Particle entanglement spectra for quantum Hall states on lattices

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TCM

GM & N. R. Cooper, Phys. Rev. Lett. **103**, 105303 (2009)

Th. Scaffidi & GM, arxiv:1207.3539 (to appear in Phys. Rev. Lett.)

A. Sterdyniak, N. Regnault, & GM, Phys. Rev. B **86**, 165314 (2012).

November 16, 2012, MPI-PKS Dresden

Entanglement Spectra in Complex Quantum Wavefunctions

Trinity Hall Cambridge

The Leverhulme Trust

Cavendish Laboratory

University of Cambridge

Outline

- Motivation: physical realizations of lattices with high flux density

- Relationship between different realizations of topological band structures:
  - Adiabatic continuation from Fractional Chern Insulators to Fractional Quantum Hall states

- Particle entanglement spectrum as a probe for quantum Hall states in lattices

Magnetic flux through periodic potentials

- magnetic field in presence of crystal potentials in solid state systems $\quad \ell_0 \gg a \qquad n \simeq 1$

Cold Atoms

- cold atoms in `rotating' optical lattices: $\quad \ell_0 \sim a \qquad n \gg 1$

- Simulating Aharonov-Bohm flux by complex hopping in tight binding optical lattices $\quad \ell_0 \simeq a \qquad n \simeq 1$

- Simulating Aharonov-Bohm flux by Berry phases in real space
  - Simulation of Landau-gauge (continuum) $\ell_0 \sim L^{1/2}$
  - "Optical Flux Lattices" $\quad \ell_0 \simeq a \qquad n \simeq 1$

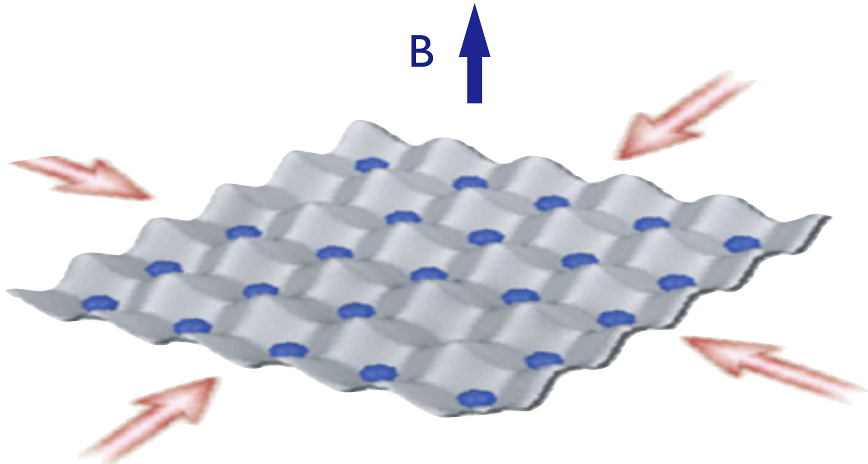
Solid State / Cold Atoms

- Chern bands: Berry flux in reciprocal space

$\ell_0 \simeq a \qquad n \simeq 1$

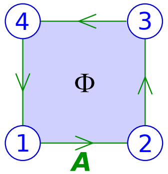
Cold Atoms I: Optical Lattices with Complex Hopping

experimental realisation: optical lattice + Raman lasers



⇒ possibility to simulate Aharonov-Bohm effect of magnetic field by imprinting phases for hopping via Raman transitions

$$\sum_{\square} A_{\alpha\beta} = 2\pi n_{\rm V}$$



⇒ Bose-Hubbard with a magnetic field (→ Lorentz force)

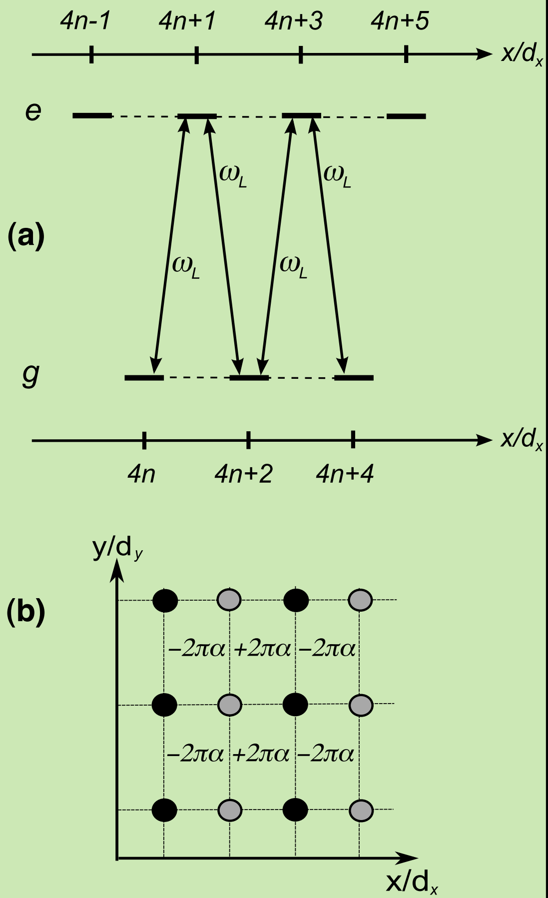
$$\mathcal{H} = -J \sum_{\langle\alpha,\beta\rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.\right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha(\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

particle density n vortex/flux density $n_{\rm V}$ interaction U/J

J. Dalibard, et al. Rev. Mod. Phys. 83, 1523 (2011)

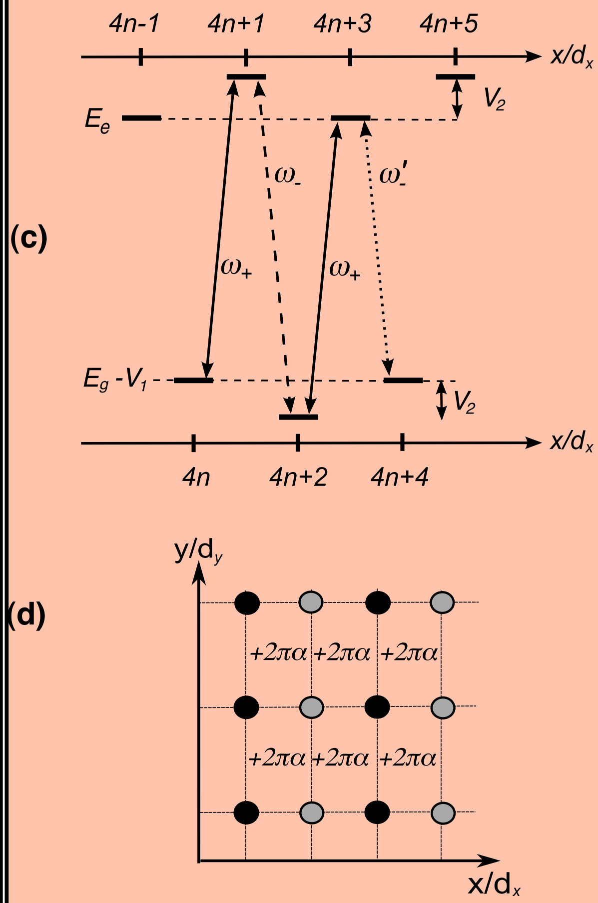
Cold Atoms I: Optical Lattices with Complex Hopping

Staggered Flux



Realized by group of I. Bloch:
Phys. Rev. Lett. **107**, 255301 (2011).

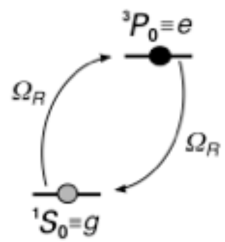
'Rectified' Flux

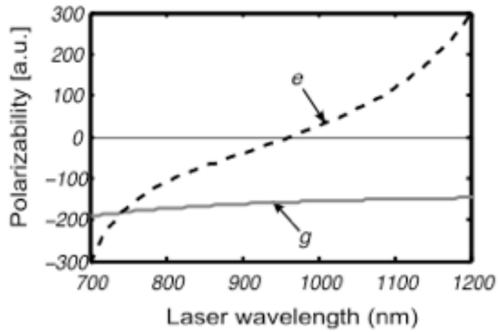


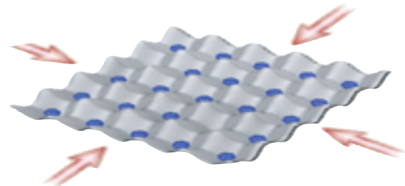
Can tune flux density!

Experimental realization outstanding

'anti-magic' optical lattice for Yb





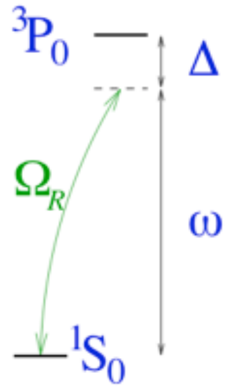


Gerbier & Dalibard, NJP 2010

Cold Atoms II: Berry phases of optically dressed states

J. Dalibard, F. Gerbier, G. Juzeliunas, P. Öhberg, RMP 2011

spacially varying optical coupling of *N* internal states (consider N=2)



Yb

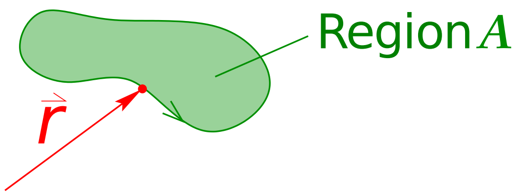
$$\mathcal{H} = \frac{\mathbf{p}^2}{2m}\hat{1} + \frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega_R(\mathbf{r}) \\ \Omega_R(\mathbf{r}) & \Delta \end{pmatrix}$$

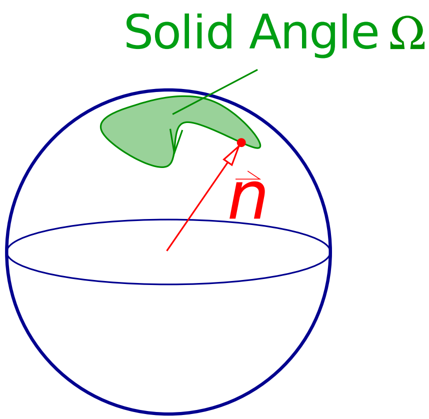
local spectrum $E_n(\boldsymbol{r})$ and dressed states: $|\Psi_{\mathbf{r}}\rangle$

→ adiabatic motion of atoms *in space* in the optical potential generates a Berry phase

for *N*=2, consider Bloch sphere of unit vector $\vec{n} = \langle \Psi_{\mathbf{r}} | \hat{\vec{\sigma}} | \Psi_{\mathbf{r}} \rangle$

Berry flux generated: $n_\phi = \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{\mu\nu} n_i \partial_\mu n_j \partial_\nu n_k$





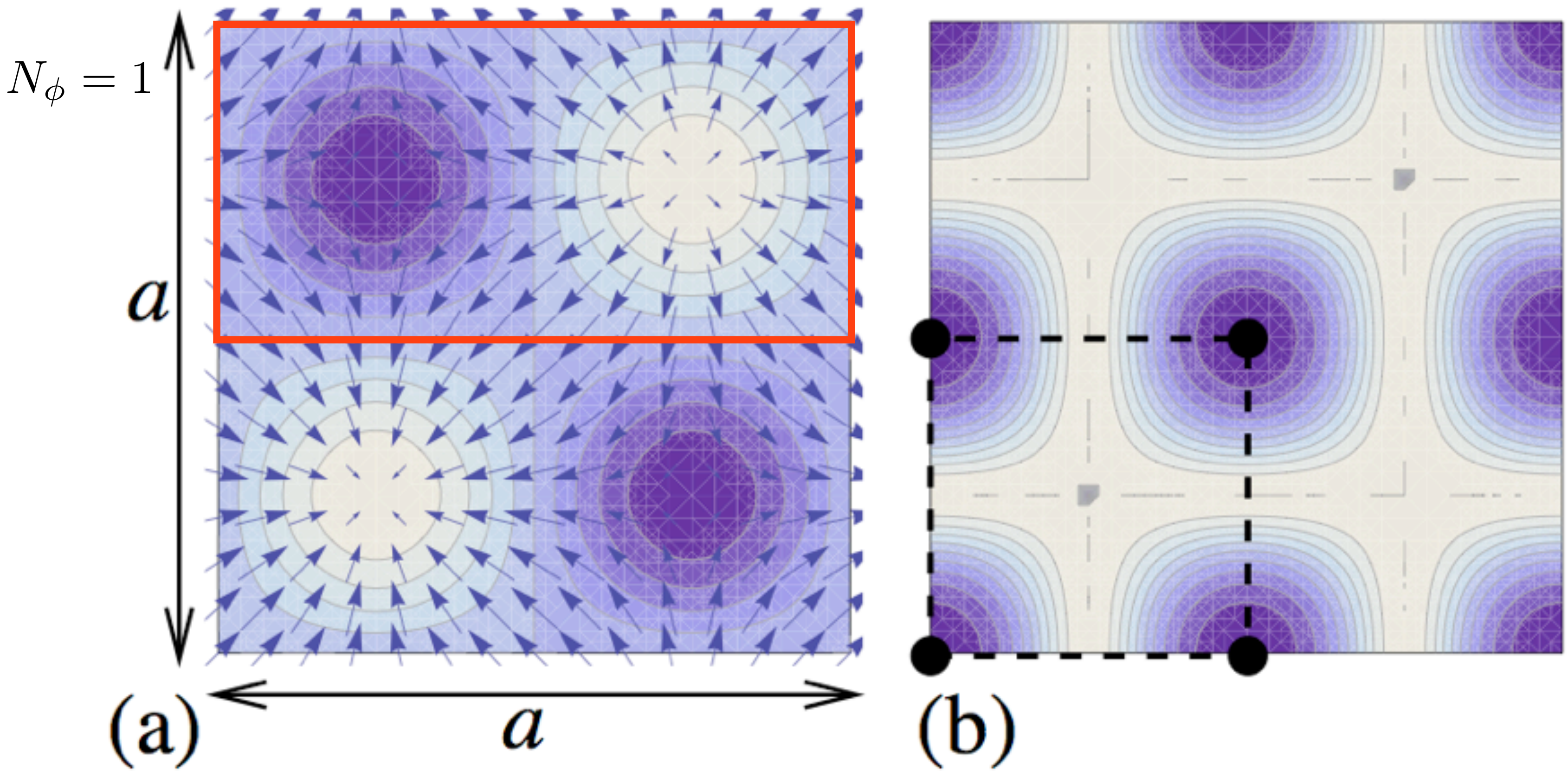
Total flux quanta
= # times Bloch
vector wraps sphere

$$\int_A n_\phi \, d^2r = \frac{\Omega}{4\pi}$$

Cold Atoms II: Optical Flux Lattices

Nigel Cooper, PRL (2011); N. Cooper & J. Dalibard, EPL (2011)

periodic optical Raman potentials are conveniently located by standing wave Lasers yielding an optical flux lattice of high flux density, here $n_\phi = 1/2a^2$ (fixed)



(a) Local direction of unit Bloch vector $\vec{n}$ (b) Local density of Berry Flux n_ϕ

Fractionalization in Chern bands? Chern #1 Bands = FQHE ?

Proposition: correlated states reproduce the physics of FQHE

Selected for a Viewpoint in *Physics*
PRL **106**, 236804 (2011) PHYSICAL REVIEW LETTERS week ending 10 JUNE 2011

Fractional Quantum Hall States at Zero Magnetic Field

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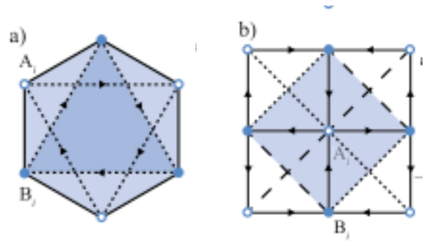
(Received 22 December 2010; published 6 June 2011)



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PRL **106**, 236803 (2011) PHYSICAL REVIEW LETTERS week ending 10 JUNE 2011

Nearly Flatbands with Nontrivial Topology

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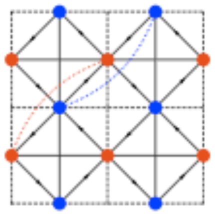
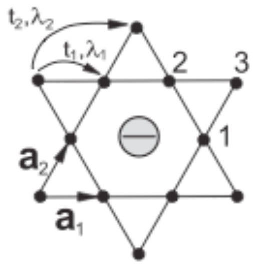
(Received 28 December 2010; revised manuscript received 19 March 2011; published 6 June 2011)



(a)



Selected for a Viewpoint in *Physics*
PRL **106**, 236802 (2011) PHYSICAL REVIEW LETTERS week ending 10 JUNE 2011

High-Temperature Fractional Quantum Hall States

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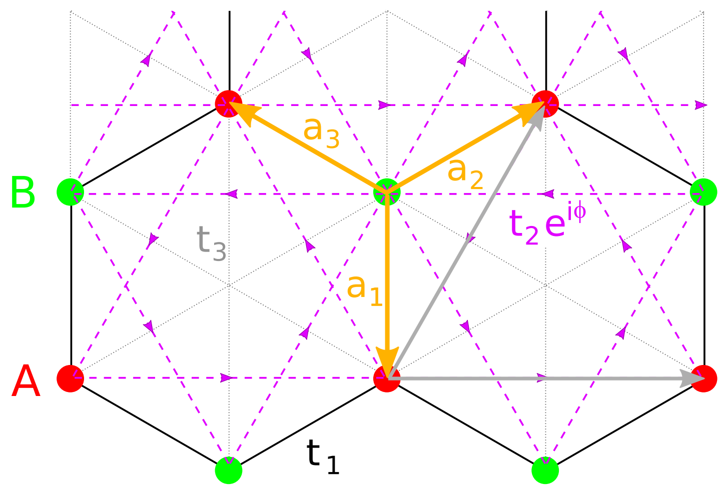
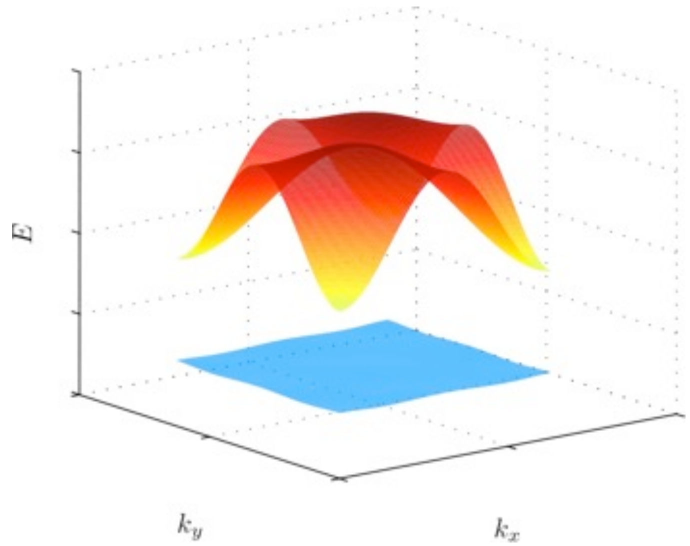
(Received 14 December 2010; published 6 June 2011)

Characteristic example: The Haldane Model

F.D.M. Haldane, PRL (1988), Neupert *et al.* PRL (2011)





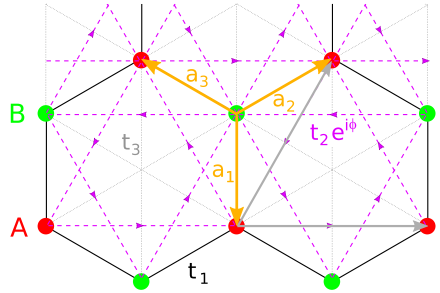
$$\mathcal{H} = -\, t_1 \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} (\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} + h.c.) - t_2 \sum_{\langle\langle \mathbf{r}\mathbf{r}' \rangle\rangle} (\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} e^{i\phi_{\mathbf{r}\mathbf{r}'}} + h.c.)$$

$$-\, t_3 \sum_{\langle\langle\langle \mathbf{r}\mathbf{r}' \rangle\rangle\rangle} (\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} + h.c.) + \frac{U}{2} \sum_{\mathbf{r}} \hat{n}_{\mathbf{r}} (\hat{n}_{\mathbf{r}} - 1)$$

- *tight binding* model in *real space* on hexagonal lattice
- with fine-tuned hopping parameters: obtain flat lower band, e.g. values [D. Sheng, PRL (2011)]

$$t_1 = 1,\ t_2 = 0.60,\ t_2 = -0.58 \text{ and } \phi = 0.4\pi$$

Topological (flat) bands in two-dimensions



- diagonalize Hamiltonian by Fourier transform

$$\mathcal{H} = \sum_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k},\alpha} h_{\alpha\beta}(\mathbf{k}) \hat{a}_{\mathbf{k},\beta}$$

Bloch states

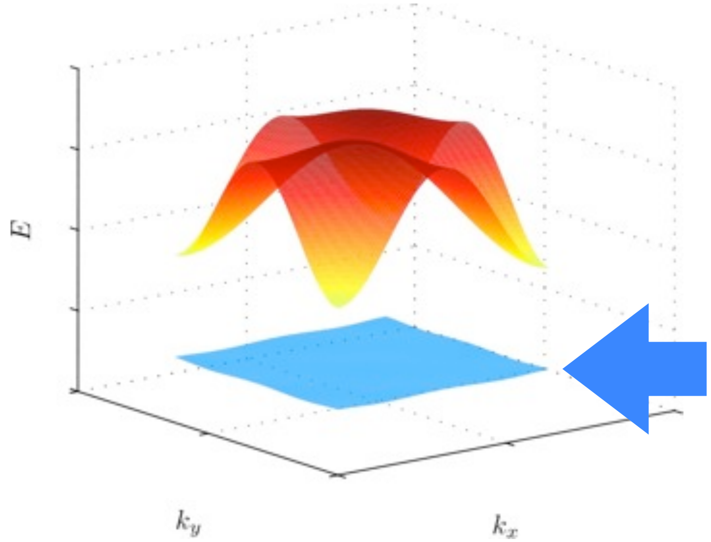
$$h_{\alpha\beta}(\mathbf{k}) u^n_\beta(\mathbf{k}) = \epsilon_n(\mathbf{k}) u^n_\alpha(\mathbf{k})$$

- study Berry curvature in n^{th} band:

Berry connection: $\mathcal{A}(n, \mathbf{k}) = -i \sum_\alpha u^{n*}_\alpha(\mathbf{k}) \nabla_{\mathbf{k}} u^n_\alpha(\mathbf{k})$

Berry curvature: $\mathcal{B}(k) = \nabla_{\mathbf{k}} \wedge \boldsymbol{\mathcal{A}}(k)$

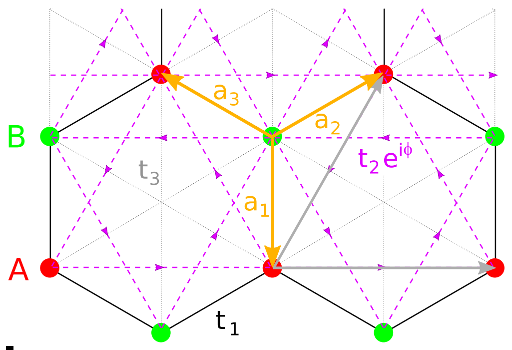
Chern number: $C = \frac{1}{2\pi} \int_{BZ} d^2\mathbf{k}\, \mathcal{B}(\mathbf{k})$

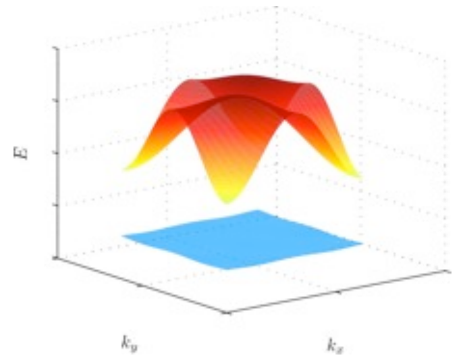


Flux Lattices vs Chern Bands

CBs = tight binding model in *real* space with topological flat bands

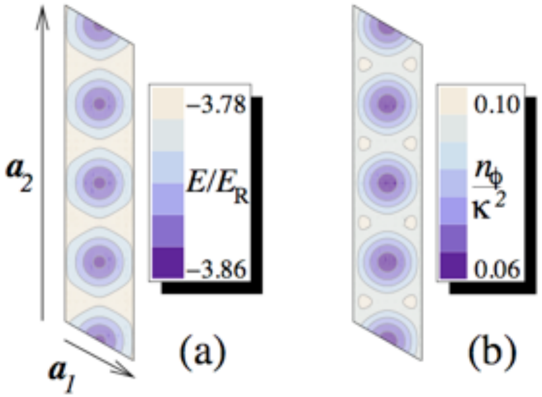
real space | *reciprocal* space

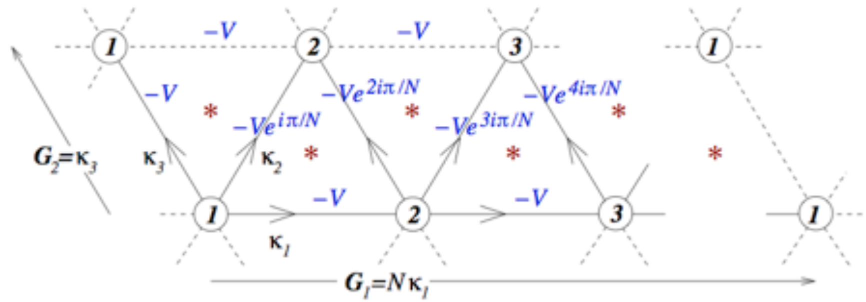




Flux Lattice = tight binding model in *reciprocal* space with topological flat bands

N. R. Cooper, R. Moessner, arxiv:1208.4579

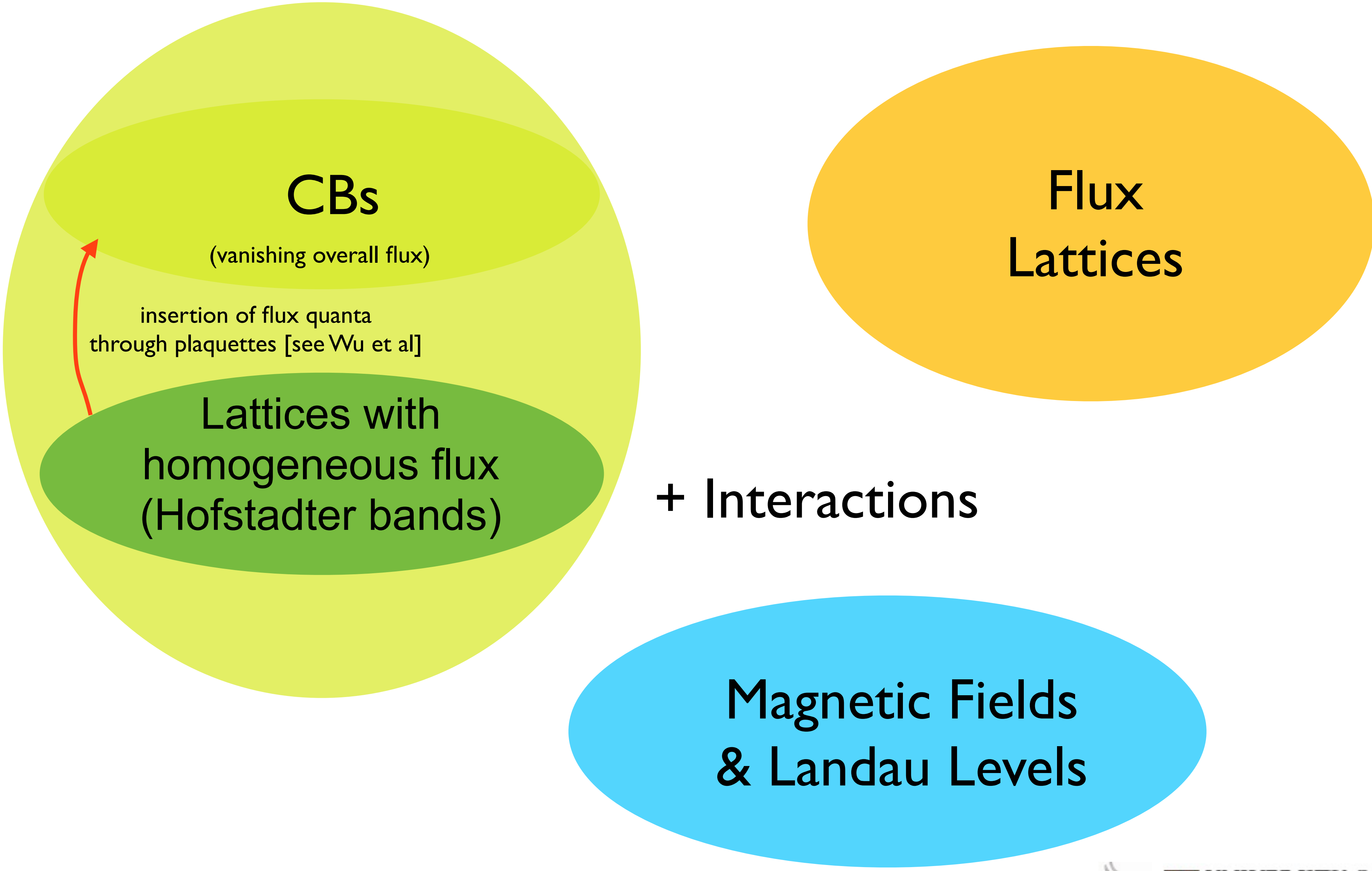




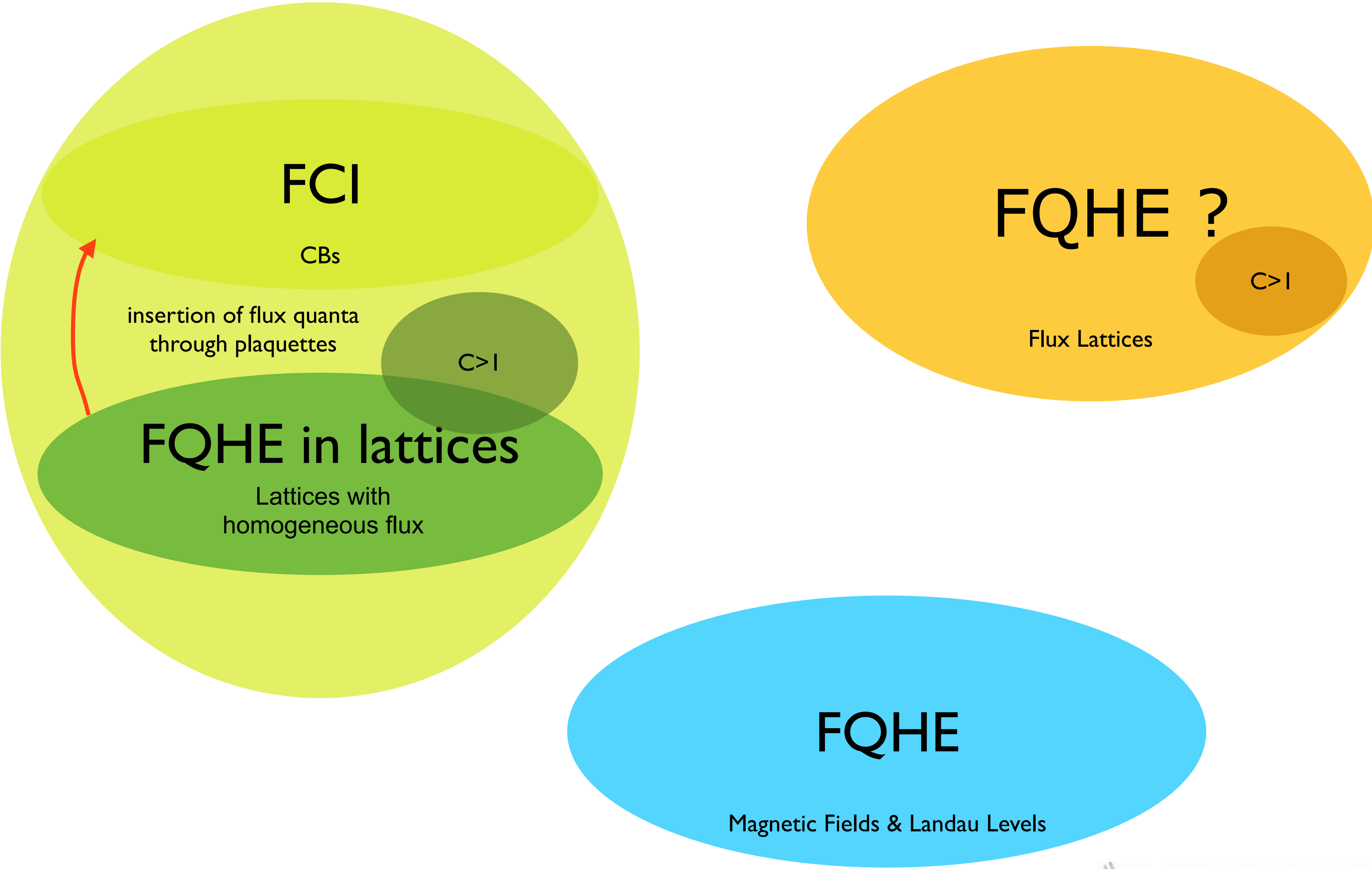
absorption of photons = momentum + state transfers

real space | *reciprocal* space

Overview of underlying band structures

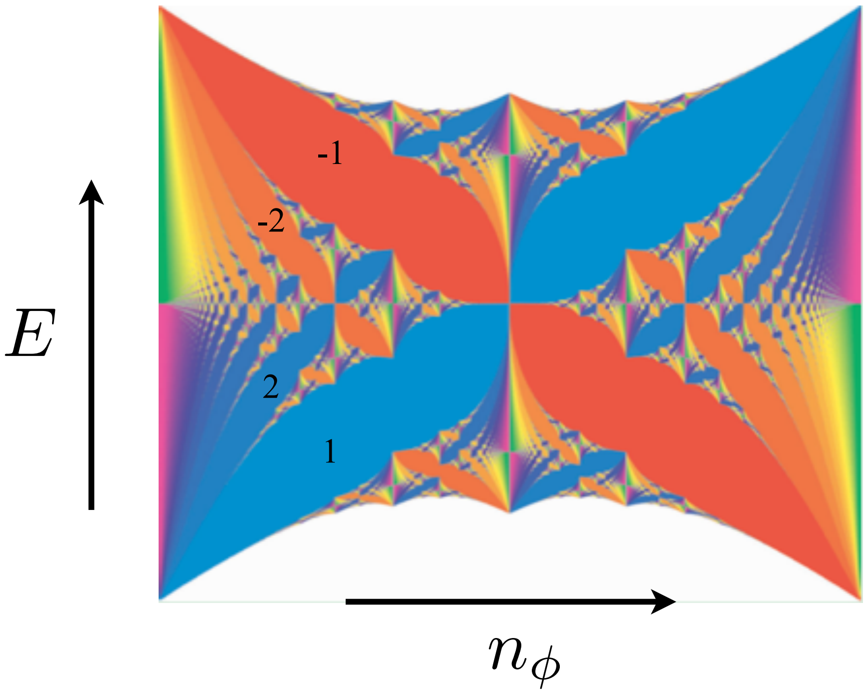


Interaction driven phases in these bands

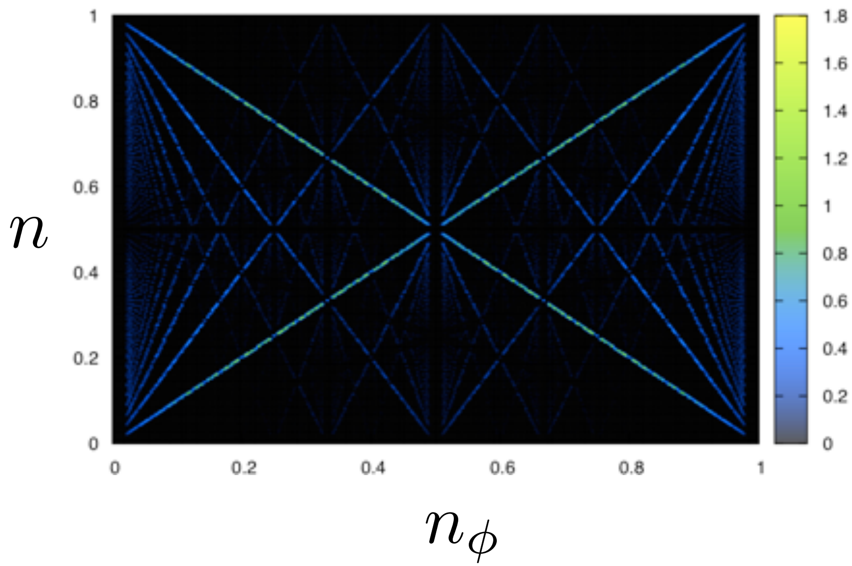


Strongly correlated states from the Hofstadter spectrum

- Hofstadter spectrum provides bands of all Chern numbers [Avron et al.]



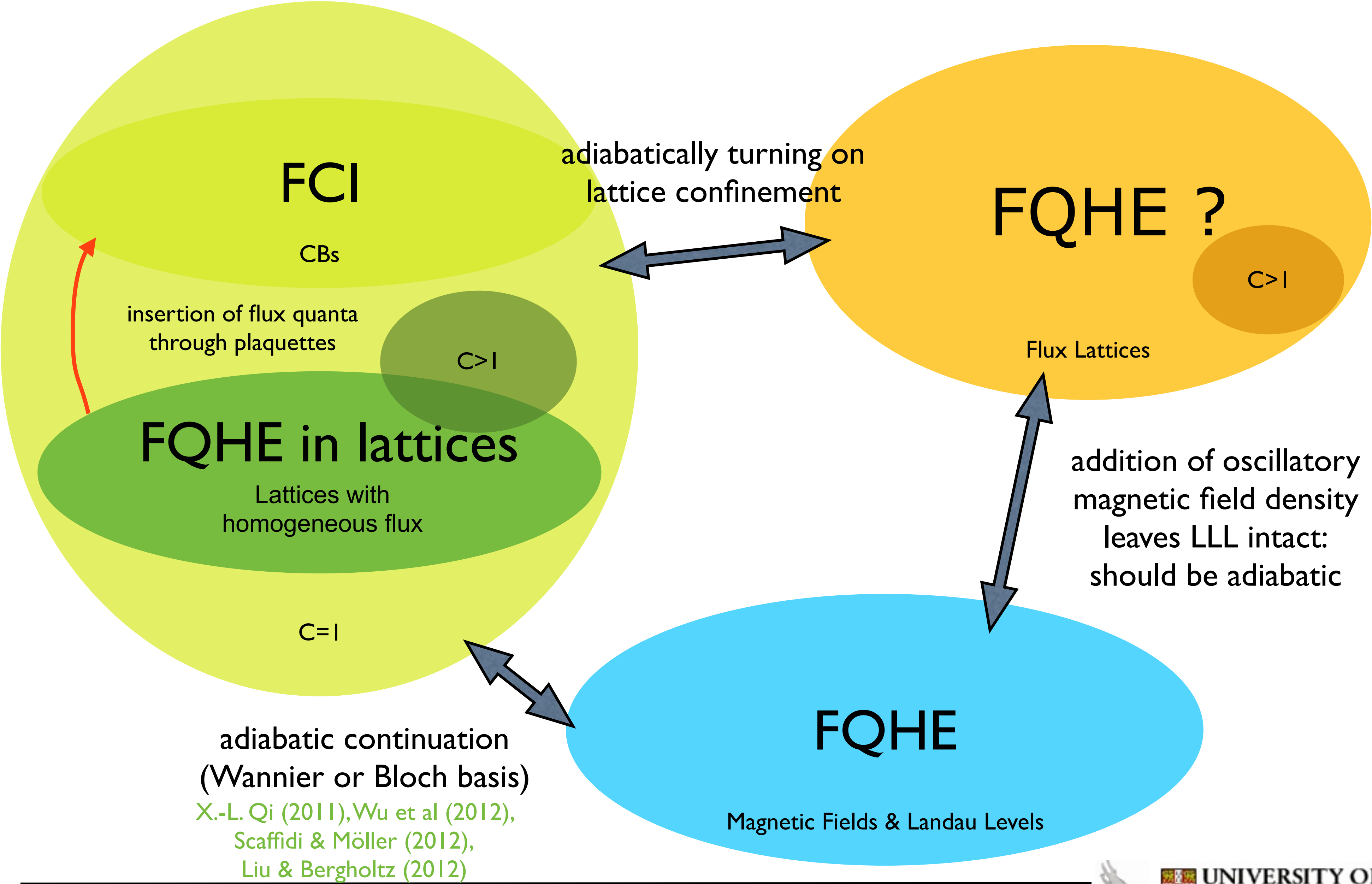
- Interactions stabilize fractional quantum Hall liquids in these bands!
- CF Theory: GM & N. R. Cooper, PRL (2009)



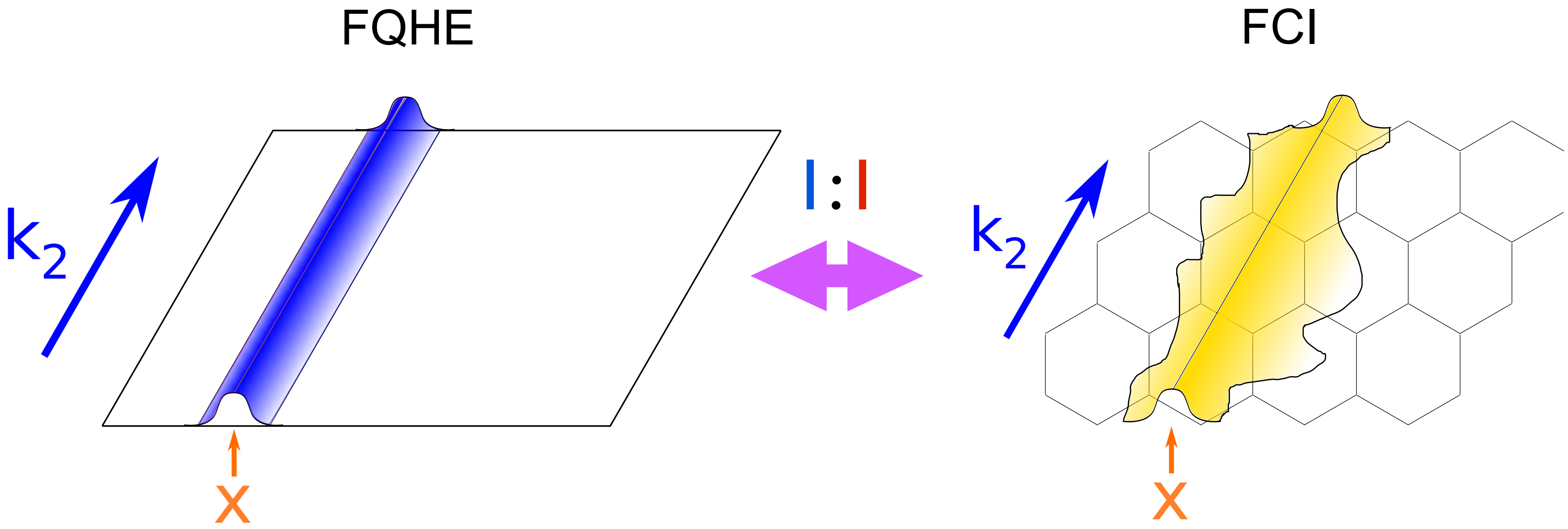
- Near rational flux density: LL's with additional pseudospin index

R. Palmer & D. Jaksch PRL 2006
L. Hormozi et al, PRL 2012

Interrelations between different interacting problems



Mapping from FQHE to FCI: Single Particle Orbitals



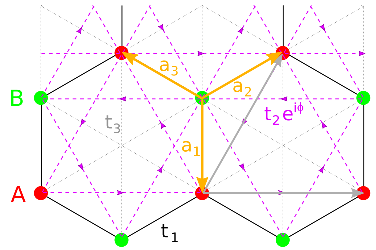
- Proposal by X.-L. Qi [PRL '11]: Get FCI Wavefunctions by mapping single particle orbitals

- Idea: use Wannier states which are localized in the x-direction
- keep translational invariance in y (cannot create fully localized Wannier state if C>0!)

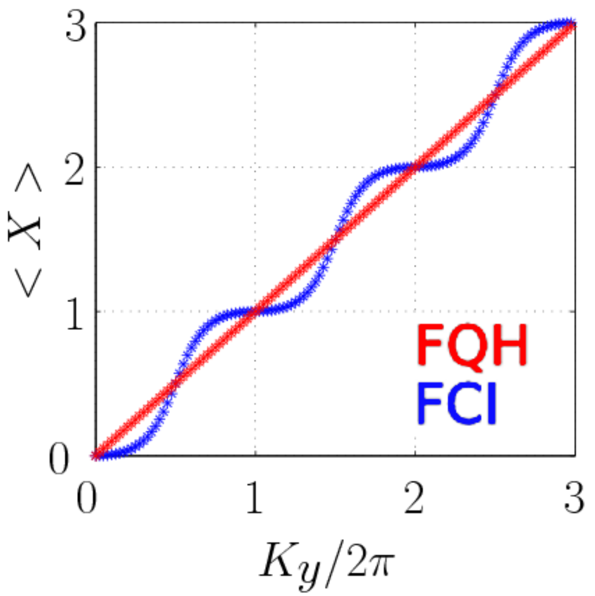
$$|W(x,k_y)\rangle = \sum_{k_x} f_{k_x}^{(x,k_y)} |k_x, k_y\rangle$$

- Qi's Proposition: using a mapping between the LLL eigenstates (QHE) and localized Wannier states (FCI), we can establish an exact mapping between their many-particle wavefunctions

Qi's Mapping



- Can introduce a canonical order of states with monotonously increasing position:



$$k_y = 2\pi n_y / L_y$$

$$\boxed{K_y} = k_y + 2\pi x = \boxed{2\pi j / L_y}$$

$$j = n_y + L_y x = 0, 1, ..., N_\phi - 1$$

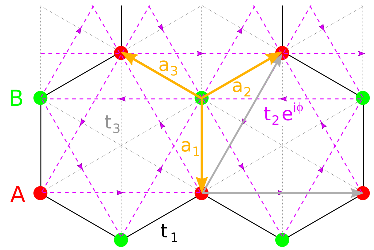
- Increase in position for $k_y \rightarrow k_y + 2\pi$ = Chern-number C, as

$$\frac{\partial}{\partial k_y} \langle \hat{X}^{cg} \rangle |_x = -\frac{1}{2\pi} \frac{\partial \theta(k_y)}{\partial k_y} = \int_0^{2\pi} \mathcal{B}(p_x, k_y) dp_x$$

More on Wannier states:

Y-L Wu, A. Bernevig, N. Regnault, PRB (2012)

Z. Liu & E. Bergholtz arxiv 2012 (see yesterday's talk)

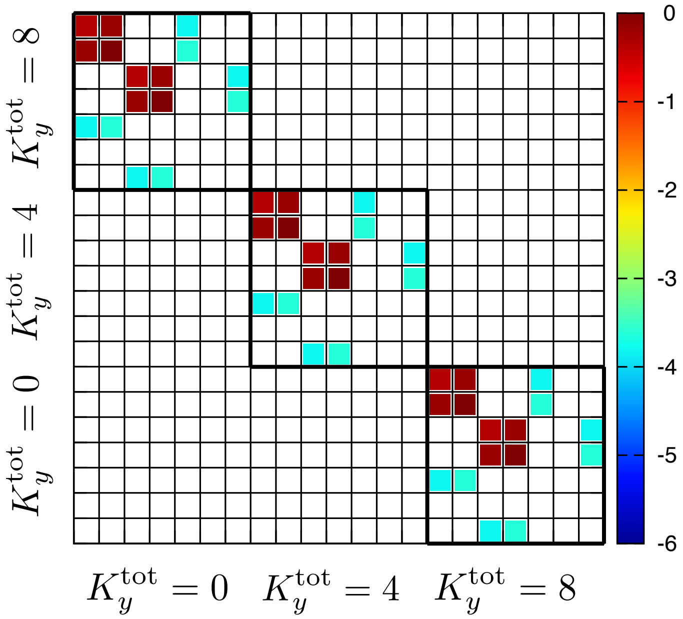


Case study: Bosons with contact interactions

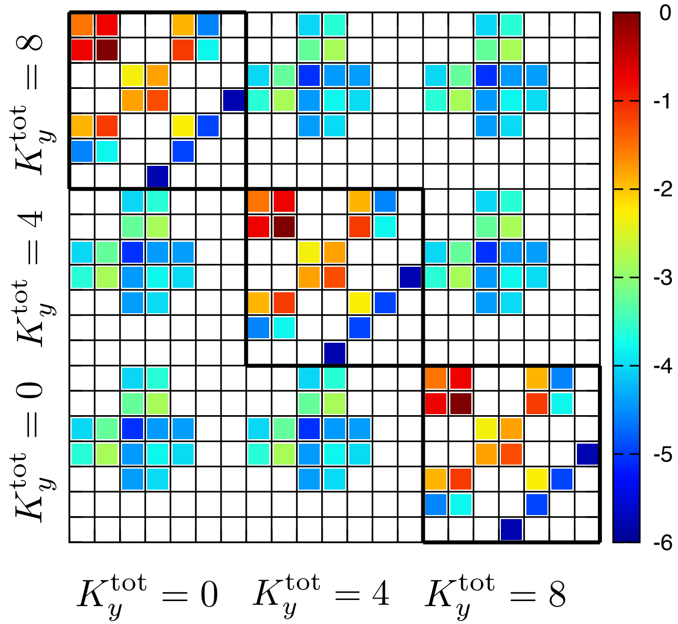
Th. Scaffidi & GM, arxiv:1207.3539 (to appear in Phys. Rev. Lett.)

- Magnitude of two-body matrix elements for delta interactions in the Haldane model

FQHE



FCI



- System shown: two-body interactions for $L_x \times L_y = 3 \times 4$
- Matrix elements differ in magnitude, but overall similarities are present
- Different block-structure due to non-conservation of linearized momentum K_y
- Lack of translational invariance of matrix elements in momentum space

Evaluating the accuracy of the Wannier states: Overlaps

- different Hamiltonians: FCI and FQHE ground states must differ
- can write both states in single Hilbert space with the same overall structure (indexed by K_y, enlarging the space for the torus)
- Can study adiabatic deformations between the two types of systems:

$$\mathcal{H}(x) = \frac{\Delta_{\mathrm{FCI}}}{\Delta_{\mathrm{FQHE}}}(1-x)\mathcal{H}^{\mathrm{FQHE}} + x\mathcal{H}^{\mathrm{FCI}}$$

Overlap

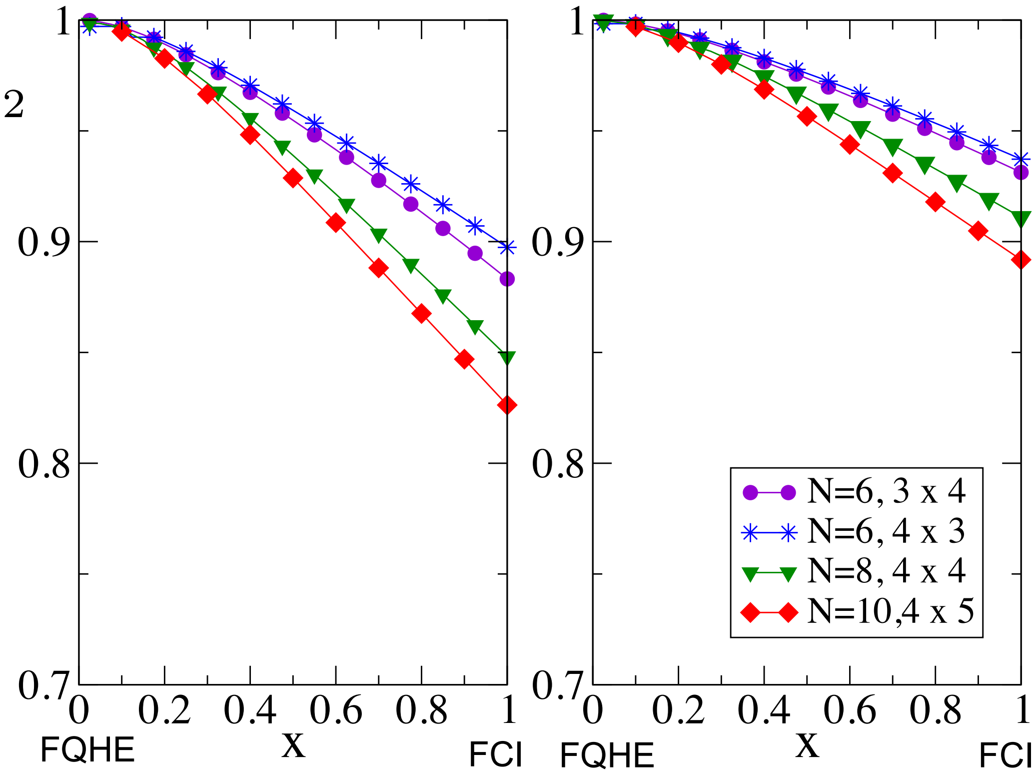
$|\langle\Psi(x)|\Phi\rangle|^2$

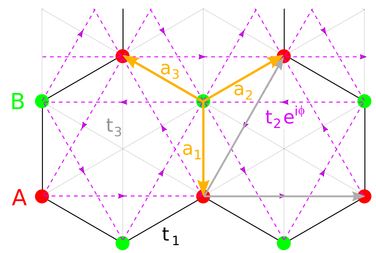
Weight in GS sector

$|\mathcal{P}_{k_{\mathrm{GS}}}|\Psi\rangle|^2$

Bosons at

$\nu = 1/2$



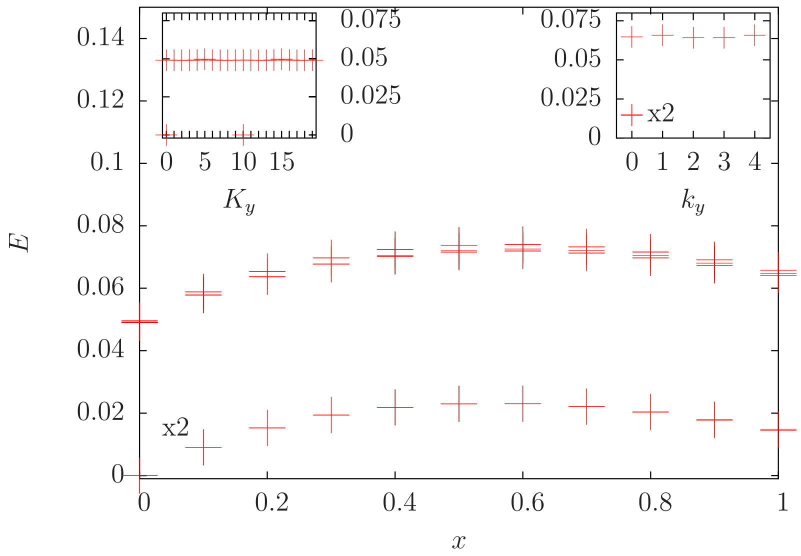


Th. Scaffidi & GM, arxiv:1207.3539 (to appear in Phys. Rev. Lett.)

Adiabatic continuation in the Wannier basis

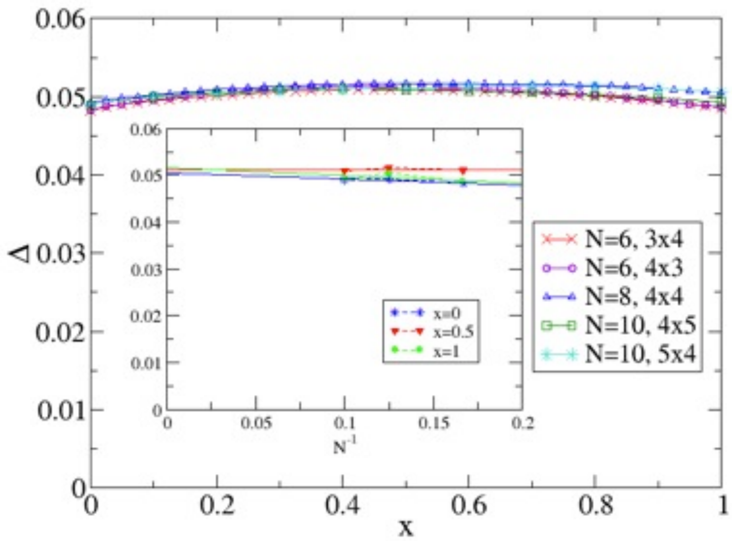
- Spectrum for N=10:

FQHE FCI



- Gap for different system sizes & aspect ratios:

FQHE FCI



- We confirm the Laughlin state is adiabatically connected to the groundstate of the half-filled topological flat band of the Haldane model

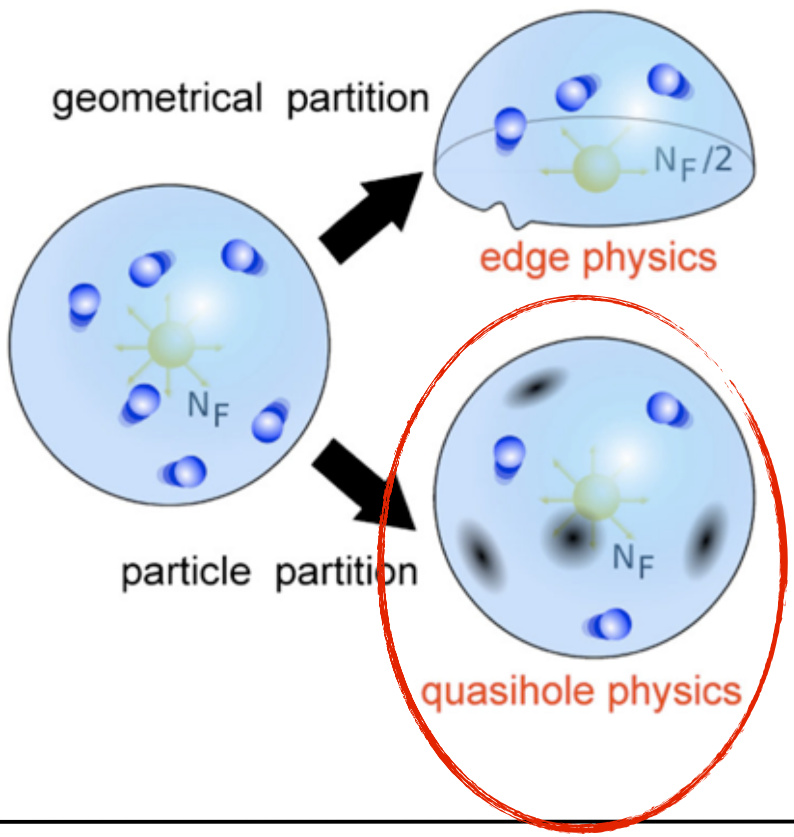
- Clean extrapolation to the thermodynamic limit - (unlike overlaps)

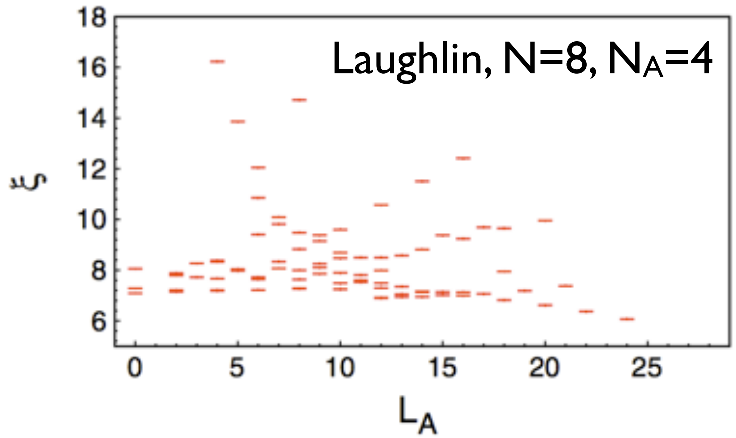
Th. Scaffidi & GM, arxiv:1207.3539 (to appear in Phys. Rev. Lett.)

Entanglement spectra and quasiparticle excitations

• Entanglement spectrum: arises from Schmidt decomposition of ground state into two groups A, B
=> Schmidt eigenvalues ξ plotted over quantum numbers for symmetries within each block

$$|\Psi\rangle = \sum_{\varpi}\sum_{i} e^{-\xi_{\varpi,i}/2}|\Psi^A_{\varpi,i}\rangle \otimes |\Psi^B_{\varpi,i}\rangle$$





Dominant (universal) eigenvalues of PES yield count of excited states - and their wavefunctions - from groundstate wavefunction only!

credit: Sterdyniak et al. PRL 2011

FCI: Adiabatic continuation of the entanglement spectrum

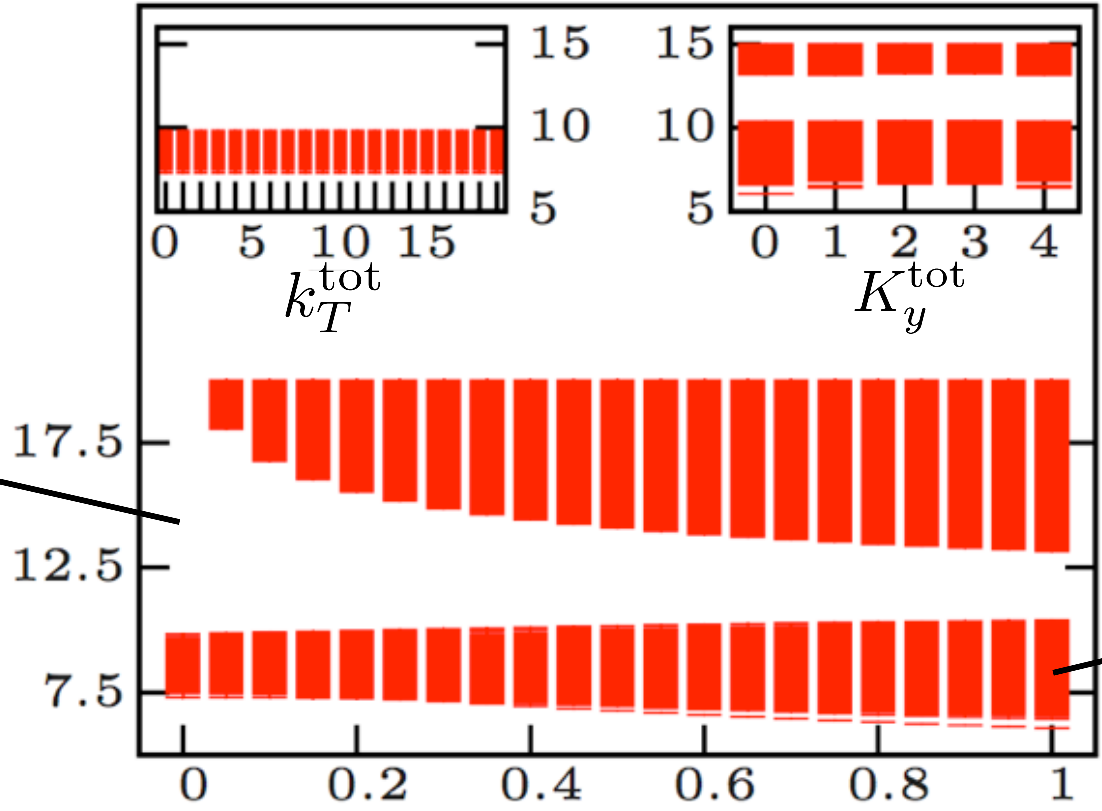
Total #eigenvalues below entanglement gap
= 4x(201 + 200 + 200 + 200 + 200)

Total #eigenvalues below entanglement gap
= 804 + 800 + 800 + 800 + 800

$N = 10$

$N_A = 5$

‘Infinite’ entanglement gap for pure Laughlin state



Same number of states for all x

Dictionary: FQHE FCI

$$|\Psi\rangle = \sum_{\varpi} \sum_{i} e^{-\xi_{\varpi,i}/2} |\Psi^A_{\varpi,i}\rangle \otimes |\Psi^B_{\varpi,i}\rangle$$

Conclusions on Mapping FCI wavefunctions

• Wavefunctions of FCI's in the Wannier basis are similar but not identical to FQH states in the Landau gauge

• We demonstrated the adiabatic continuity of the ground states at $\nu=1/2$ using Qi's mapping between Wannier basis and FQH eigenstates

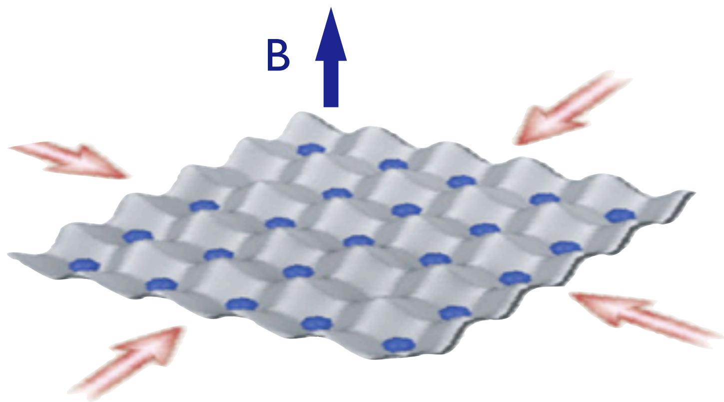
• FCI wavefunctions not very accurate for the Haldane model (higher overlaps in models with N>2 sublattices)

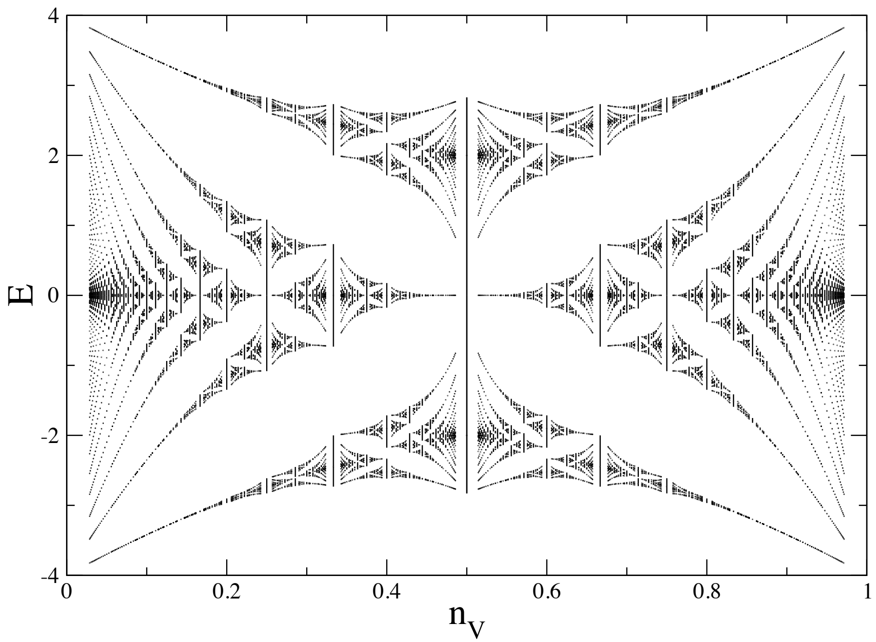
higher overlaps also by explicit gauge fixing, see:
Wu, Regnault, Bernevig, PRB (2012)

T. Scaffidi, GM, arxiv:1207.3539 (PRL, in press)

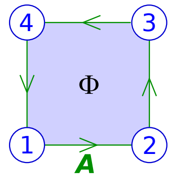
Idea: learn about PES using lattice FQHE states

- in lattices pierced by homogeneous magnetic flux, FQHE is well understood:
  - continuum limit reduces to usual FQHE
  - trial wavefunctions for continuum quantum Hall states accurately describe lattice, also.





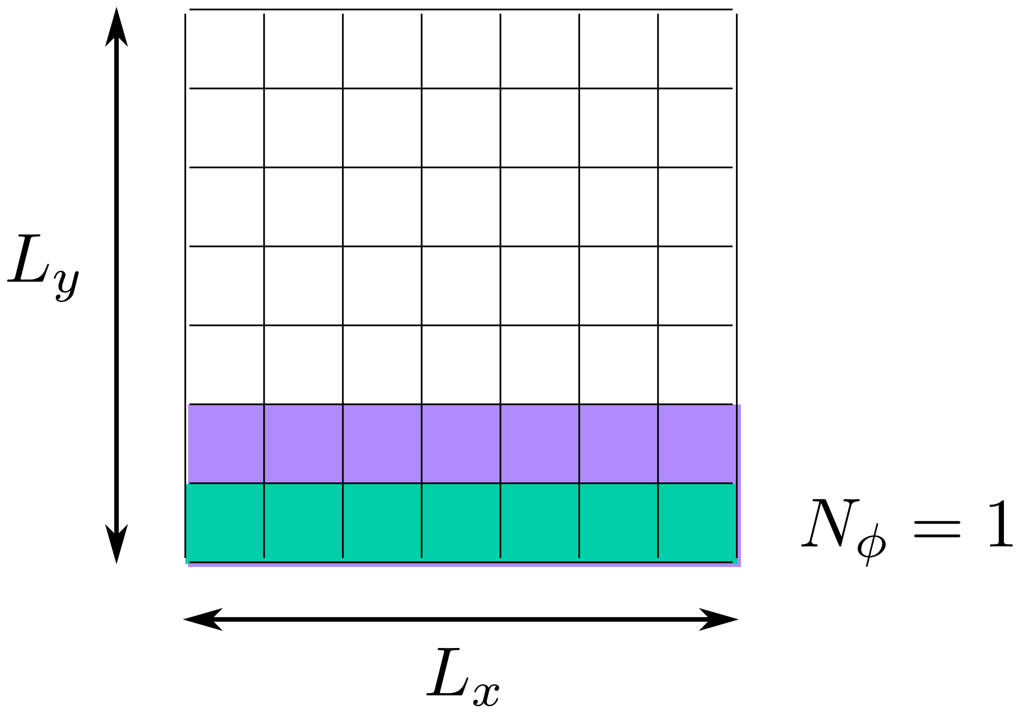
$$\sum_{\square} A_{\alpha\beta} = 2\pi n_{\mathrm{V}}$$

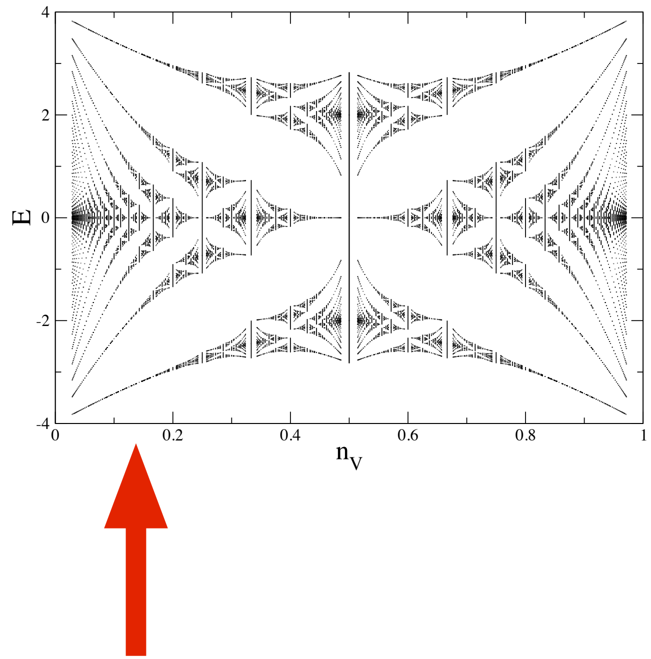


$$\mathcal{H} = -J \sum_{\langle\alpha,\beta\rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.\right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha(\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

Symmetries of lattices in magnetic flux

- finite simulation cell: L_x x L_y, periodic boundary conditions





- finite simulation cell: L_x x L_y
- translation group reduced to magnetic translations

➜ can trivially translate magnetic unit cell enclosing an integer number of flux quanta

- choose to implement momenta only along k_y (maybe reduced symmetry $k_y^{\max} < L_y$!)

Target phase 1: Laughlin state for bosons at ν=1/2

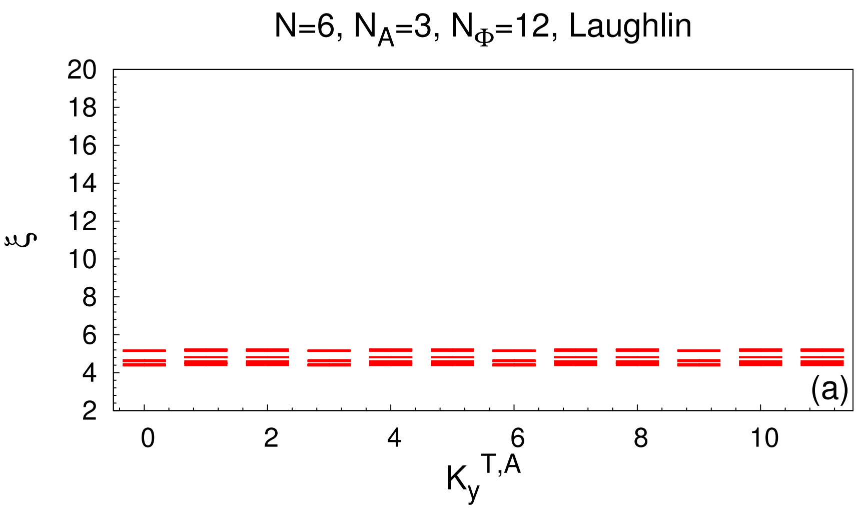
F.D.M. Haldane, PRB (1985)

- reference: Laughlin state in the continuum (torus geometry) $\nu = 1/2, N_\phi = 12$

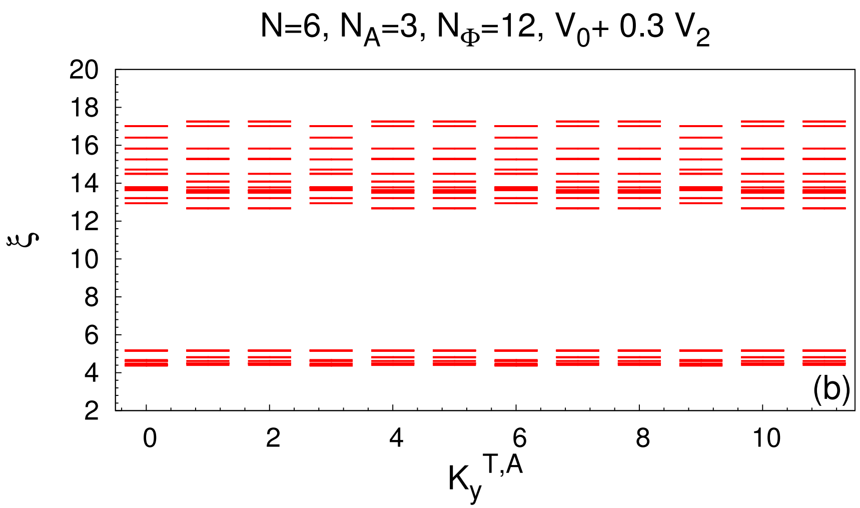
$$\Psi(z_1,\ldots,z_N) = \prod_{i<j} \vartheta\left(\frac{z_i - z_j}{L_y}\middle| i\frac{L_x}{L_y}\right)^2 F_{c.m.}(Z) e^{-\frac{1}{2}\sum_i x_i^2/l_B^2}$$

model wavefunction: $\hat{V} = \sum \delta(\vec{r}_i - \vec{r}_j) \equiv \hat{V}_0$

finite range interaction: $\hat{V} = \hat{V}_0 + 0.3\hat{V}_2$



infinite entanglement gap



finite entanglement gap

- count of entanglement eigenstates = count of QH wavefct. with $N_\phi = \nu^{-1} N_A + \underbrace{\nu^{-1} N_B}_{\Delta N_\phi}$

count per sector (N=6, N_A=3): (10,9,9,10,9,9,10,9,9,10,9,9)

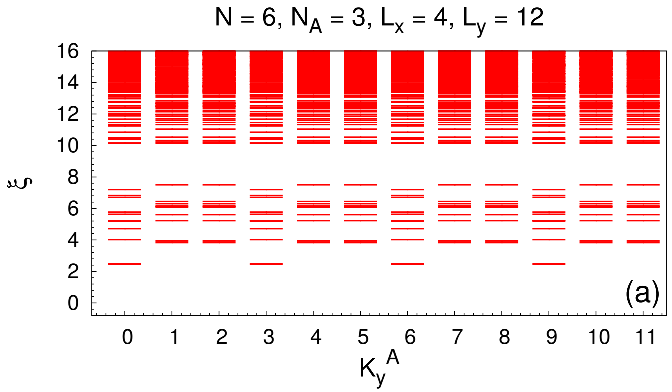
A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).

Target phase 1: Laughlin state (bosons)

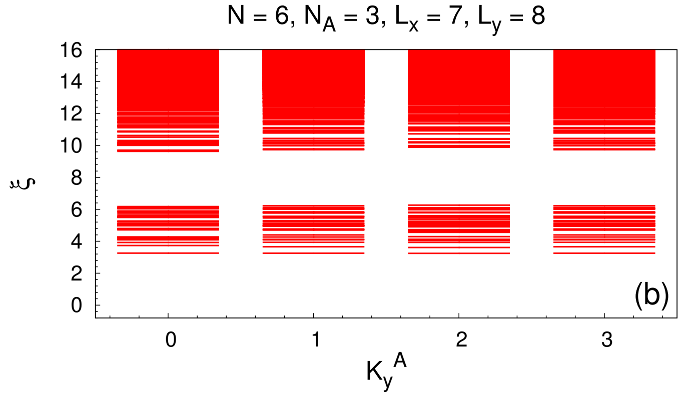
A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).

- contact interactions U on lattice, filling factor $\nu = N/N_\phi = 1/2$, vary lattice geometry

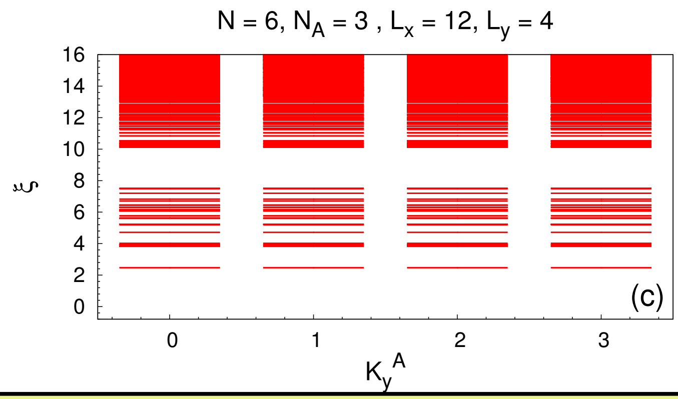
(10, 9, 9, 10, 9, 9, 10, 9, 9, 10, 9, 9)

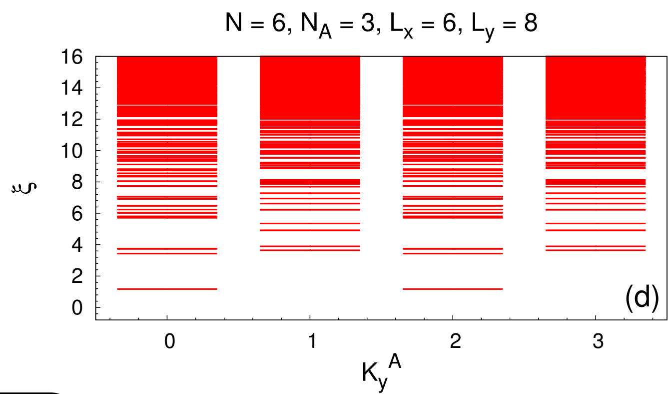


(28, 28, 28, 28)



(28, 28, 28, 28)





- PES satisfy QH counting with 'folding rule'

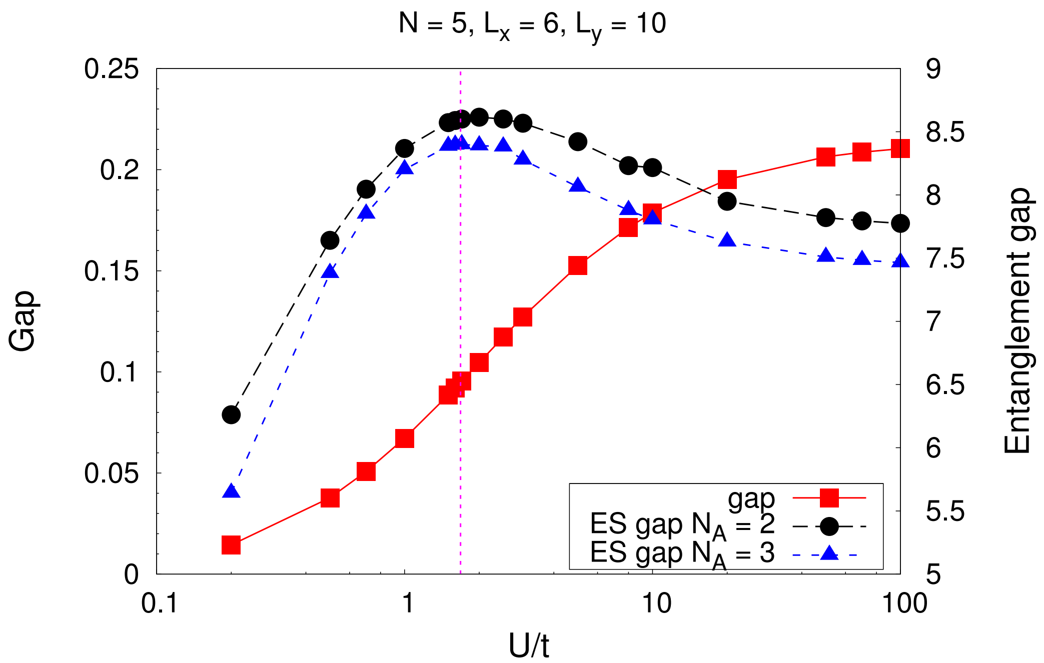
$$\mathcal{N}^L(k_y) = \sum_{\mathcal{K}_y^T} \delta_{k_y,(\mathcal{K}_y^T \bmod K_y^{\max})} \mathcal{N}^T(\mathcal{K}_y^T)$$

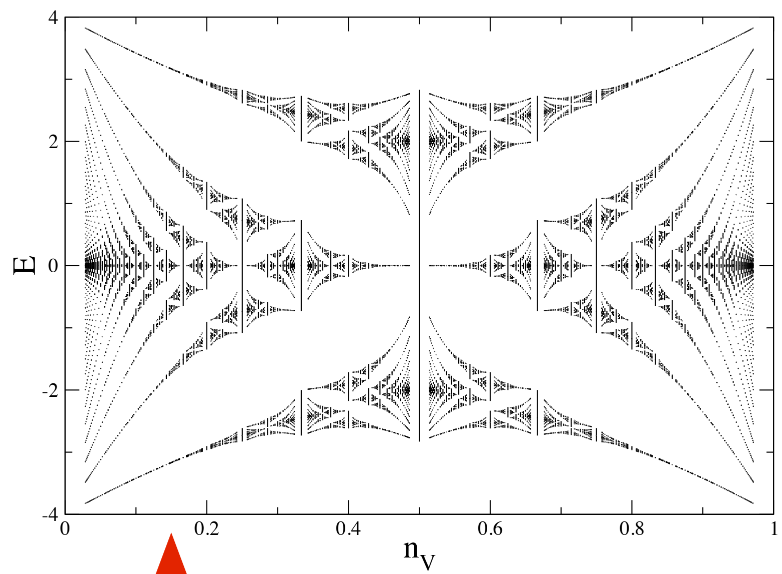
does not match
(discuss later)

Robustness of topological order

A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).

- energy gap vs entanglement gap





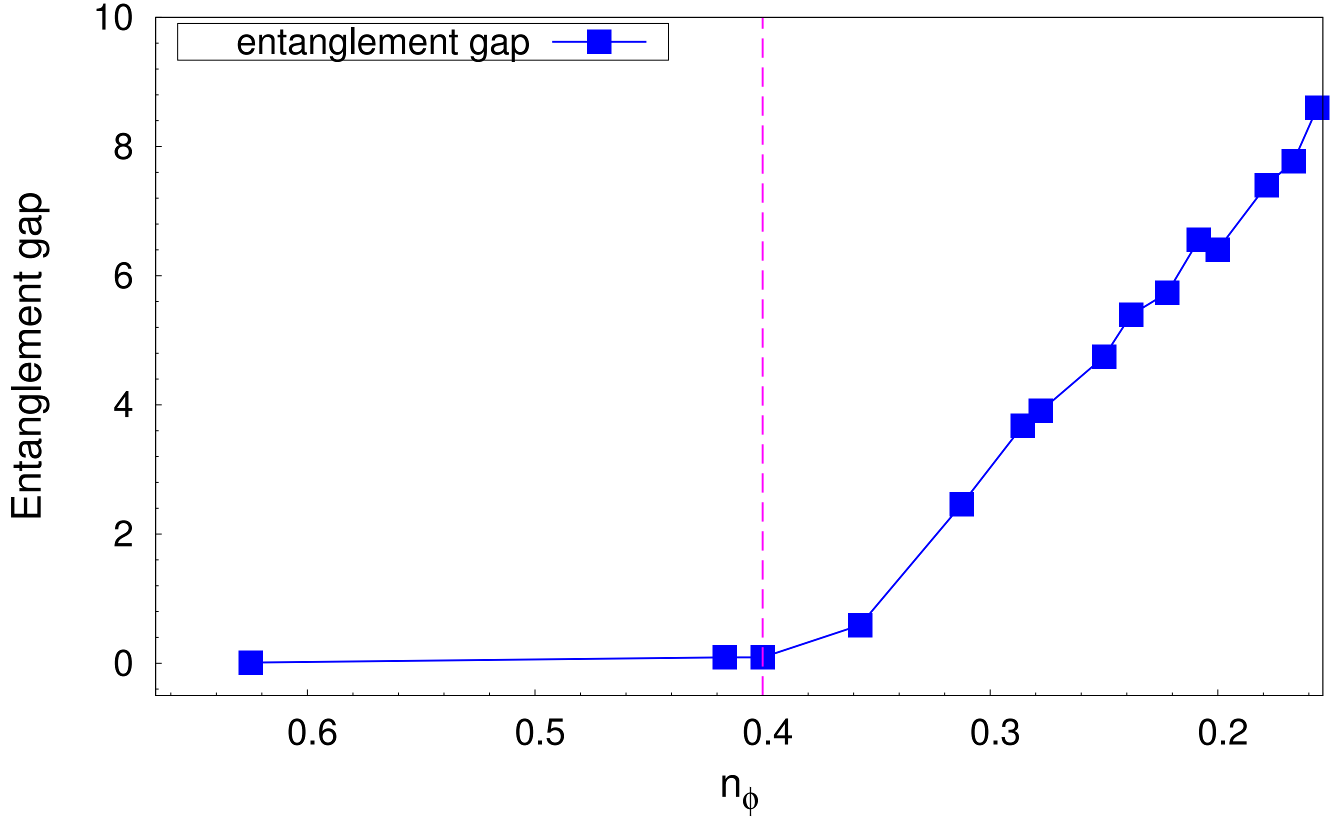
$n_\phi = 1/6$

- decrease of entanglement gap at large U from admixture of states in higher bands!

Stability of the Laughlin state

A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).

• previous study of Chern number C=1/2+1/2 of groundstate manifold as indicator of Laughlin state: difficult to calculate M. Hafezi, A. S. Sørensen, E. Demler, and M. D. Lukin, PRA (2007)

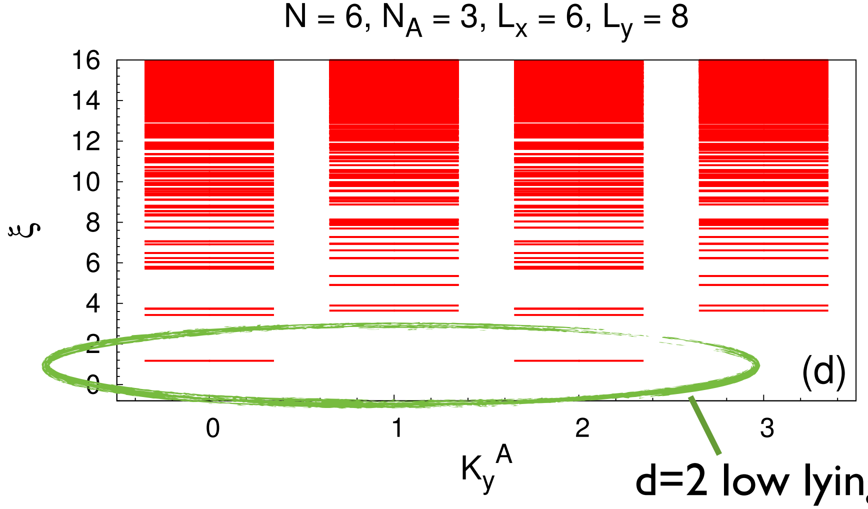


• N=5 fixed
• vary geometry

• entanglement gap: same answer, but quantitative + much easier to calculate

PES as a signature of competing states

A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).

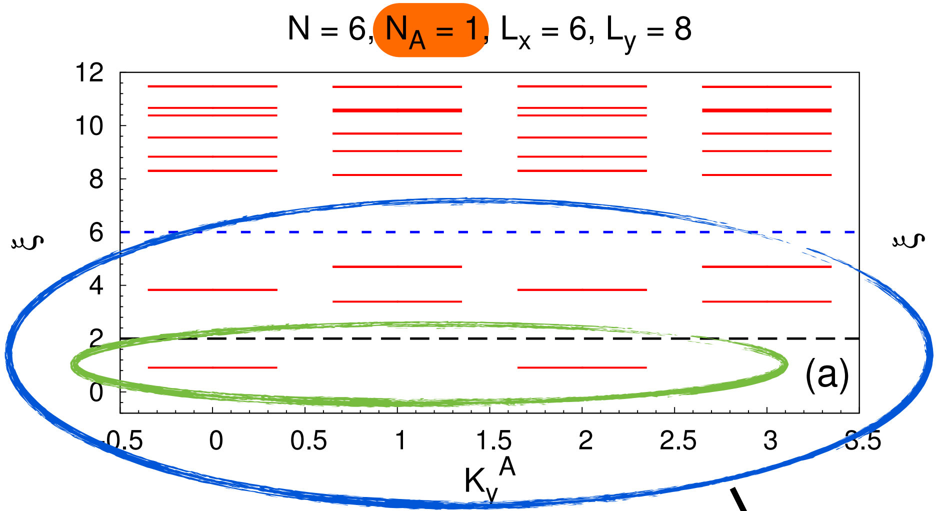


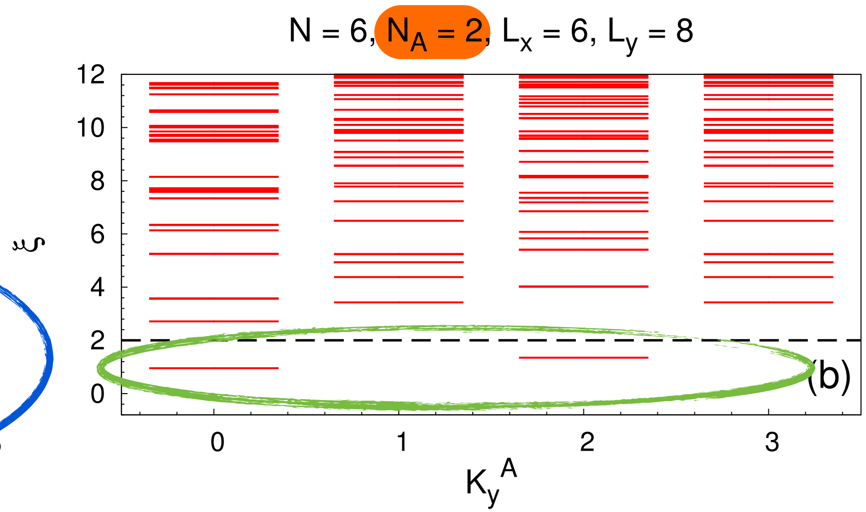
- need to understand 'outlier', still at

$$\nu = N/N_\phi = 1/2$$

study variation of PES under number N_A of particles in particle partition

d=2 low lying states

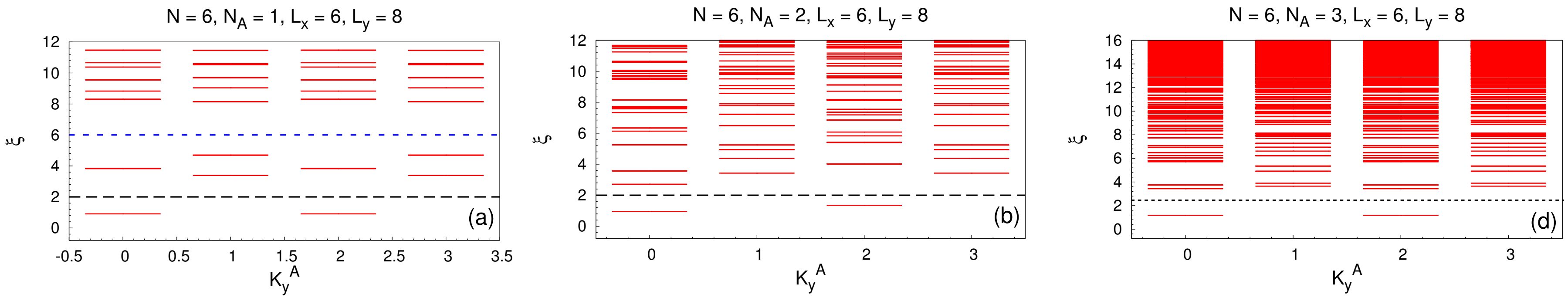




second gap encompassing d'=N_ϕ states: LLL basis!

Signatures of Condensed States in the PES

A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).



- invariance of low-lying level degeneracy in PES under variation of N_A: d=2
- this is a signature of a condensed state, e.g. pure condensate:

$$\Psi_N = \left(\sum_i \phi_i \hat{a}_i^\dagger\right)^N |\text{vac.}\rangle$$

➜ same wavefunction for all values of N = condensate wavefunction ϕ_i

➜ special case of $N_A = 1$: reduced density matrix = single particle DM $\rho_1 = \langle \hat{c}_i^\dagger \hat{c}_j \rangle$

- (near) degenerate low-lying entanglement eigenvalues for condensates

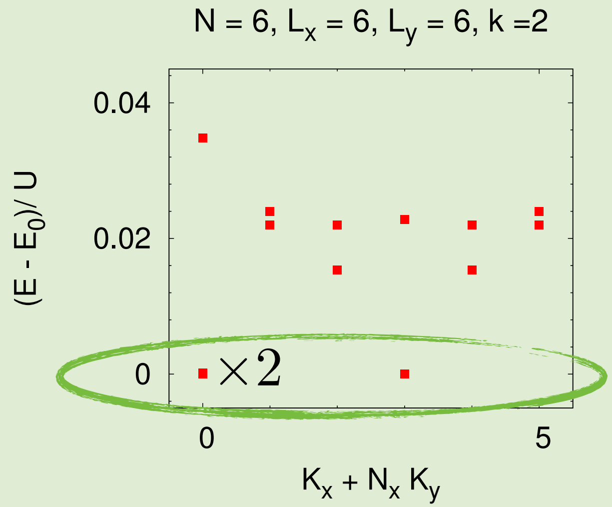
➜ signature of discrete symmetry breaking! See GM & N. R. Cooper, PRA (2010)

- here: translational symmetry breaking = density wave state

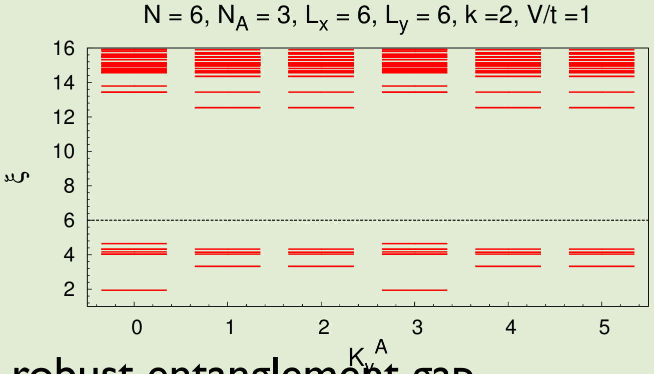
Target phase 2: Moore-Read state for bosons at $\nu=1$

A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).

- easy to stabilize with 3-body interactions



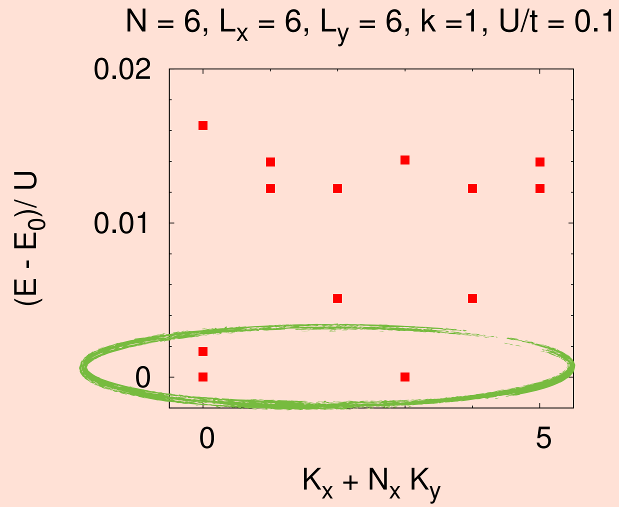
Energy



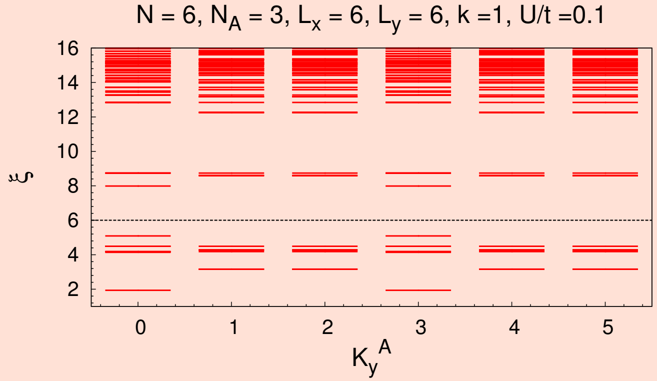
PES

- robust entanglement gap
- match count of MR: $(7,6,6,7,6,6)$

- 2-body contact interactions:



Energy



PES

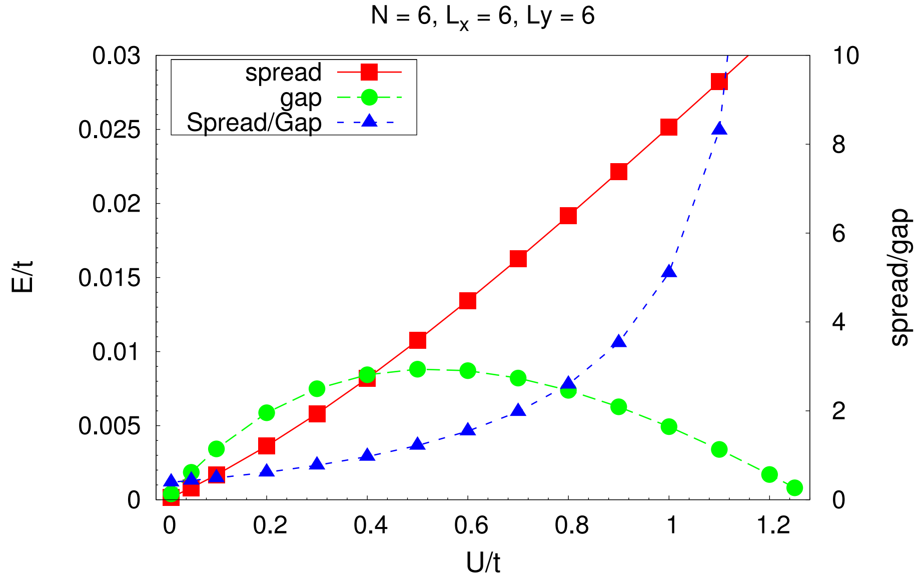
- count of MR qh's reproduced

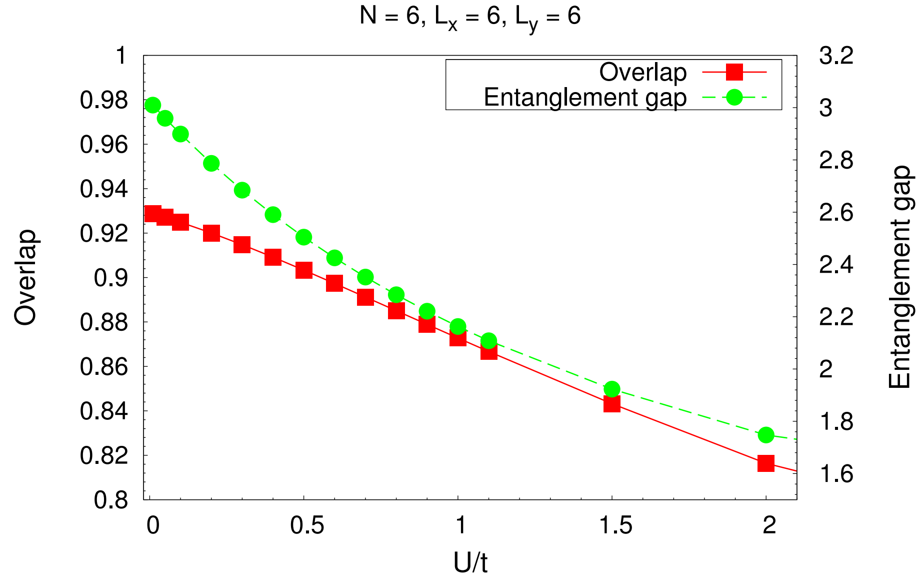
$(7,6,6,7,6,6)$

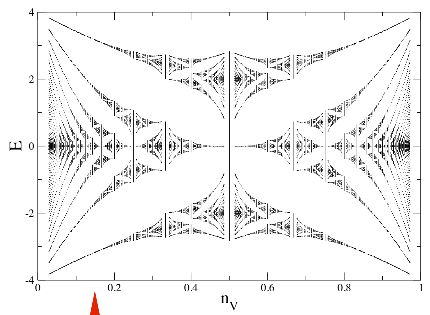
Extent of Moore-Read phase for 2-body interactions

A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).

- study behaviour when two-body interaction strength U is tuned







$n_\phi = 1/6$

- entanglement gap provides proxy for overlap
- robust entanglement gap gives indication that Moore-Read phase can be stabilized by two-body interactions
- finite size effects of the energy spectrum are strong: Energy spectrum not nearly as clear as PES
- collapse of energy gap at Uc ~ 1.25t ➜ LL mixing

Conclusions

- Flux lattices of cold atomic gases provide multiple opportunities to realise flat Chern bands in the near future

- Adiabatic continuation yields a robust tool for identifying correlated phases in Chern bands by association with FQHE physics

- PES is a reliable tool for identifying correlated phases:
  - topological states: universal quasihole count
  - entanglement gap clear signature of the stability of topological properties
  - condensed phases: conservation of low-lying structure for different cuts

T. Scaffidi, GM, arxiv:1207.3539 (PRL, in press)

A. Sterdyniak, N. Regnault & GM, Phys. Rev. B **86**, 165314 (2012).