Lecture 3: Matrix-Product states.

idea: - use representation of eigenstates that focuses on the entanglement.
- typical physical states have mostly local entanglement, so using the full basis of the Hilbert-space is not necessary!
- want to use this to circumvent exponential growth of complexity in N.

Let us first think about spin-systems, again:

local Hilbert-space $V_i = \{|\uparrow\rangle, |\downarrow\rangle\}$

total Hilbert-space $\mathcal{H} = \bigotimes_{n=1}^{N} V_n$

Hamiltonian $= \sum_{\langle ij \rangle} \hat{h}_{ij} \leftarrow$ local terms.

general form of the ground state

$$|\psi\rangle = \sum_{\substack{s_1 \dots s_N \\ s_k \in \{\uparrow,\downarrow\}}} \psi_{s_1 s_2 s_3 \dots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots |s_N\rangle$$

$\uparrow$ 2^N coefficients = total Hilbert-space

Here, read $\psi_{s_1 \dots s_N}$ as a matrix / tensor with N indices $s_1 \dots s_N$.

Let's learn to write these efficiently

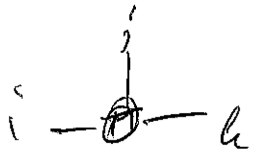
Diagrammatic notation

- each tensor is represented by a box / shape 
- external legs represent indices

 = C-number c

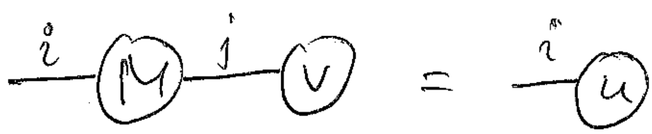
 = vector v_i

 = matrix M_{ij}

 = rank 3 tensor T_{ijk}

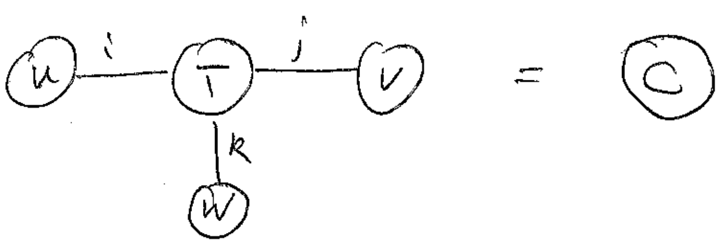
⋮

- summation-convention: lines connecting different tensors correspond to indices to be summed over

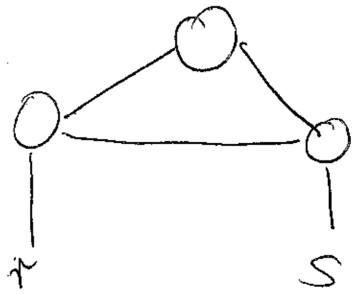
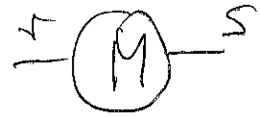


$$\sum_j M_{ij} v_j = u_i$$

or

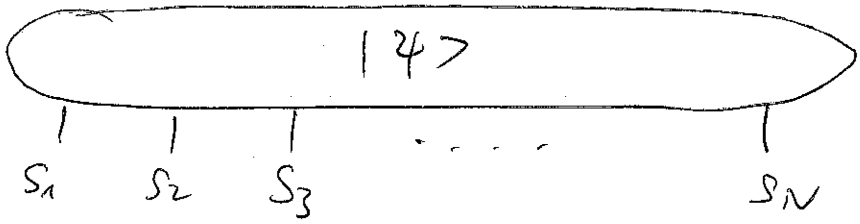


or

 = 

(Show big example on slide)

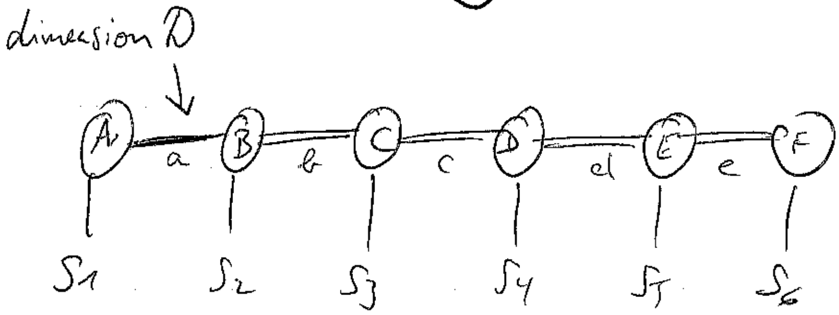
Representing the wave-function : spin systems



$= \psi_{s_1 \cdots s_N}$

2^N coefficients $\Rightarrow$
$\sim \exp(N)$

$\Downarrow$ Ansatz



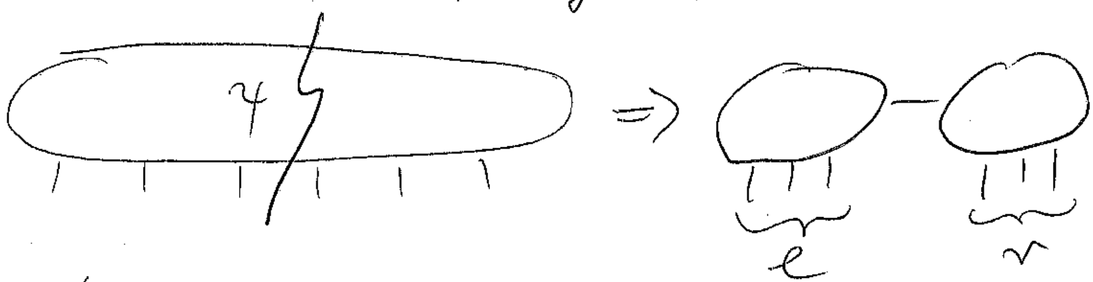
$\approx \tilde{\psi}_{s_1 \cdots s_N}$

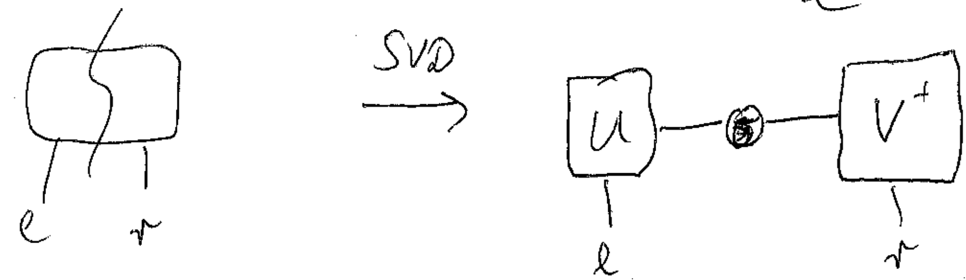
$$= \sum_{a \cdots e} A^{a}_{s_1} B^{ab}_{s_2} C^{bc}_{s_3} \cdots F^{e}_{s_6}$$

\# parameters $\simeq 2 \cdot N \cdot D^2 \approx \text{poly}(N, D)$.

<u>Why would such an Ansatz work?</u>

let's think about splitting tensors





$$\psi_{lr} = \sum_{l,r} \psi_{lr} |l\rangle|r\rangle = \sum_{lr} \sum_{k} U_{lk} \sigma_{kk} V^{+}_{rk} |l\rangle|r\rangle$$

$$= \sum_{k} \sigma_{kk} |u_k\rangle |v_k\rangle$$

Examples: pair of spins

- spin-triplet

$$|\psi\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 = \begin{array}{cc} & S_1 \\ S_2 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

$$= \sum_{\sigma} S_\sigma \, |\sigma\rangle_1 |\sigma\rangle_1 \qquad \text{with } S_+ = 1, \; S_- = 0$$

unentangled state, only one singular value!

- spin-singlet

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \right)$$

$$= \begin{pmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1/\sqrt{2} & \\ & 1/\sqrt{2} \end{pmatrix}}_{\sigma} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{V^+}$$

$$= \frac{1}{\sqrt{2}} |u_1\rangle |v_1\rangle + \frac{1}{\sqrt{2}} |u_2\rangle |v_2\rangle$$

maximally entangled state → ~~# singular values = # DOF.~~ all singular values have equal magnitude

Realistic states:

entanglement grows with the length of the circumference, not volume.

- show example from ED:

reduced density matrix.

Examples of Tensor Networks ⇒ continued on slides