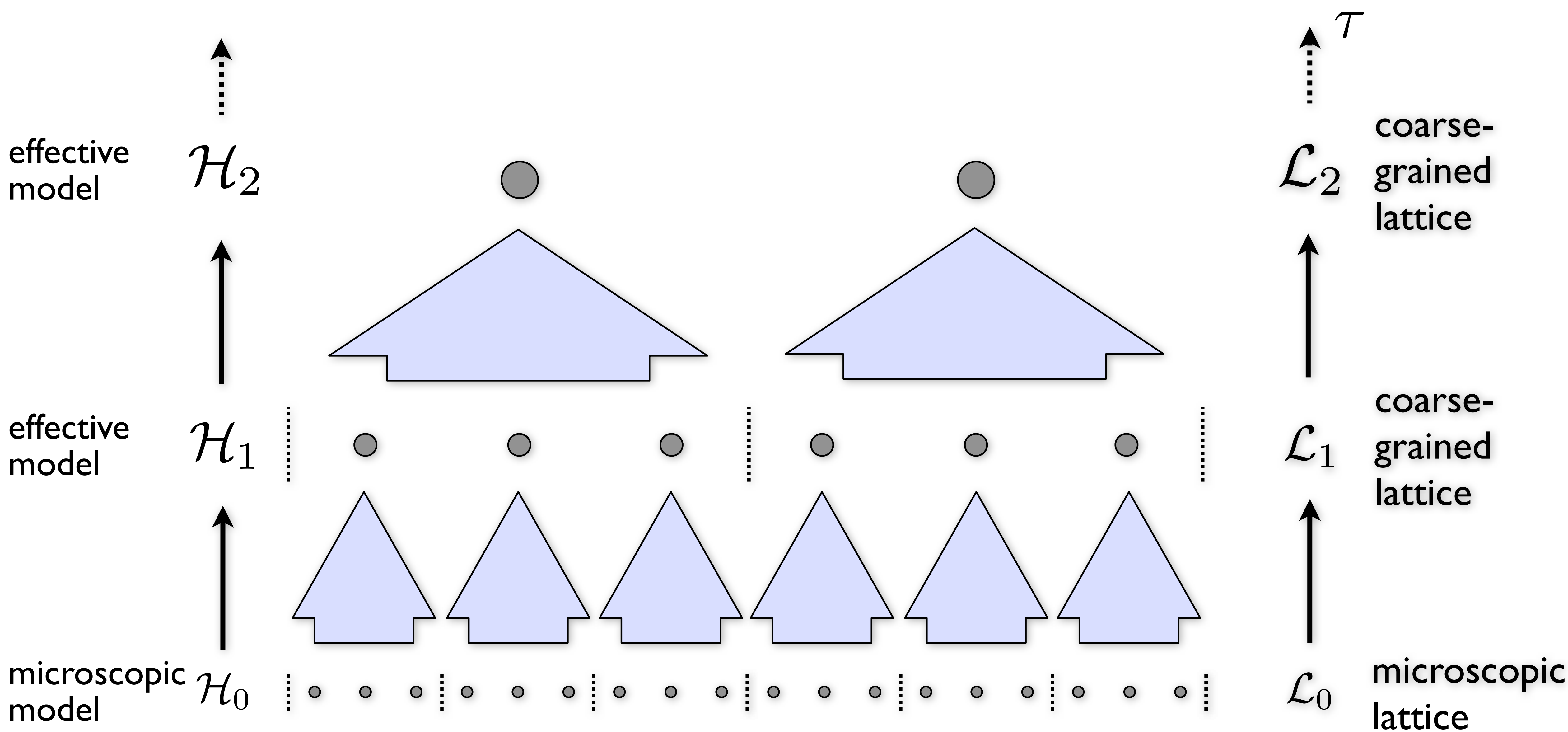
“flat”

**MERA**




tensors at different length scales

# Real-space renormalization group transformation

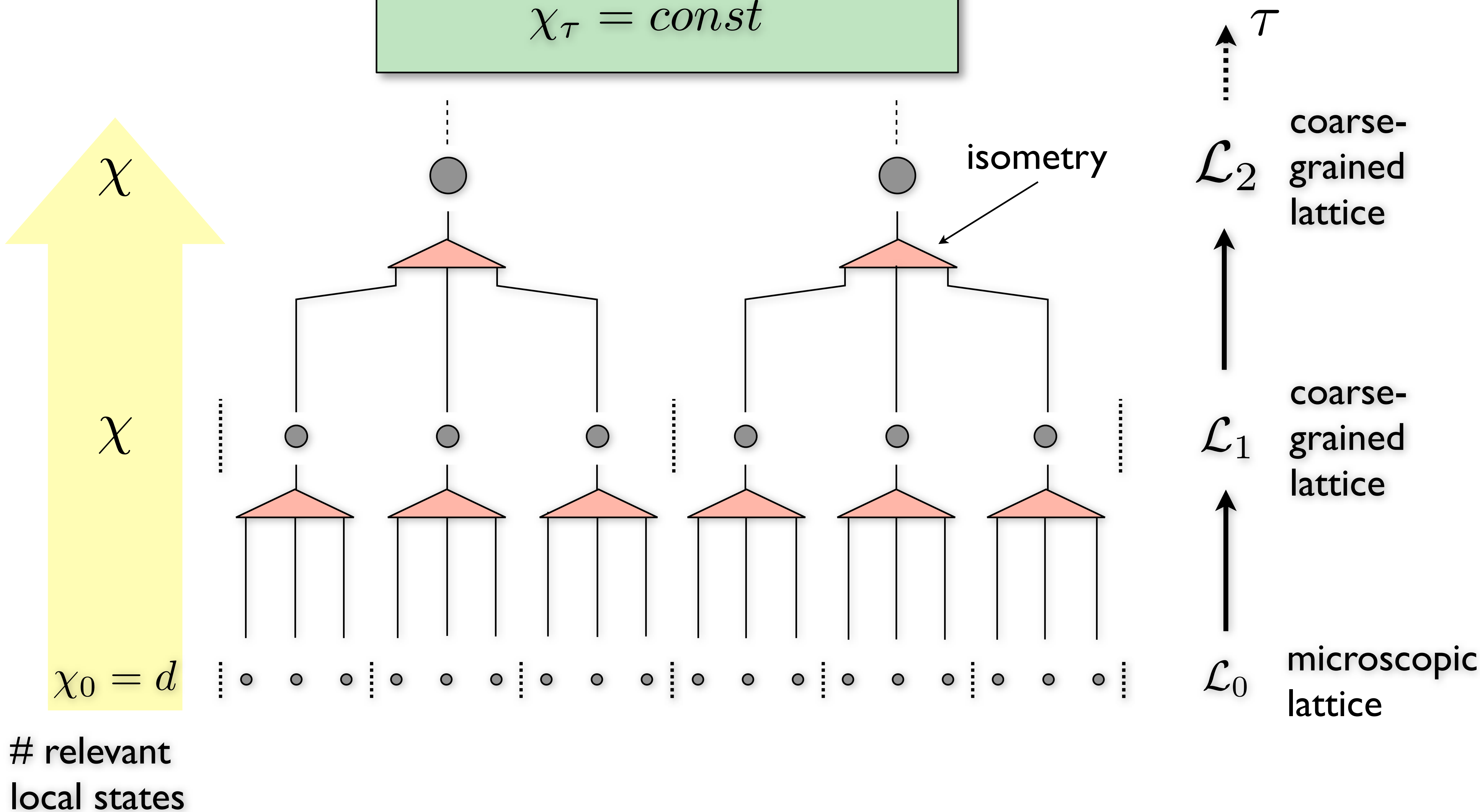


# Tree Tensor Network (1D)

1D systems (non-critical)

$$S(L) = const$$

$$\chi_\tau = const$$

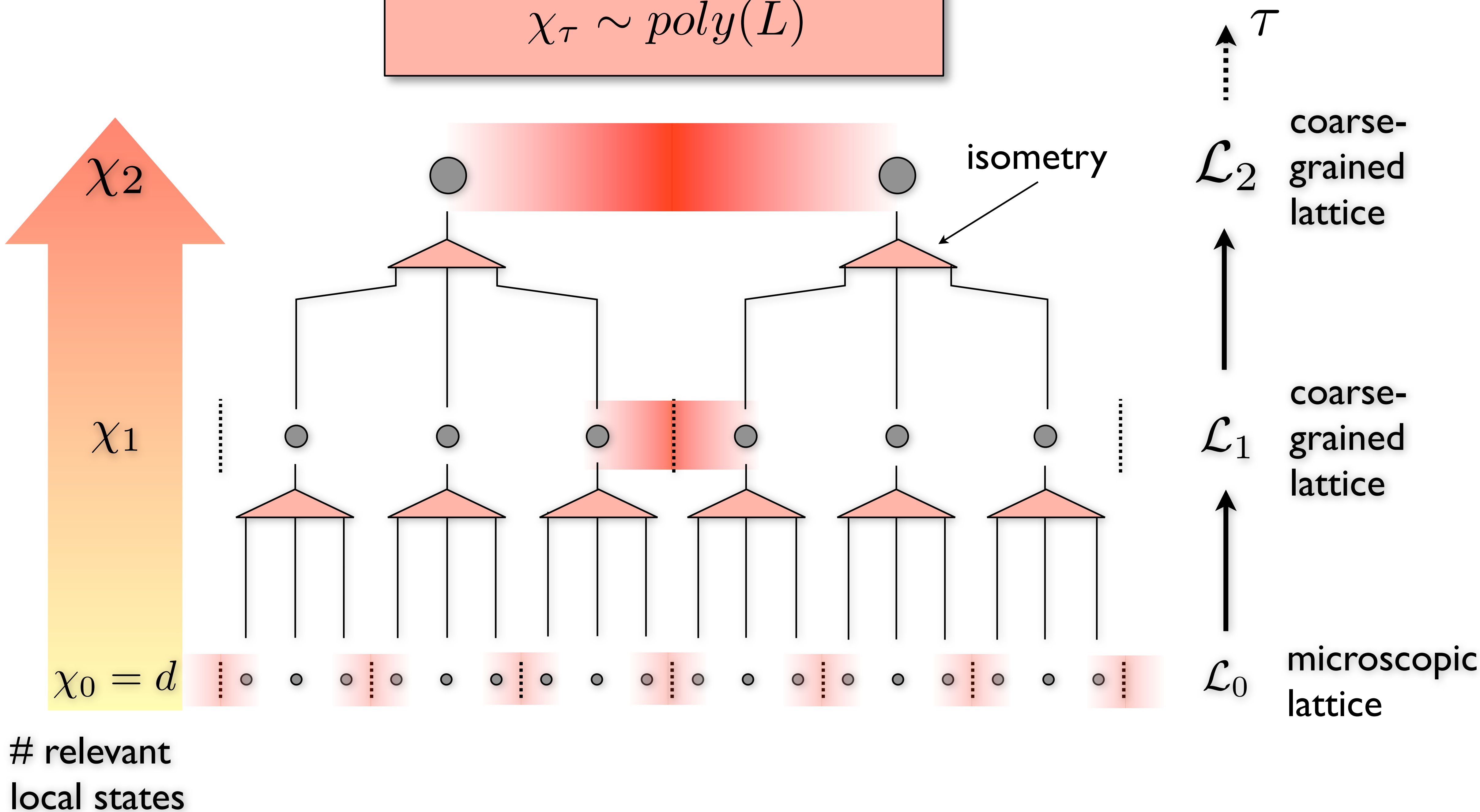


# Tree Tensor Network (1D)

1D critical systems

$$S(L) \sim log(L)$$

$$\chi_\tau \sim poly(L)$$



# The MERA (The multi-scale entanglement renormalization ansatz)

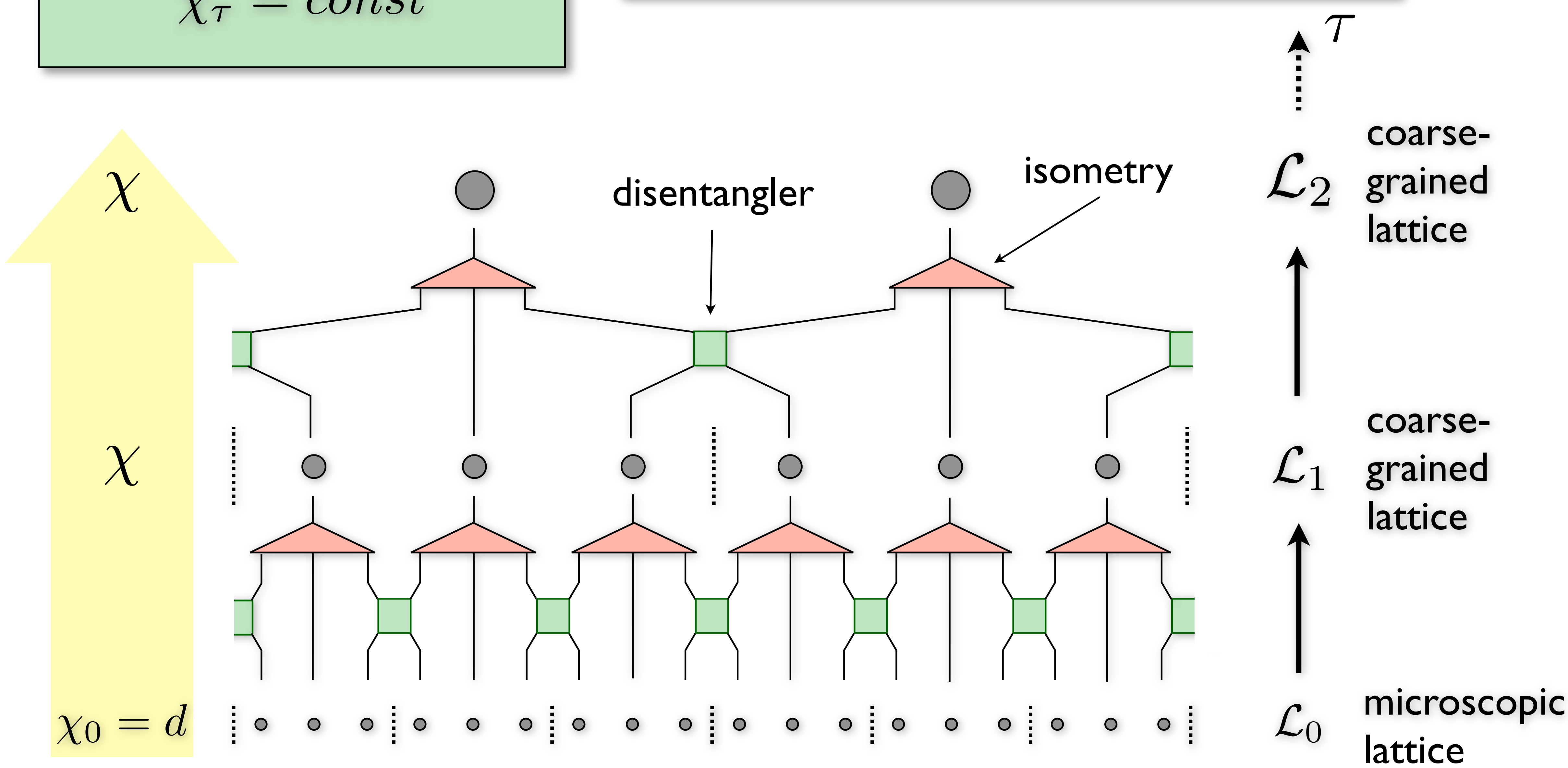
G. Vidal, PRL 99, 220405 (2007)
G. Vidal, PRL 101, 110501 (2008)

1D systems (critical)

$$S(L) \sim log(L)$$

$$\chi_\tau = const$$

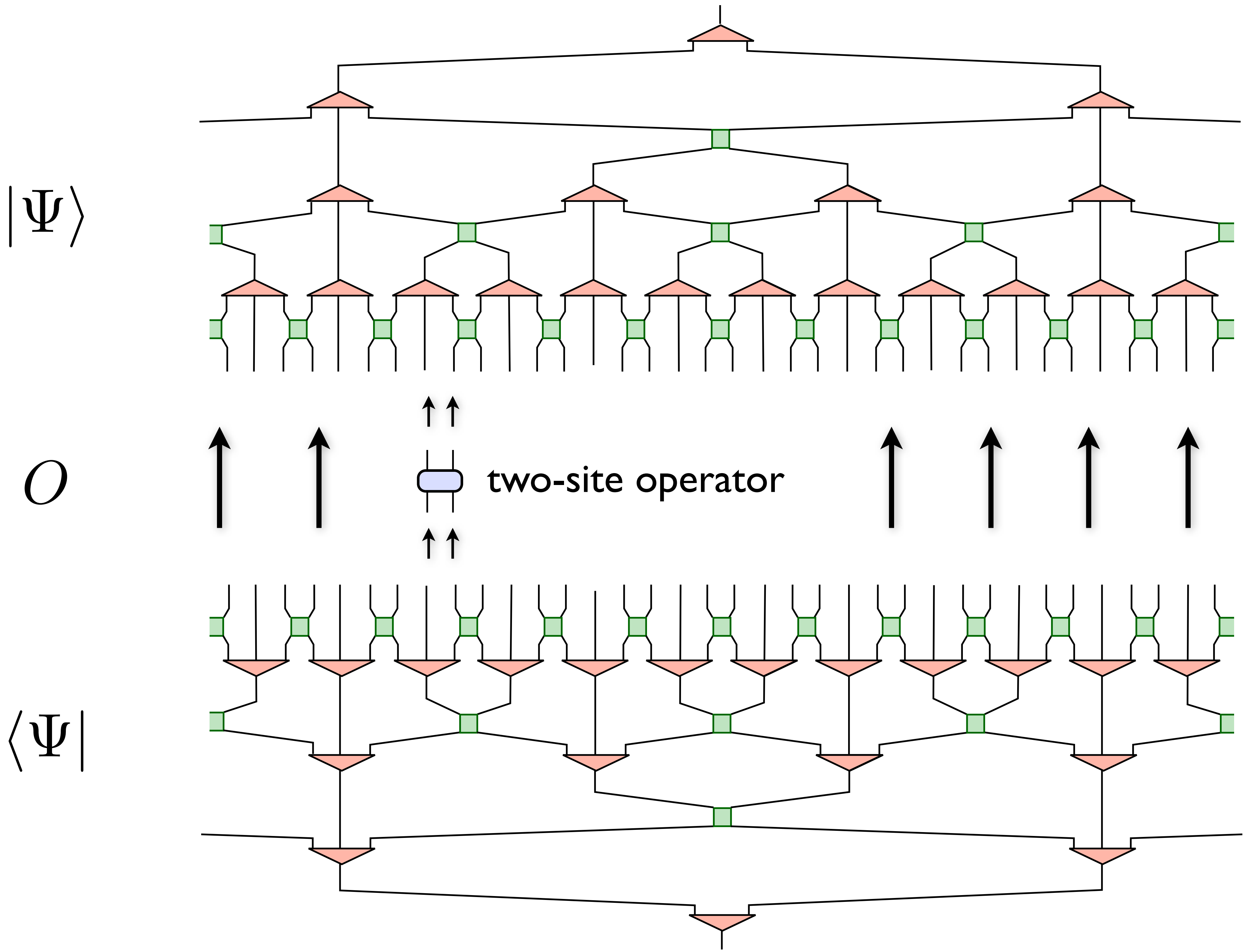
**KEY**: disentanglers reduce the amount of short-range entanglement



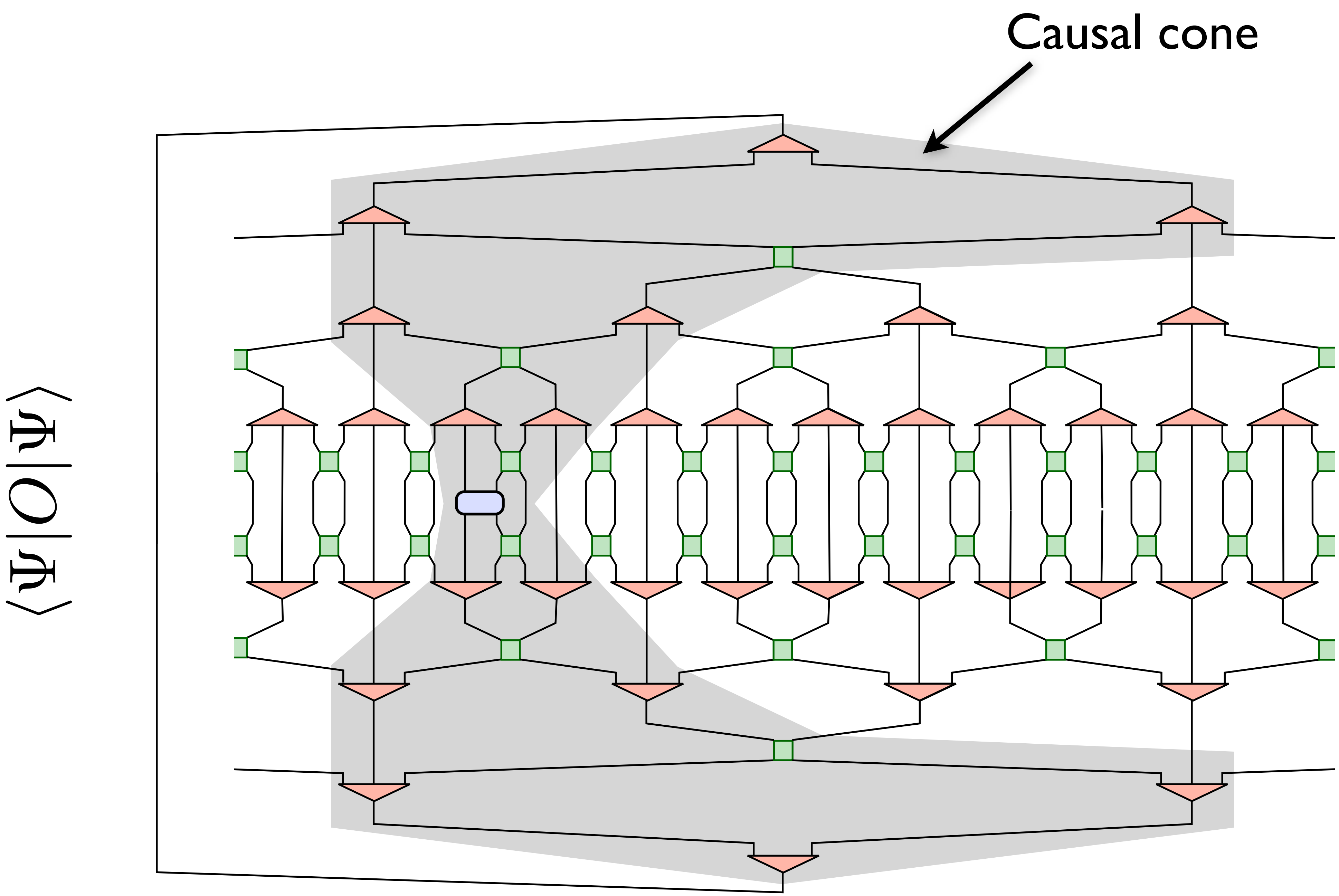

# relevant local states

# MERA: Properties

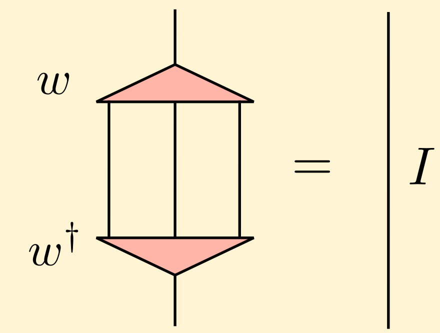
Let's compute $\langle \Psi | O | \Psi \rangle$ $\quad O$: two-site operator



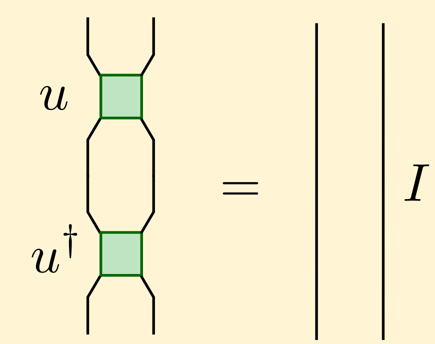
# MERA: Contraction




Isometries are *isometric*

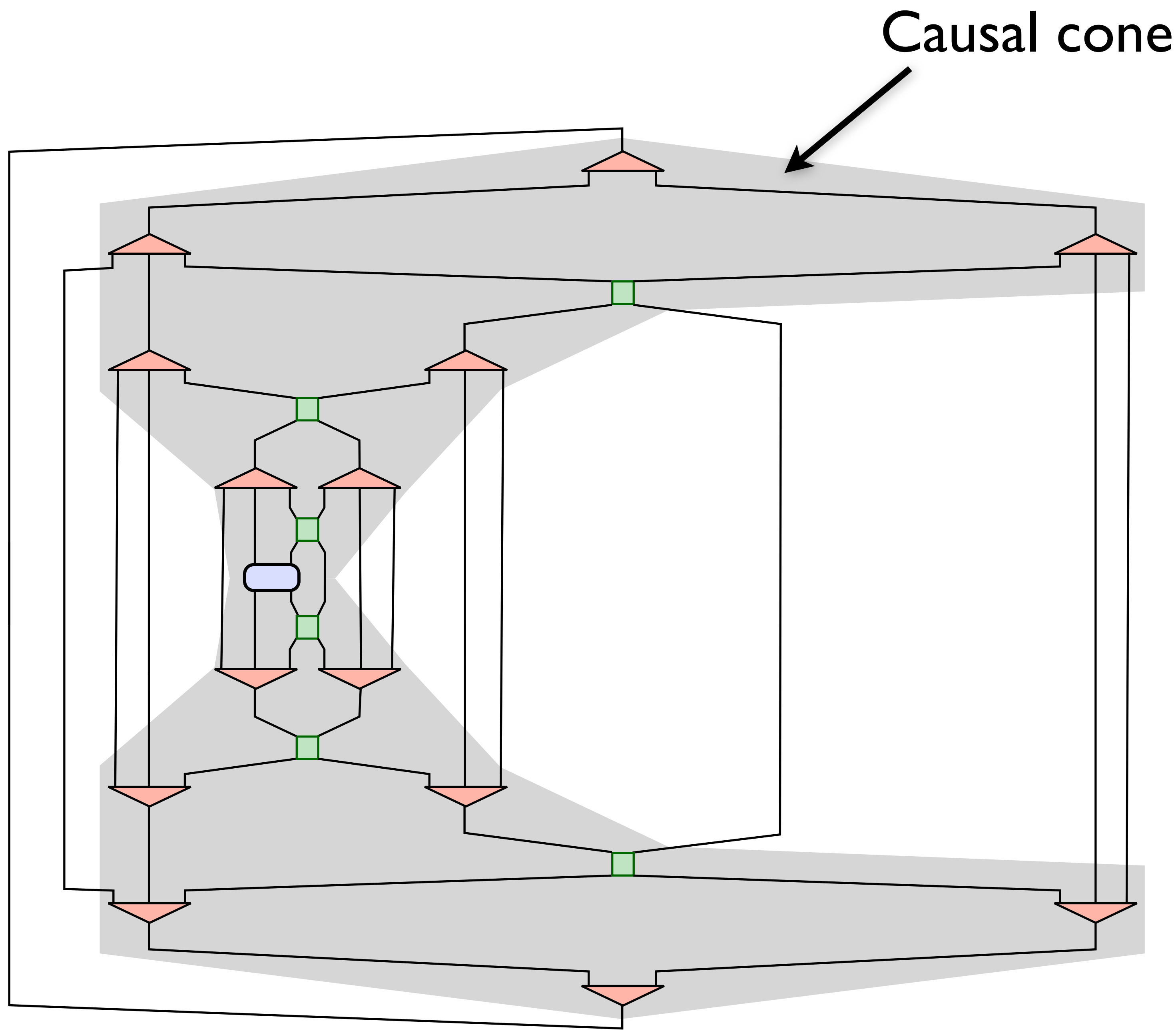



Disentanglers are *unitary*

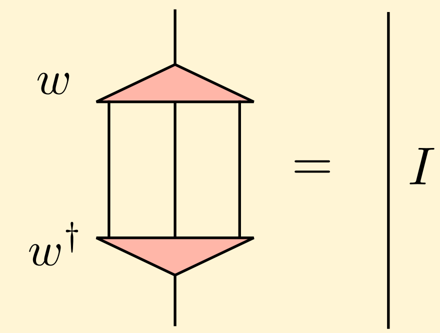


# MERA: Contraction

$\langle \Psi | O | \Psi \rangle$




Isometries
are *isometric*

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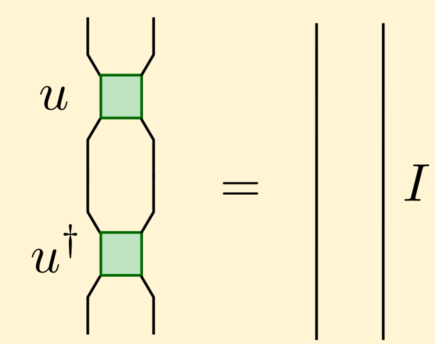
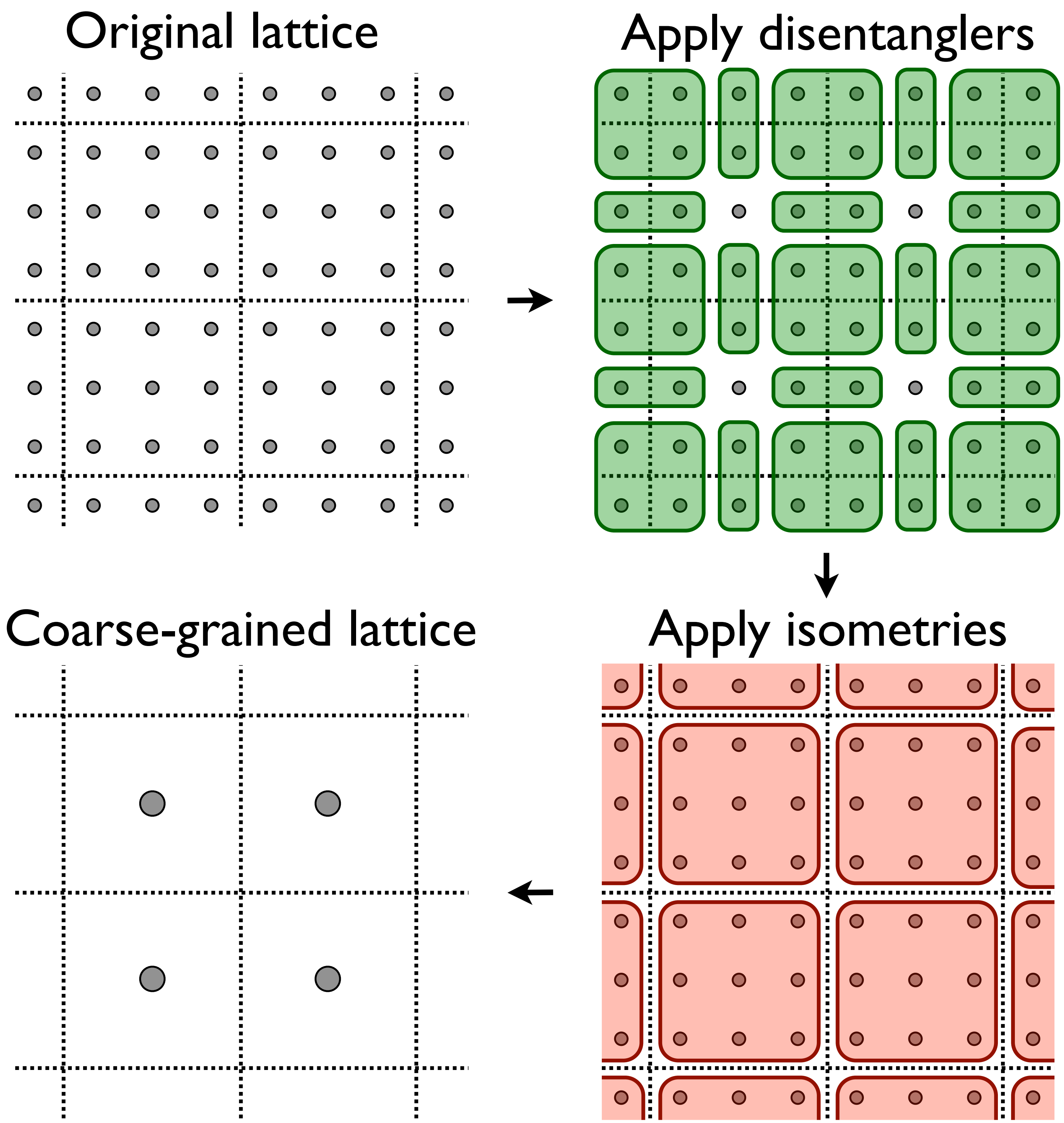



**Efficient** computation of expectation values of observables!

# 2D MERA (top view)

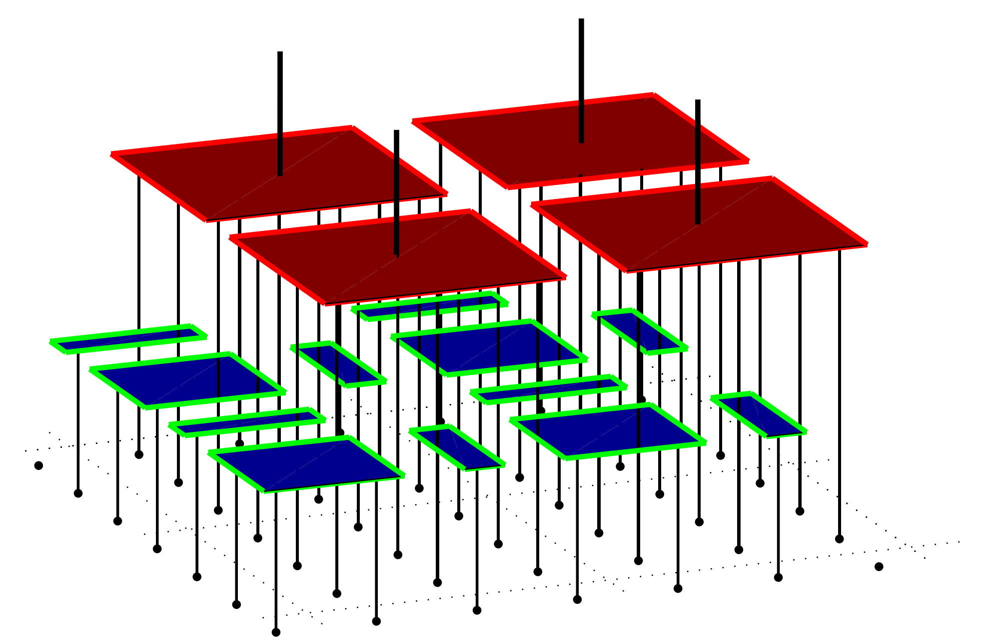
Evenbly, Vidal. PRL 102, 180406 (2009)




✓ Accounts for area-law in 2D systems

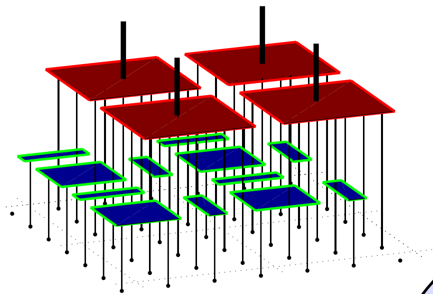
$$S(L) \sim L$$

$$\chi_{\tau} = const$$

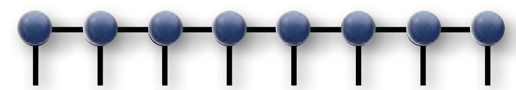


# Summary: Tensor network ansatz

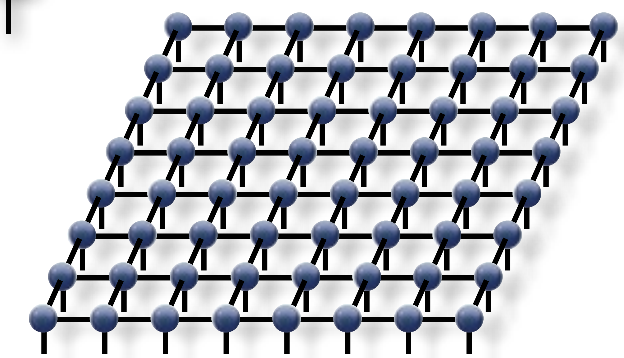
**2D MERA**

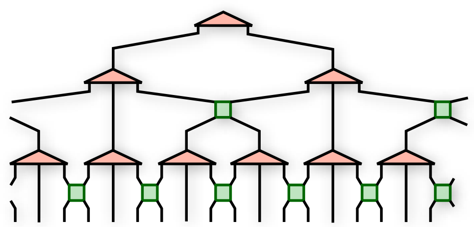


**MPS**



**PEPS**





**1D MERA**

**Structure**
**Variational ansatz**

➡ A tensor network ansatz is an efficient variational ansatz for "physical" states (GS of local H) where the accuracy can be systematically controlled with the bond dimension

➡ Different tensor networks can reproduce different entanglement entropy scaling:

- ★ MPS: area law in 1D
- ★ MERA: log L scaling in 1D (critical systems)
- ★ PEPS/iPEPS: area law in 2D
- ★ 2D MERA: area law in 2D
- ★ branching MERA: beyond area law in 2D (e.g. L log L scaling) (see Evenbly&Vidal)

**Reminder:**
variational wave-function

$$\langle\Psi|H|\Psi\rangle \geq E_0$$

the lower the energy the better!

# Computational cost

- Leading cost: $\mathcal{O}(D^k)$

MPS: $k = 3$

PEPS: $k \approx 10$

$N_{var} \sim D^4$

$cost \sim (N_{var})^{2.5}$

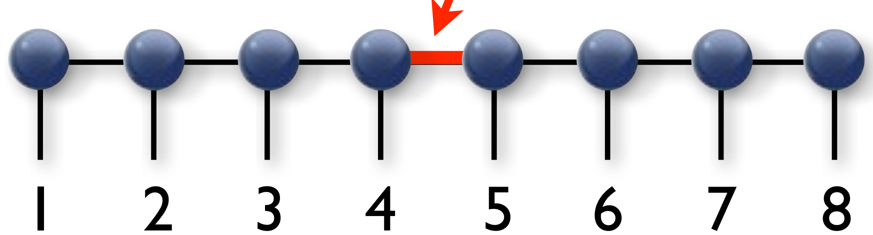
polynomial scaling but large exponent!

- How large does $D$ have to be?

It depends on the amount of entanglement in the system!

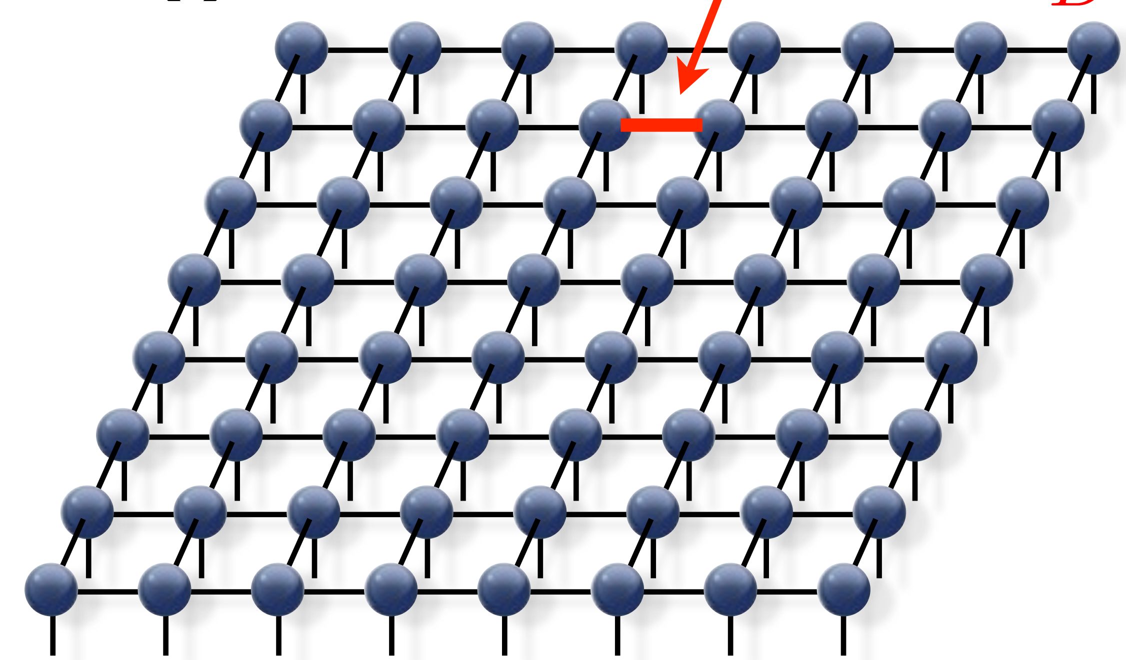
**MPS**

Bond dimension $D$




**(i)PEPS**

Bond dimension $D$



MPS for 2D system: D~exp(W)
accurate for cylinders
up to a width W~10

Typical size (2D): D=3000, O($10^7$) params.

D=10, O($10^4$) params.

3 orders of magnitude smaller!

# Summary: Tensor network algorithms