Fractional Chern Insulators in Harper-Hofstadter Bands with Higher Chern Number

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TCM

GM & N.R. Cooper, Phys. Rev. Lett. (2015), in press; arXiv:1504.06623
and T. Jackson, GM, R. Roy, Nature Communications (2015), in press; arxiv:1408.0843

Topological phases in Condensed Matter and Cold Atoms systems
Institut d'Etudes Scientifiques de Cargèse

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Overview

- Background: Topology and interactions in tight-binding models

Part I

- FCI beyond Chern number one: composite fermion theory

- Numerical Results for the Hofstadter model

➡ confirmation of series of states: $\nu = \dfrac{r}{r|Ck|+1}$

Part II

- The role of band geometry for stability of FCI: Single Mode Approximation to quantum Hall liquids / fractional Chern insulators

Quantum Hall Effect: Blueprint of Topological Order

- a macroscopic quantum phenomenon: magnetoresistance in 2D electron gases

Where?

- in semiconductor hetero-structures with clean two-dimensional electron gases
- at low temperatures (~0.1K) and in strong magnetic fields

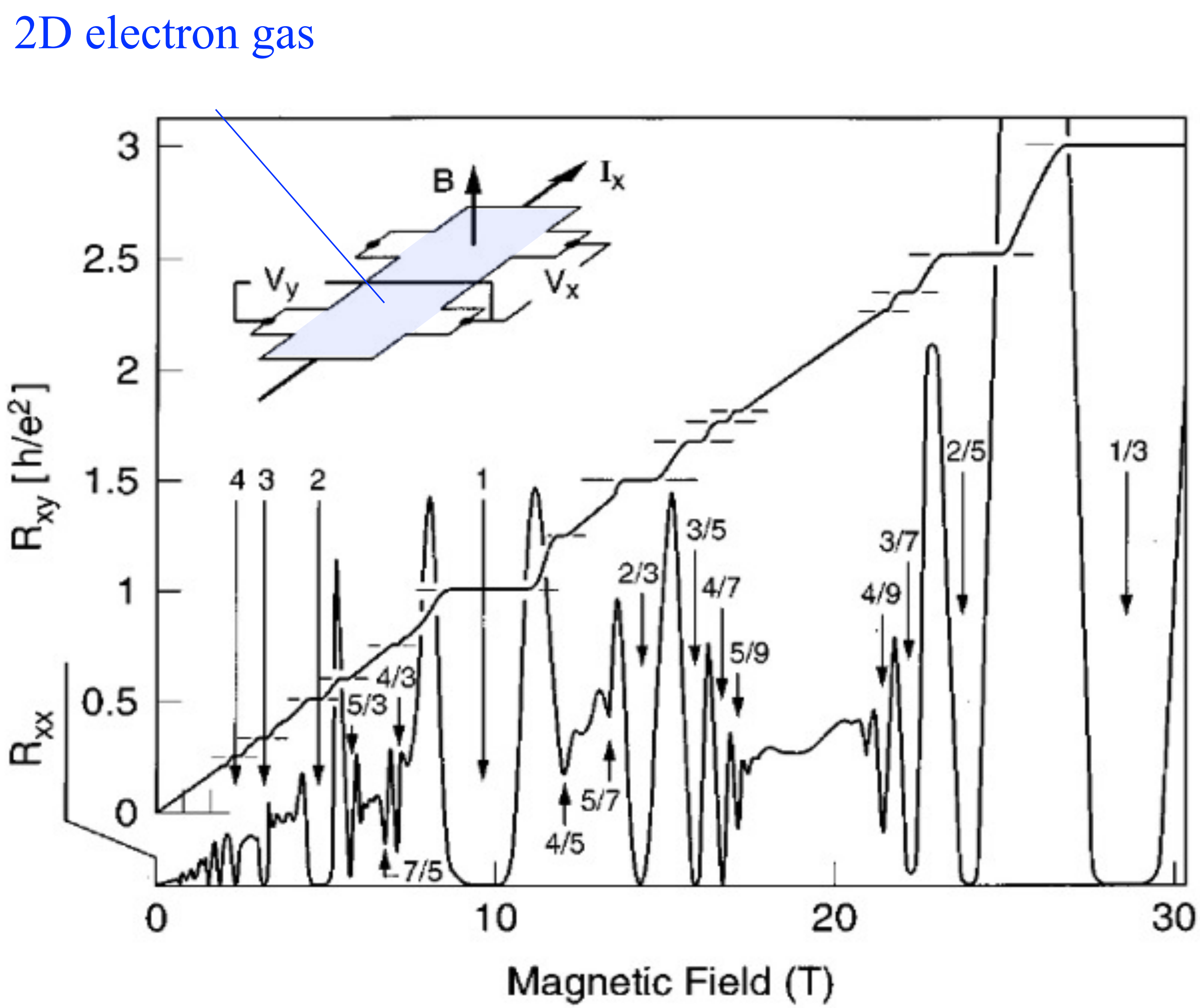
$$k_B T \ll \hbar\omega_c = \hbar e B/m_e$$

What?

- plateaus in Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

- simultaneously: (near) zero longitudinal resistance



- many different phases: different order at each Hall plateau
- no local order parameter for any of these phases!

Integer Quantum Hall Effect

$$\mathcal{H} = \frac{(\vec{p} + e\vec{A})^2}{2m}$$

$$\vec{A} = Bx\vec{e}_y$$

- single-particle eigenstates (=bands) in a homogeneous magnetic field: degenerate Landau levels with spacing $\hbar\omega_c$

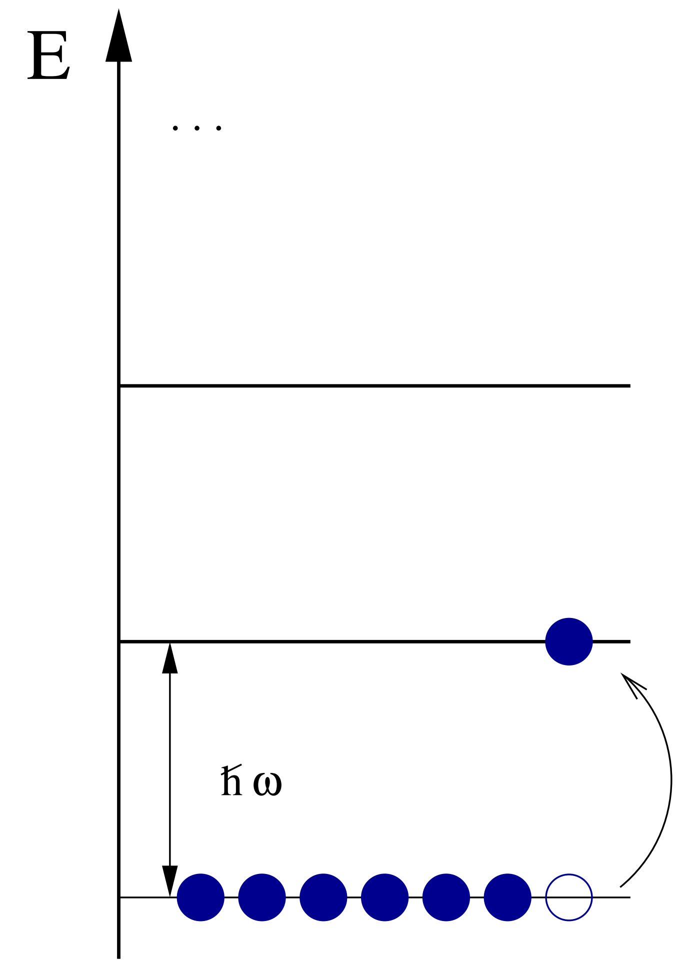
$\omega_c = eB/m_e$ cyclotron frequency

- degeneracy per surface area: $d_{LL} = eB/h$

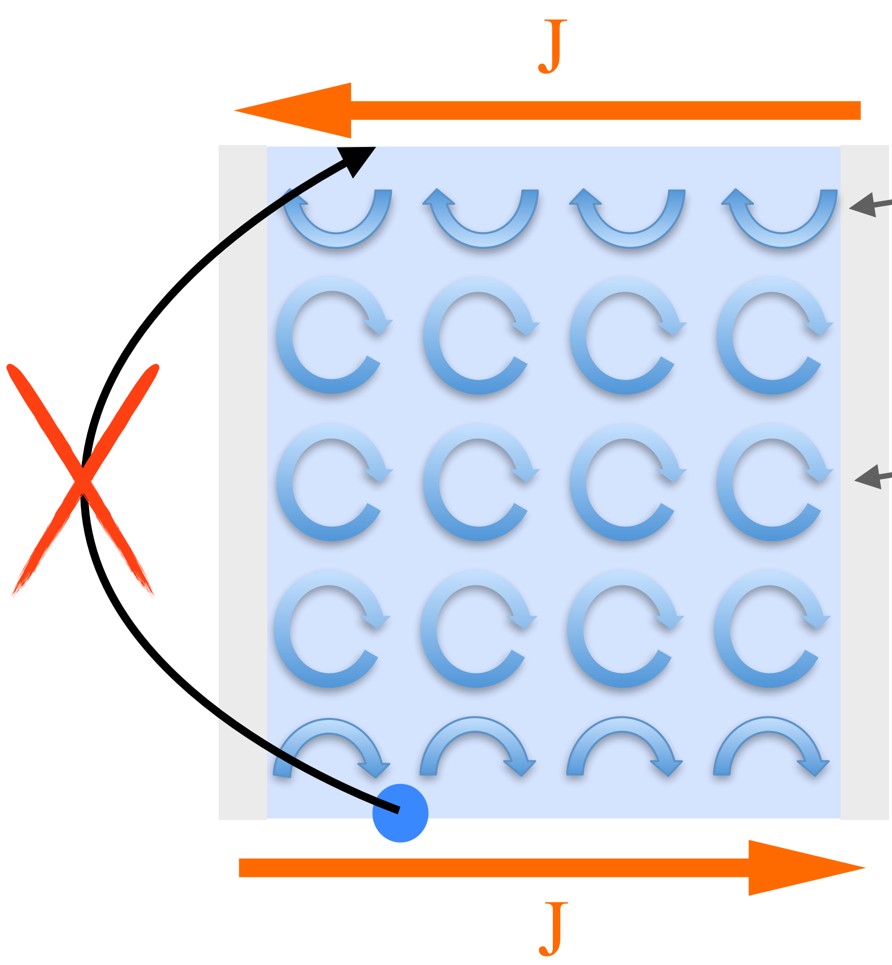
- fill a number of bands = integer filling factor $\nu = n/d_{LL}$

> ⇒ large gap Δ for single particle excitations:
>
> naively, we should have a band insulator

- There must be something special about Landau-levels!



Semiclassical picture: skipping orbits



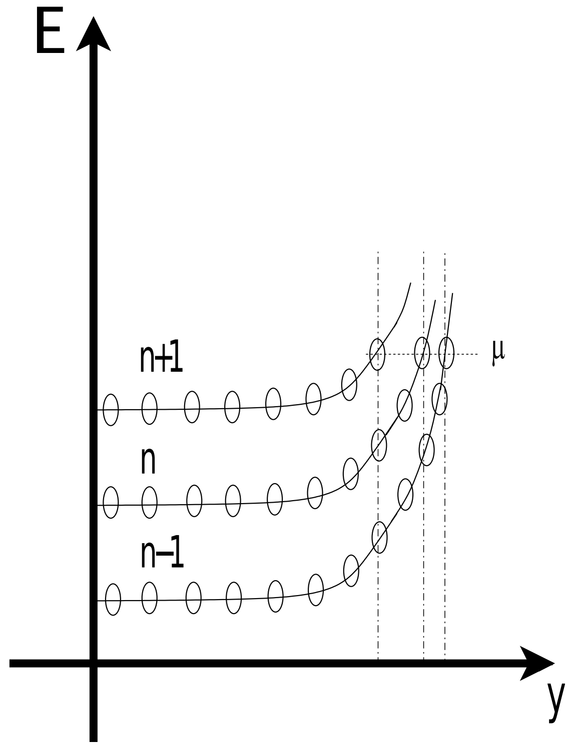
at edge of sample, 'skipping orbits' contribute a uni-directional current

cyclotron motion produces no net current in bulk of sample

picture for quantum transport:
absence of backscattering
⇒ dissipationless current

⇒ no voltage drop along lead!

several LLs filled:

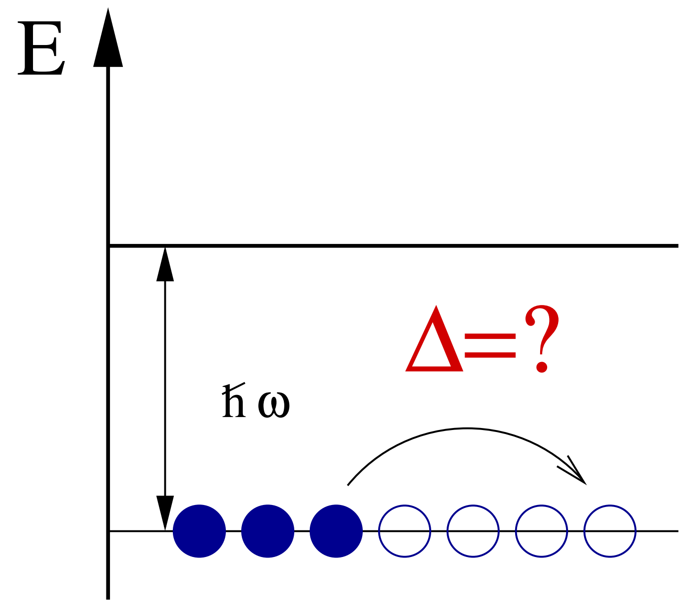


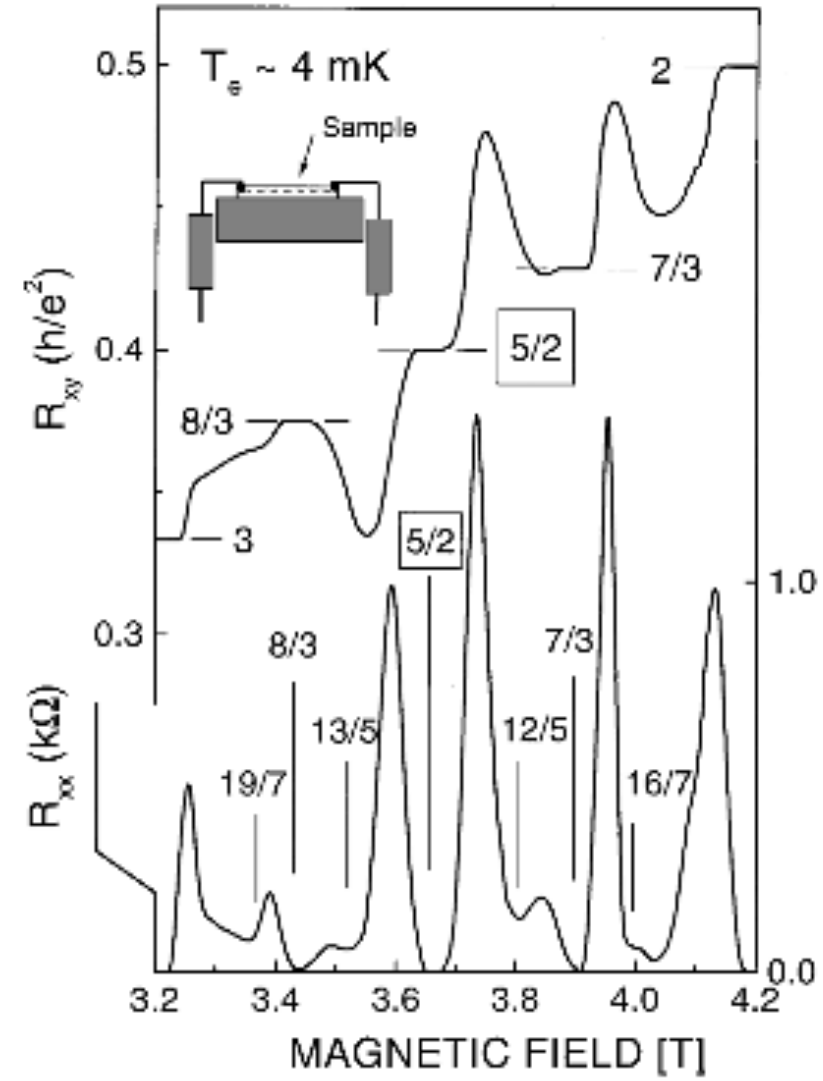
low-energy or 'gapless' excitations present near boundary

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

Fractional Quantum Hall Effect (FQHE)

- plateaus seen also for non-integer ν
- not filled bands - but similar phenomenology as integer filling:



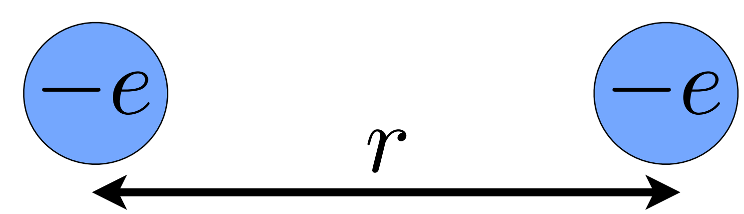


- nature of interactions determines how the system behaves:

$$\mathcal{H} = \sum_i \frac{1}{2m}(\vec{p}_i - e\vec{A}_i)^2 + \sum_{i<j} V(|\vec{r}_i - \vec{r}_j|)$$

within Landau-level: Kinetic Energy=constant

interactions determine quantum state



$$V(\mathbf{r}) = \frac{e^2}{4\pi\epsilon_0 |r|}$$

⇒ FQHE is an inherently many-body phenomenon

- each Hall plateau represents a kind of topological order

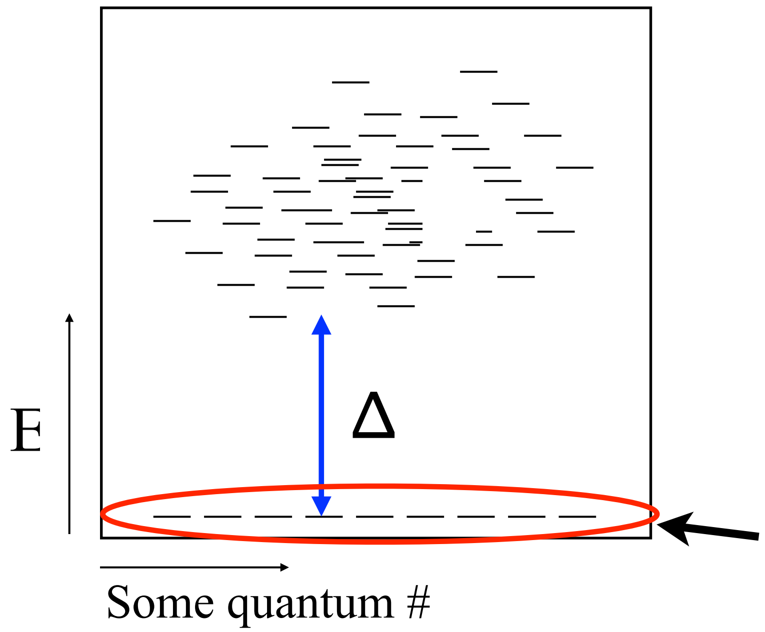
Why is the fractional quantum Hall effect important?

source of very unusual physics, for example:

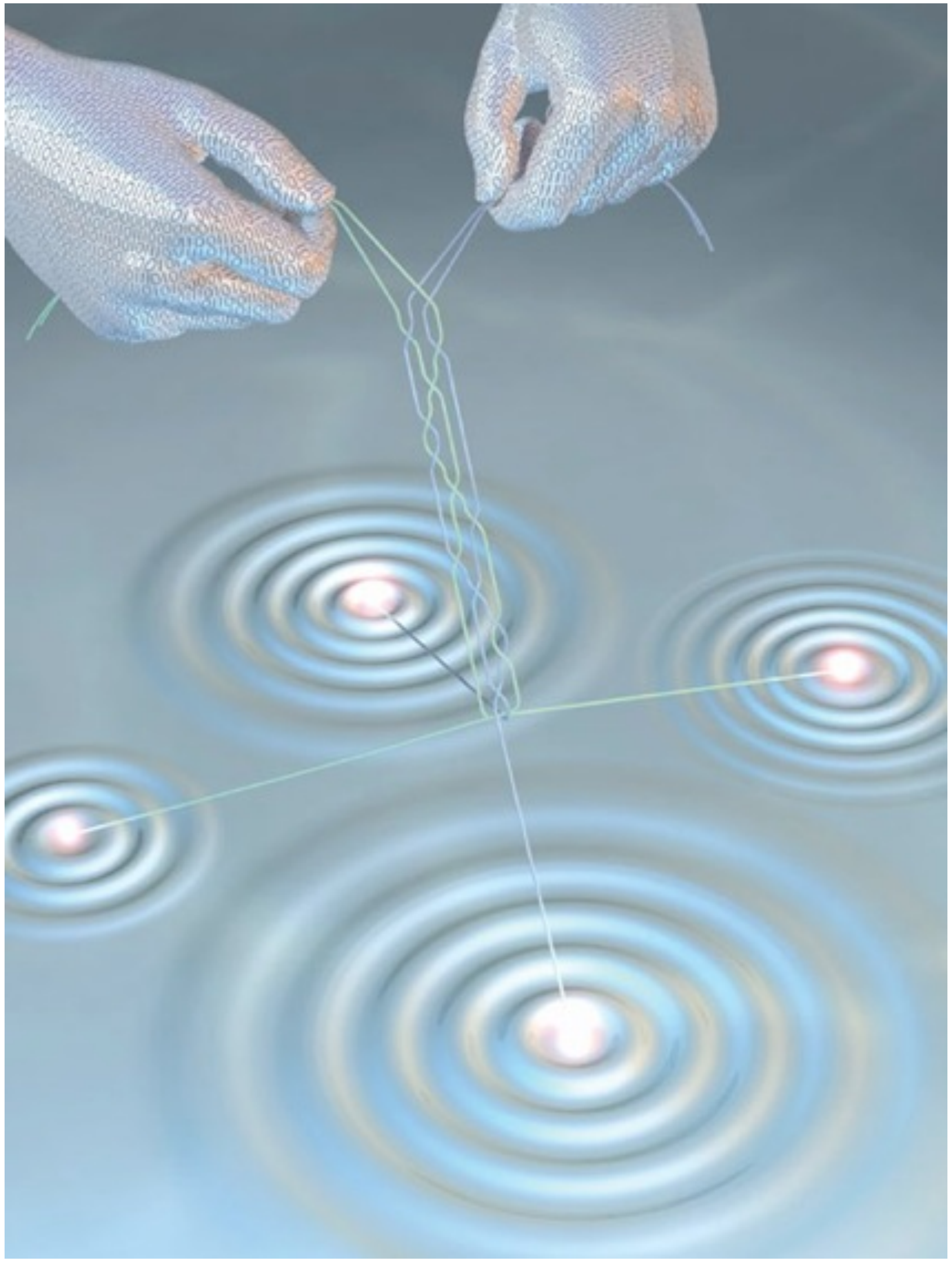
- quasi-particles with fractional electronic charge

$$e.g.,\ q = e/3$$

- manipulations of quasiparticles could provide the basis for a quantum computer that is protected from errors!



topological degeneracy of low-lying states



- quantum operations by braiding quasiparticles

Quantum Hall effect without magnetic fields

The fractional quantum Hall effect is observed under extreme conditions

- strong magnetic fields of several Tesla
- very low temperatures
- clean / high mobility semiconductor samples

Opportunities for creating novel types of quantum Hall systems

1. Cold Atomic Gases
   - both bosons and fermions
   - highly tuneable: density, interactions, tunnelling strengths, (effective) mass, ...
   - different types of experimental probes: local density, velocity distribution, correlations

2. Novel classes of materials
   - strained graphene / 2D crystals
   - materials with strong spin orbit coupling, such as topological insulators

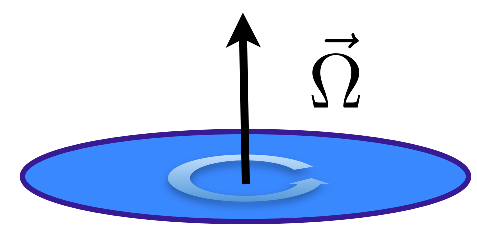
3. Photons

…

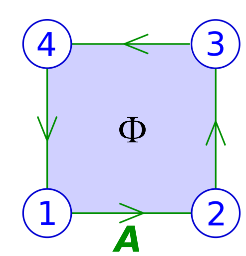
Strategies for simulating magnetic fields

- Simulate a physical effect that a magnetic field B exerts particle of charge q

Signature		Simulated by
Lorentz Force	$F_L = q\,\vec{v} \times \vec{B}$	Coriolis Force in Rotating System



Aharonov-Bohm Effect	$\Psi \propto \exp\left\{ i\frac{q}{\hbar}\int \vec{A}\cdot d\vec{\ell} \right\}$	Complex Hopping Amplitudes A in Optical Lattices



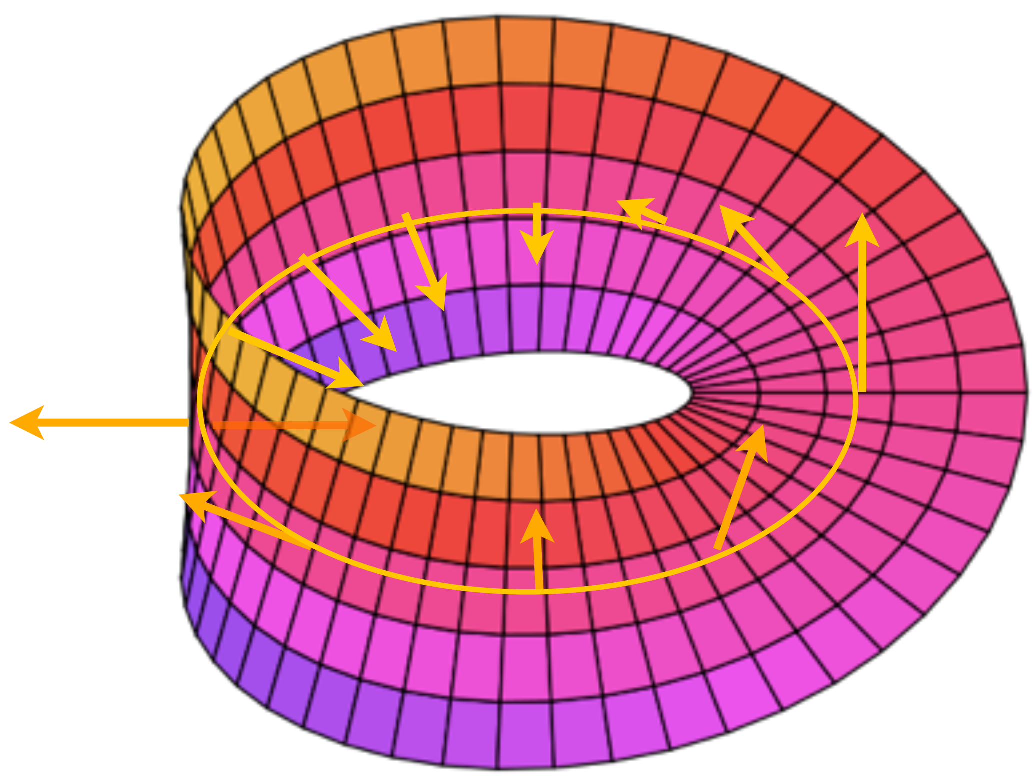
$$\sum_{\square} A_{\alpha\beta} = 2\pi n_\phi$$

Berry Curvature of Landau levels	Same physics seen in reciprocal space…

Landau-levels as a topological band-structure

- Can we see in which way Landau-levels are special, just by looking at the wavefunctions?

Start with an analogy:



Recipe for calculating the twist in this Möbius band:

- choose a closed path around the surface
- construct normal vector to the surface at points along the curve
- add up the twist angle while moving along this contour

Calculating the Berry phase

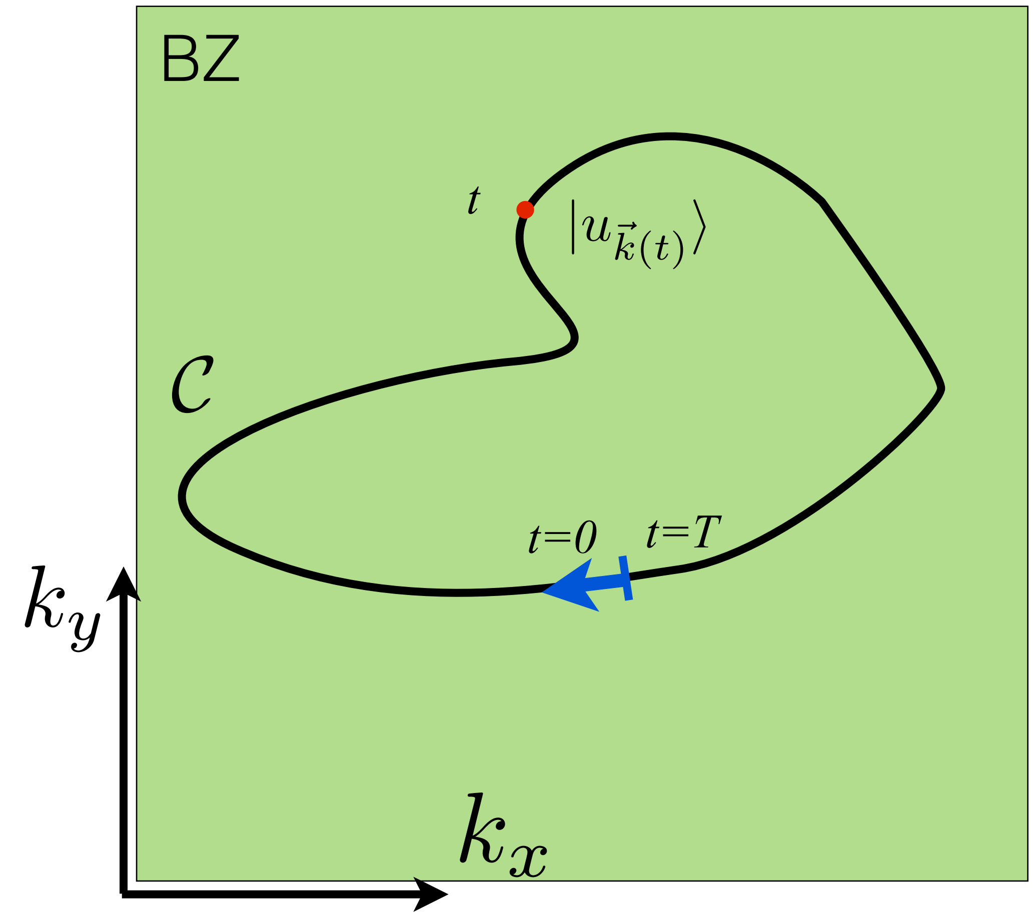
Michael Berry (1984)

Calculate how wavefunction evolves while moving adiabatically through curve $C : \mathbf{k}(t),\ t=0\ldots T$

Local basis $\quad \tilde{\mathcal{H}}|u_{\vec{k}}\rangle = \epsilon_{\vec{k}}|u_{\vec{k}}\rangle$

Phase evolution has two components:

$$|U(t)\rangle = \underbrace{\exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon_{\vec{k}(t')}dt'\right\}}_{\text{dynamical time evolution}} \underbrace{\exp\{i\gamma(t)\}}_{\text{'twist'}} |u_{\vec{k}(t)}\rangle$$



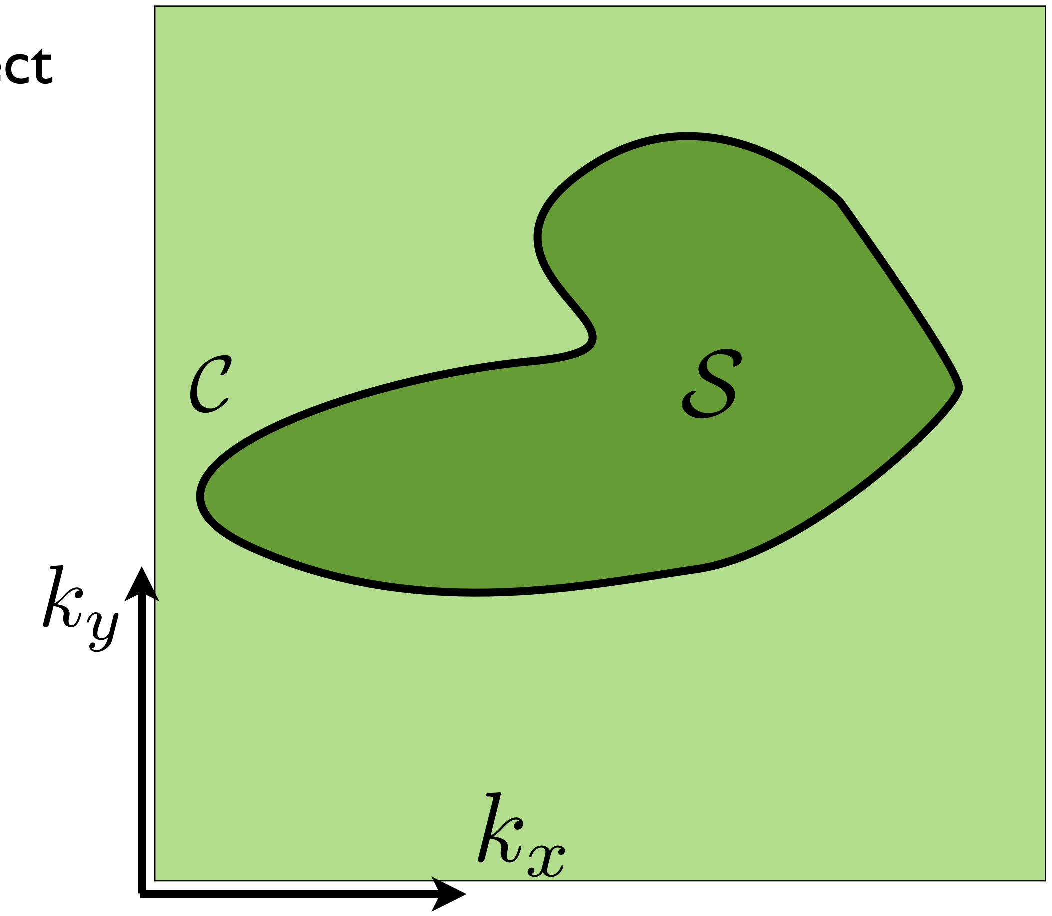
Berry curvature and Chern number

Geometrical phase analogous to Aharonov-Bohm effect

$$\gamma(\mathcal{C}) = i\int_{\mathcal{C}} \langle u_{\vec{k}}|\frac{d}{d\vec{k}}|u_{\vec{k}}\rangle d\vec{k} \equiv \int_{\mathcal{C}} \vec{\mathcal{A}}(\vec{k})d\vec{k}$$

Effective 'vector potential' called *Berry connection*

$$\vec{\mathcal{A}}(\vec{k}) = i\int_{\mathrm{UC}} u_{\vec{k}}(\vec{r})^* \vec{\nabla}_k u_{\vec{k}}(\vec{r})\, d^2 r$$



Using Stokes' theorem:

$$\gamma(\mathcal{C}) = \int_{\mathcal{C}} \vec{\mathcal{A}}(\vec{k})d\vec{k} = \int_{\mathcal{S}} \vec{\nabla}_k \times \vec{\mathcal{A}}(\vec{k})d\vec{\sigma}$$

$\mathcal{C} = \partial\mathcal{S}$

Berry curvature: $\vec{\mathcal{B}} = \vec{\nabla}_k \times \vec{\mathcal{A}}(\vec{k})$ ⇦ is a property of the band eigenfunctions, only!

Chern number: $C = \frac{1}{2\pi}\int_{BZ} d^2\mathbf{k}\, \mathcal{B}(\mathbf{k})$ ⇦ takes only integer values!

- Chern number provides classification of all possible single-particle bands (class A)

Quantum Hall Effect in Periodic Potentials

- quantized Hall response in filled bands:

$$\sigma_{xy} = \frac{e^2}{h} \sum_{\text{filled bands}} C_n$$

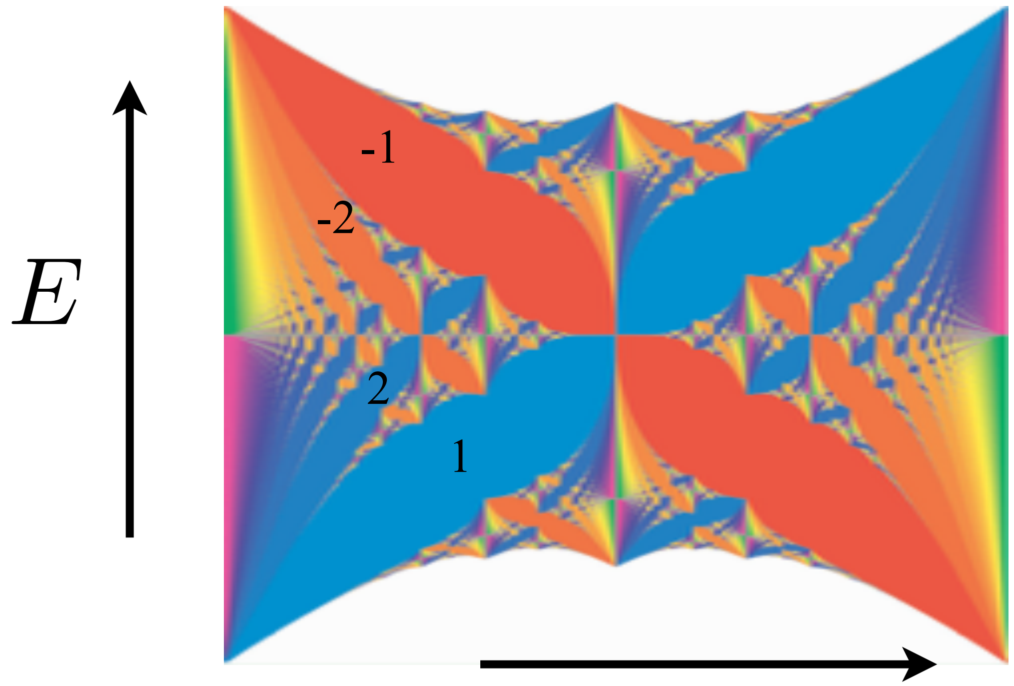
Thouless, Kohmoto, Nightingale, de Nijs 1982

- Chern-number for periodic systems

$$C = \frac{1}{2\pi} \int_{BZ} d^2\mathbf{k}\, \mathcal{B}(\mathbf{k})$$

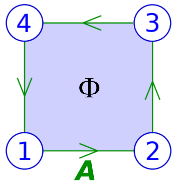
- Harper-Azbel-Hofstadter model:
tight-binding model for electrons in magnetic field ⇒ bands with finite Chern number

$$\mathcal{H} = -J \sum_{\langle\alpha,\beta\rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.\right]$$



n_ϕ

figure: Avron *et al. (2003)*



$$\sum_{\square} A_{\alpha\beta} = 2\pi n_\phi$$

- the Hofstadter spectrum provides bands of all Chern numbers
- filled bands in this spectrum yield a quantized Hall response

Fractional Quantum Hall Effect in Periodic Potentials

- quantized Hall response in *partially* filled bands?
- THEORY: Kol & Read (1993)

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}_\alpha^\dagger \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \sum V_{ij} \hat{n}_i \hat{n}_j$$

PHYSICAL REVIEW B VOLUME 48, NUMBER 12 15 SEPTEMBER 1993-II

Fractional quantum Hall effect in a periodic potential

A. Kol and N. Read*

Departments of Physics and Applied Physics, P. O. Box 2157, Yale University, New Haven, Connecticut 06520

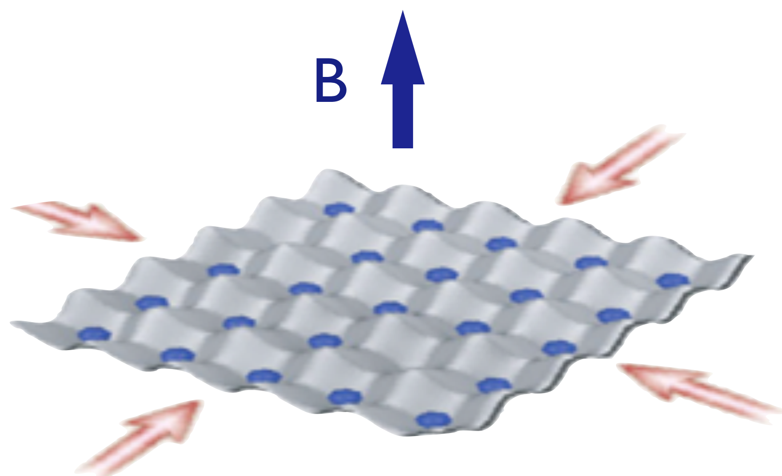
(Received 28 May 1993)

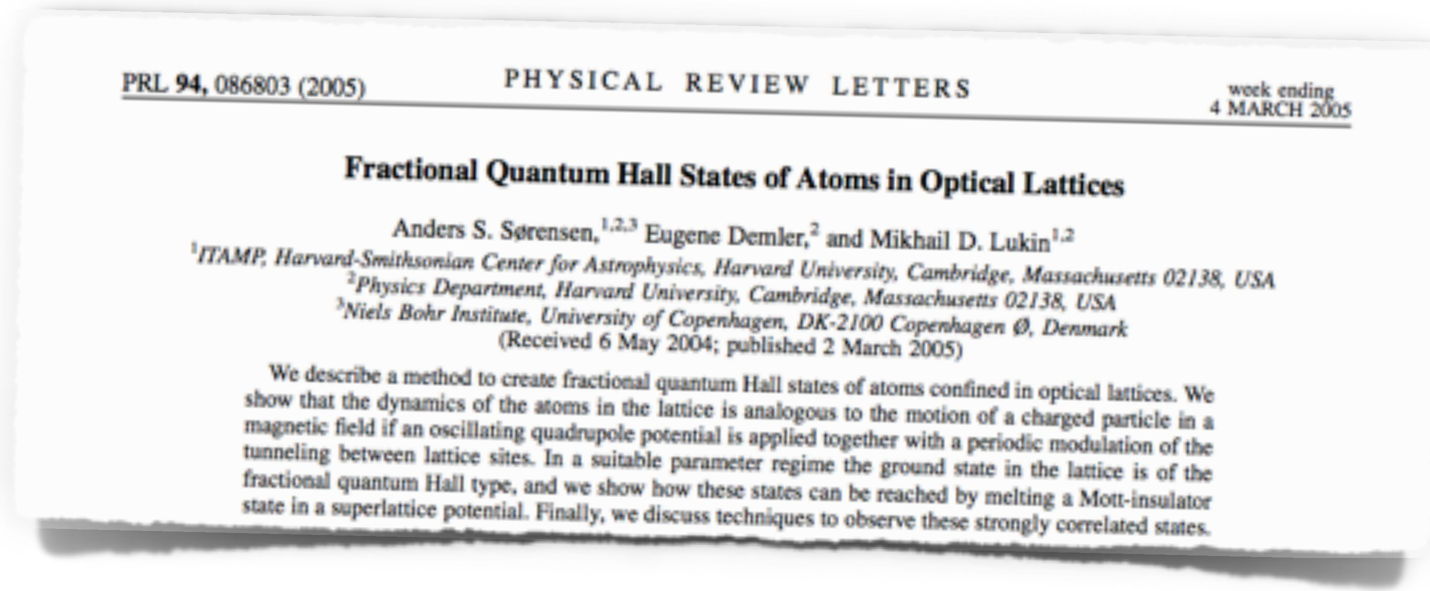
The fractional quantum Hall effect in a periodic potential or modulation of the magnetic field is studied by symmetry, topological, and Chern-Simons field-theoretic methods. With periodic boundary conditions, the Hall conductance in a finite system is known to be a fraction whose denominator is the degeneracy of the ground state. We show that in a finite system, translational symmetry predicts a degeneracy that varies periodically with system size and equals 1 for certain commensurate cases which we argue are physically representative. However, this analysis may overlook gaps due to finite-size effects that vanish in the thermodynamic limit. This possibility is addressed using a fermionic Chern-Simons field theory in the mean-field approximation. In addition to solutions describing the usual Laughlin or Jain states whose properties are only weakly modified by the periodic background, we also find solutions whose existence depends on the presence of the background. In these incompressible states, the Hall conductance is a fraction not equal to the filling factor, and its denominator is the same as that of the fractional charge and statistics of the elementary quasiparticle excitations.

- Confirmations for such states?

Fractional Quantum Hall on lattices: Numerical Evidence

- interest in cold atom community 2000's:
- realisations of tight-binding models with complex hopping from light-matter coupling:





PRL **94,** 086803 (2005) PHYSICAL REVIEW LETTERS week ending 4 MARCH 2005

Fractional Quantum Hall States of Atoms in Optical Lattices

Anders S. Sørensen,[1,2,3] Eugene Demler,[2] and Mikhail D. Lukin[1,2]

[1]*ITAMP, Harvard-Smithsonian Center for Astrophysics, Harvard University, Cambridge, Massachusetts 02138, USA*

[2]*Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA*

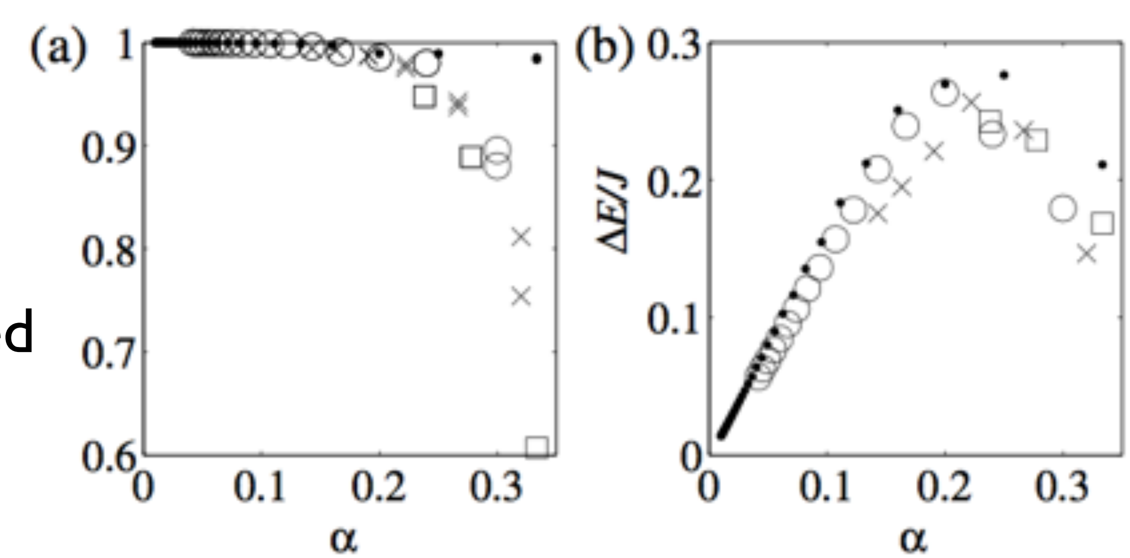
[3]*Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark*

(Received 6 May 2004; published 2 March 2005)

We describe a method to create fractional quantum Hall states of atoms confined in optical lattices. We show that the dynamics of the atoms in the lattice is analogous to the motion of a charged particle in a magnetic field if an oscillating quadrupole potential is applied together with a periodic modulation of the tunneling between lattice sites. In a suitable parameter regime the ground state in the lattice is of the fractional quantum Hall type, and we show how these states can be reached by melting a Mott-insulator state in a superlattice potential. Finally, we discuss techniques to observe these strongly correlated states.

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}_\alpha^\dagger \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1)$$

- bosons with onsite U: many-body gap in the half-filled "synthetic Landau-level" persists to large flux density



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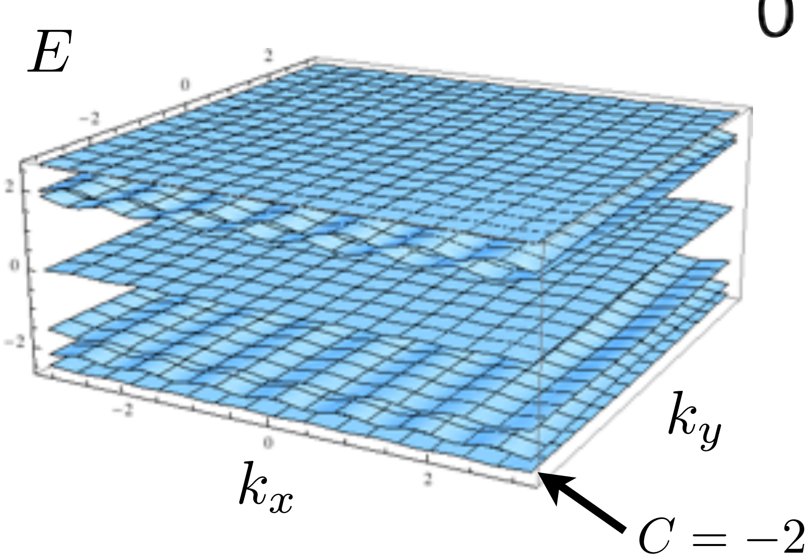
Fractional Quantum Hall on lattices with higher Chern-# bands

- bands of the Hofstadter model go *beyond* the continuum limit and support *new classes* of quantum Hall states

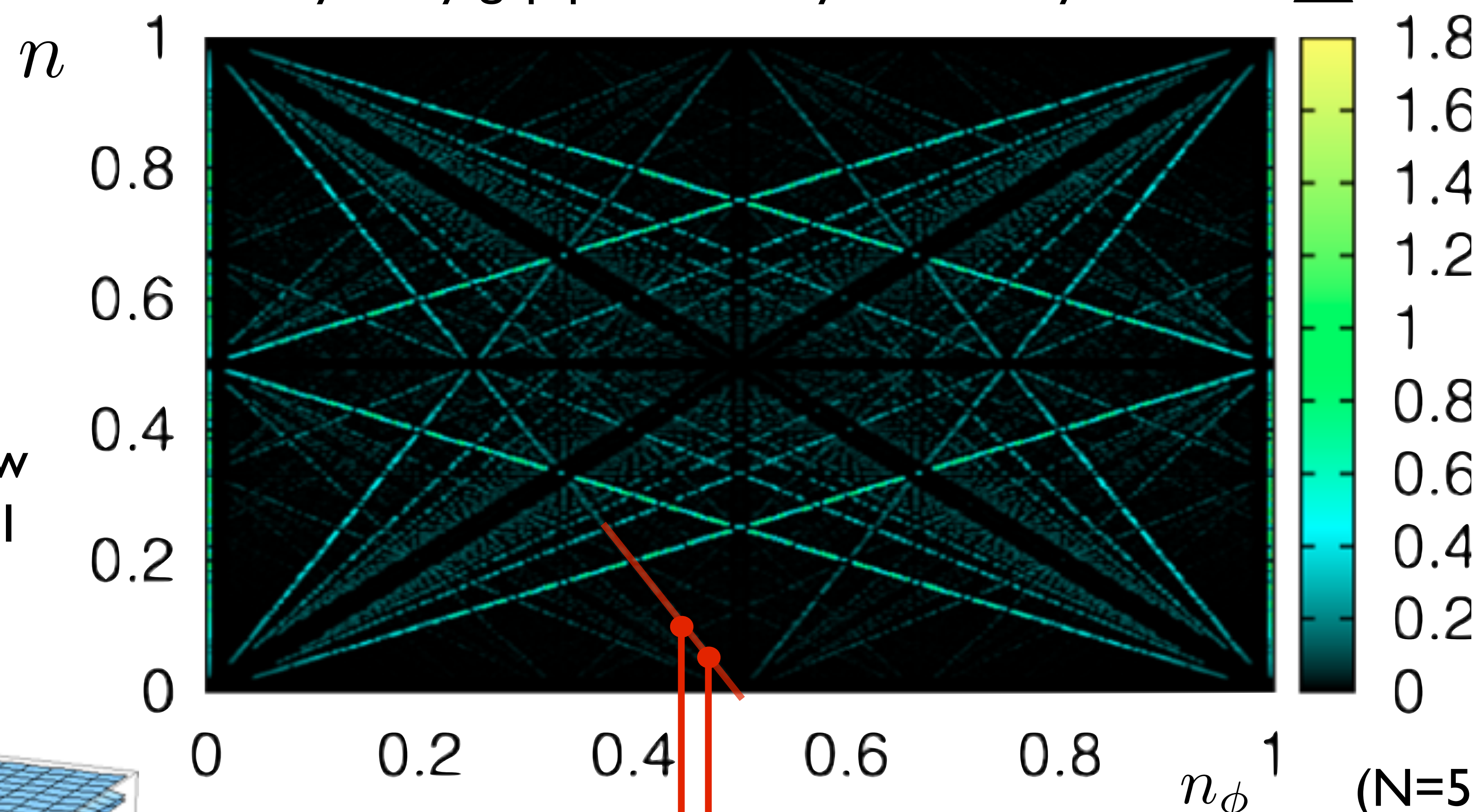
theory:
bosonic Hall states on the lattice

numerical verification
for what we would now call FCI states with ν=1
- C=2 band
- hardcore bosons



many-body gap predicted by CF theory

Δ



(N=5 particles)

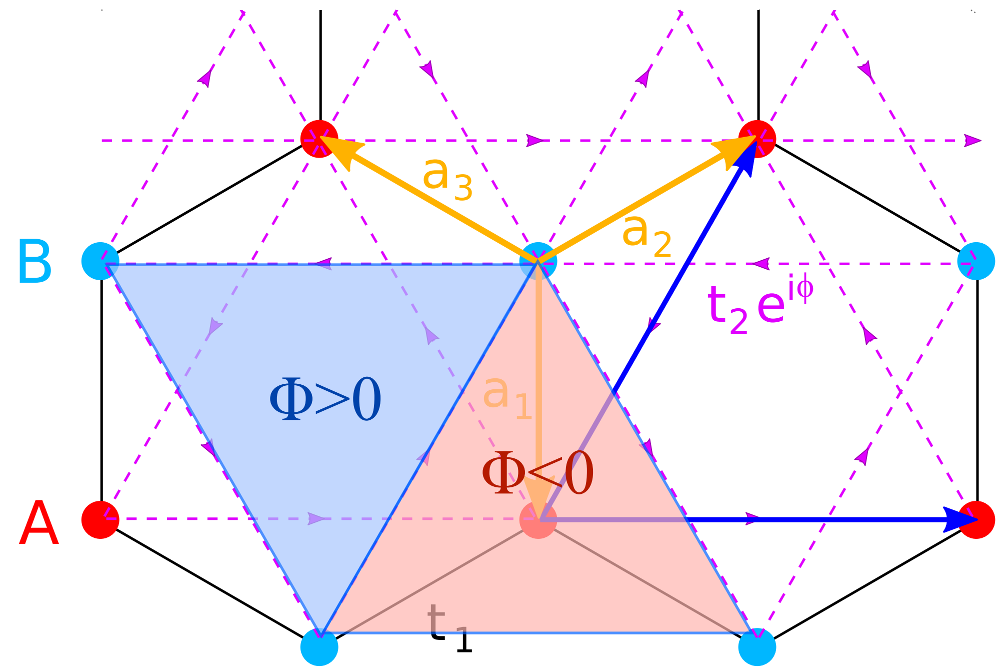
$n = 1/9$: $\mathcal{O} = |\langle \Psi_{\mathrm{CF}} | \mathrm{GS} \rangle|^2 \simeq 0.46$

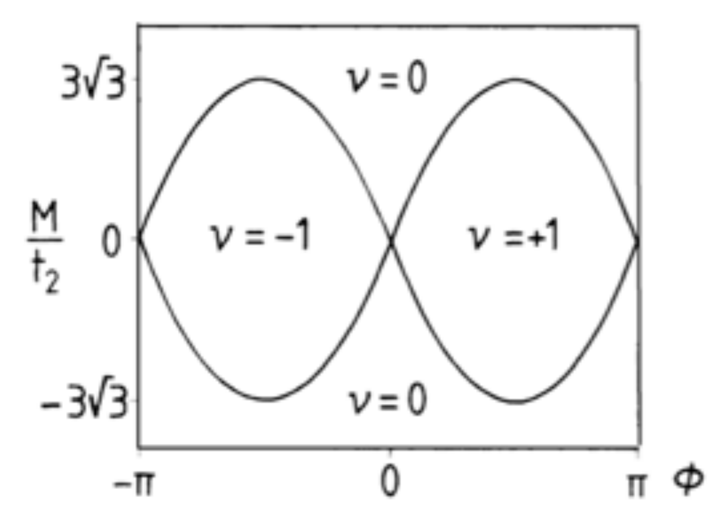
$n = 1/7$: $\mathcal{O} = |\langle \Psi_{\mathrm{CF}} | \mathrm{GS} \rangle|^2 \simeq 0.56$

GM & NR Cooper, PRL 2009

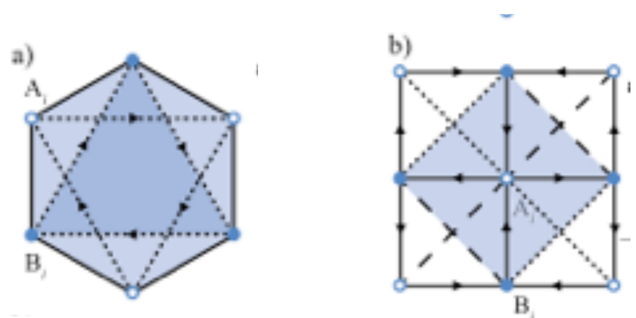
Chern bands in more general tight binding models

- Original proposal for IQHE without magnetic fields: Haldane (1988)

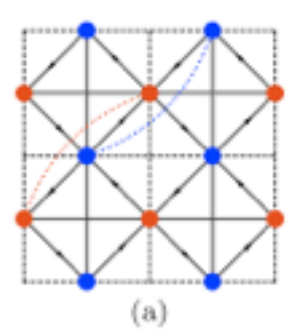




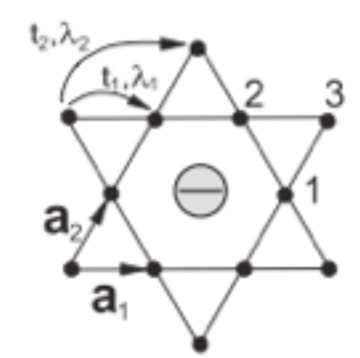
- 2011: FQHE expected in models with spin-orbit coupling + interactions



T. Neupert et al.



K. Sun et al.

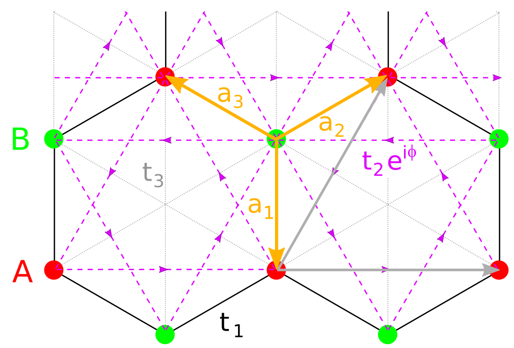


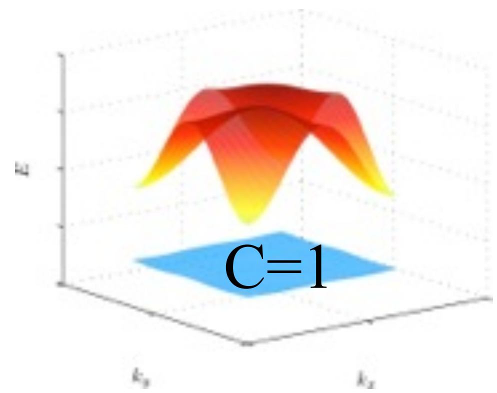
E. Tang et al.

Numerical confirmation: D. Sheng; C. Chamon; N. Regnault & A. Bernevig, …

Fractional Quantum Hall Effect in Topological Band Models

- Question: quantum Hall effect in general lattice models?





\+ Interactions = FQHE ?

- Analogy for topological order of many-body states:

- Topological order is invariant under continuous / adiabatic deformations!

> - Approach: Continuously deform a fractional quantum Hall state into a fractional Chern insulator *without closing the gap*.

Adiabatic Continuation of QH liquids in different systems

- use Hilbert spaces with the same overall structure (based on Wannier states) to study the low-lying spectrum numerically (exact diagonalization)
- adiabatically deform many-body Hamiltonian of FQHE to a fractionally filled Chern band:

$$\mathcal{H}(x) = \frac{\Delta_{\mathrm{FCI}}}{\Delta_{\mathrm{FQHE}}}(1-x)\mathcal{H}^{\mathrm{FQHE}} + x\mathcal{H}^{\mathrm{FCI}}$$

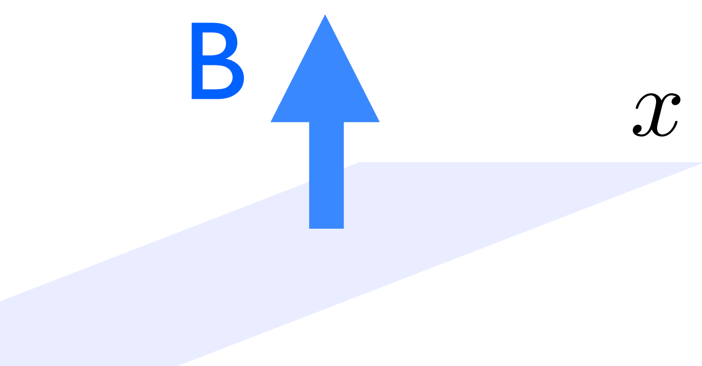
- E.g.: half-filled band for bosons & *contact repulsion*

FQHE of Bosons at

$\nu = 1/2$

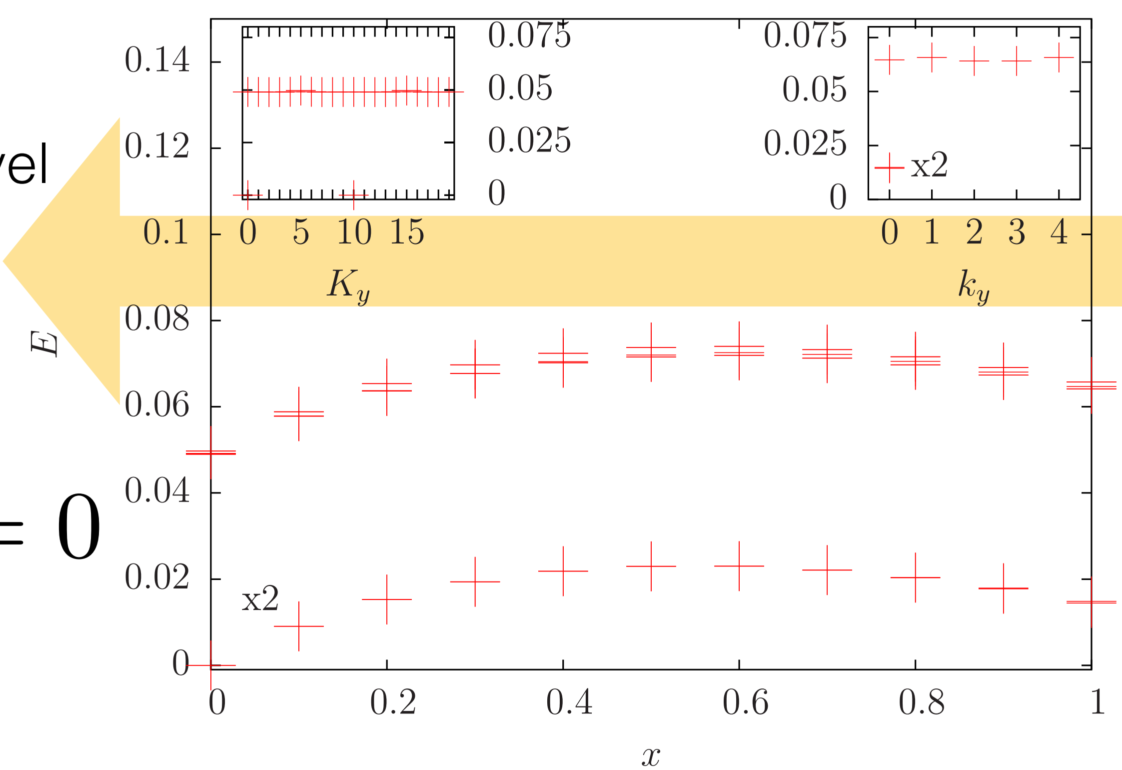
continuum Landau-level

Laughlin state



$x = 0$

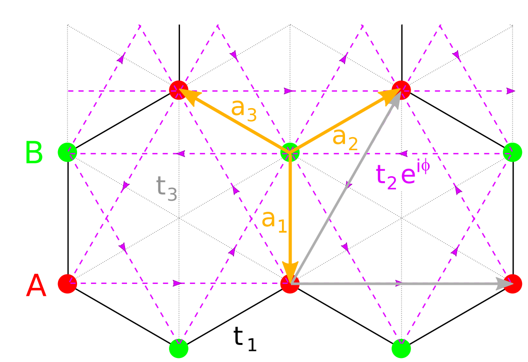
(N=10)



Half filled band of the (flattened) Haldane-model

Same topological phase!

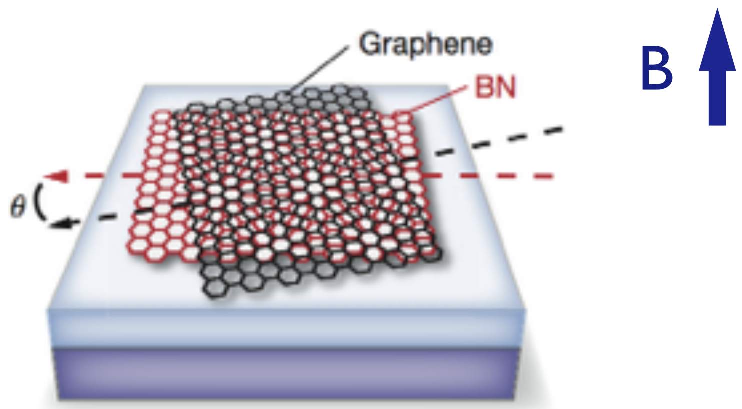
$x = 1$

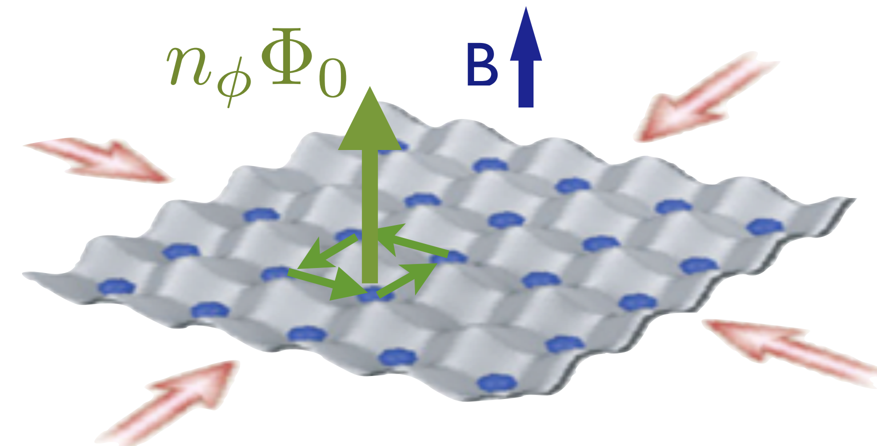


Th. Scaffidi & GM, Phys. Rev. Lett. (2012)

Chern numbers |C|>1 *in the Hofstadter Problem*

- Harper / Hofstadter: systems with Magnetic Field and periodic potentials





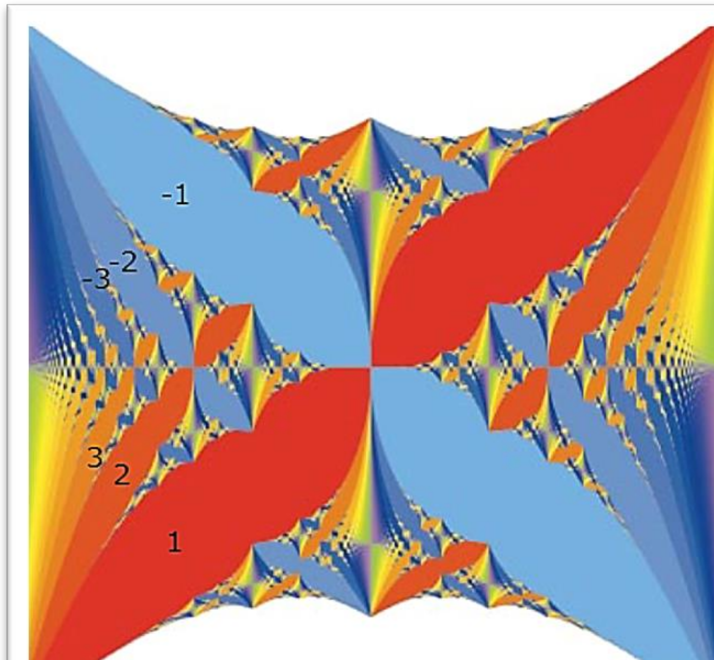
- twisted graphene bilayers: Kim *et al. (2013)*
- optical flux lattices: MIT / Munich (2014)

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

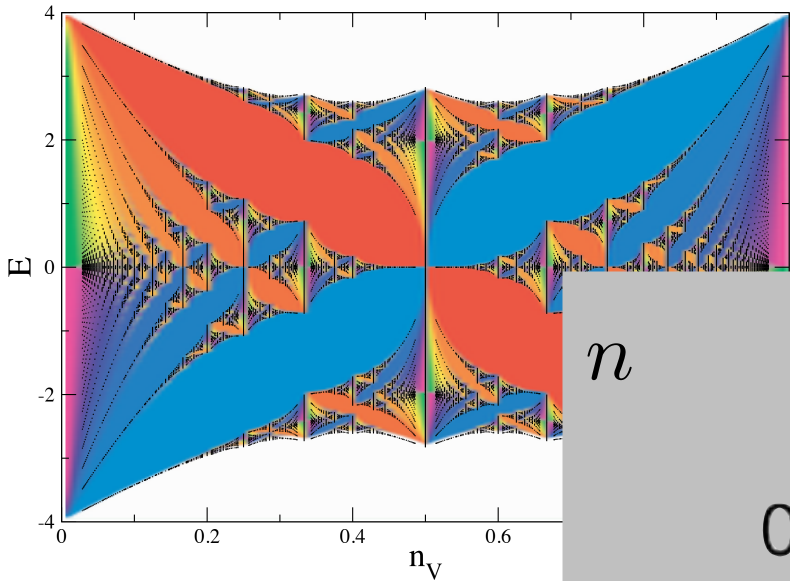
$$\sigma_H = \frac{ie^2}{2\pi h}\sum\int d^2k\int d^2r\left(\frac{\partial u^*}{\partial k_1}\frac{\partial u}{\partial k_2}-\frac{\partial u^*}{\partial k_2}\frac{\partial u}{\partial k_1}\right)$$
$$= \frac{ie^2}{4\pi h}\sum\oint dk_j\int d^2r\left(u^*\frac{\partial u}{\partial k_j}-\frac{\partial u^*}{\partial k_j}u\right), = \frac{e^2}{h}\sum_n C_n$$



Osadchy and Avron, J. Math. Phys. 42, 5665 (2001)

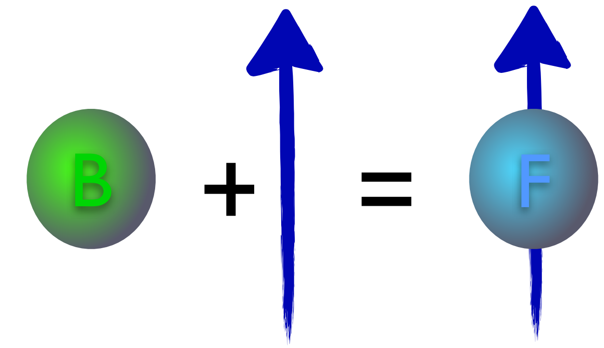
New Universality classes of FQH states

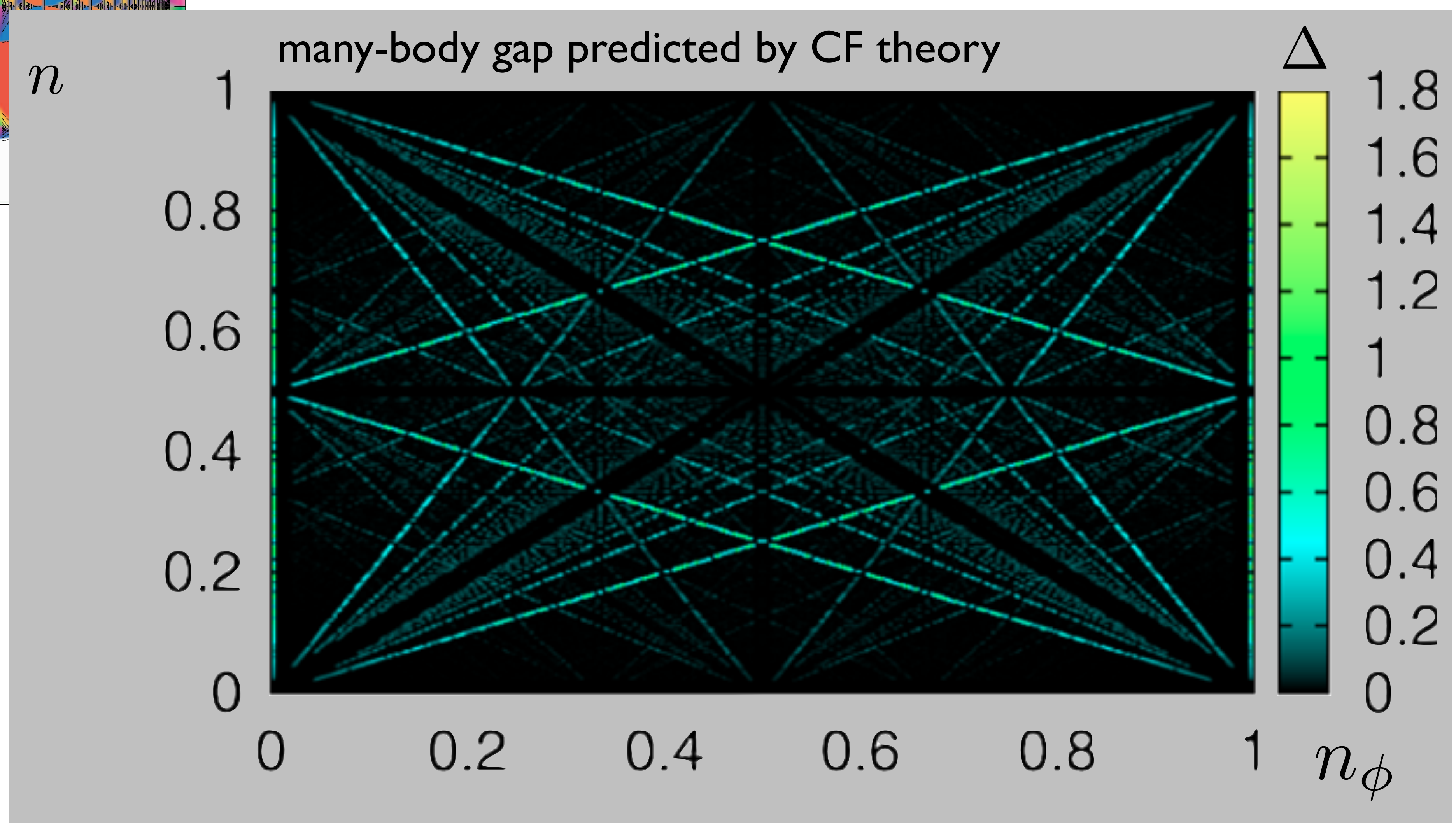
single-particle spectrum



Simple Heuristics: Composite Fermions

$$n_\phi = \pm n + n_\phi^*$$



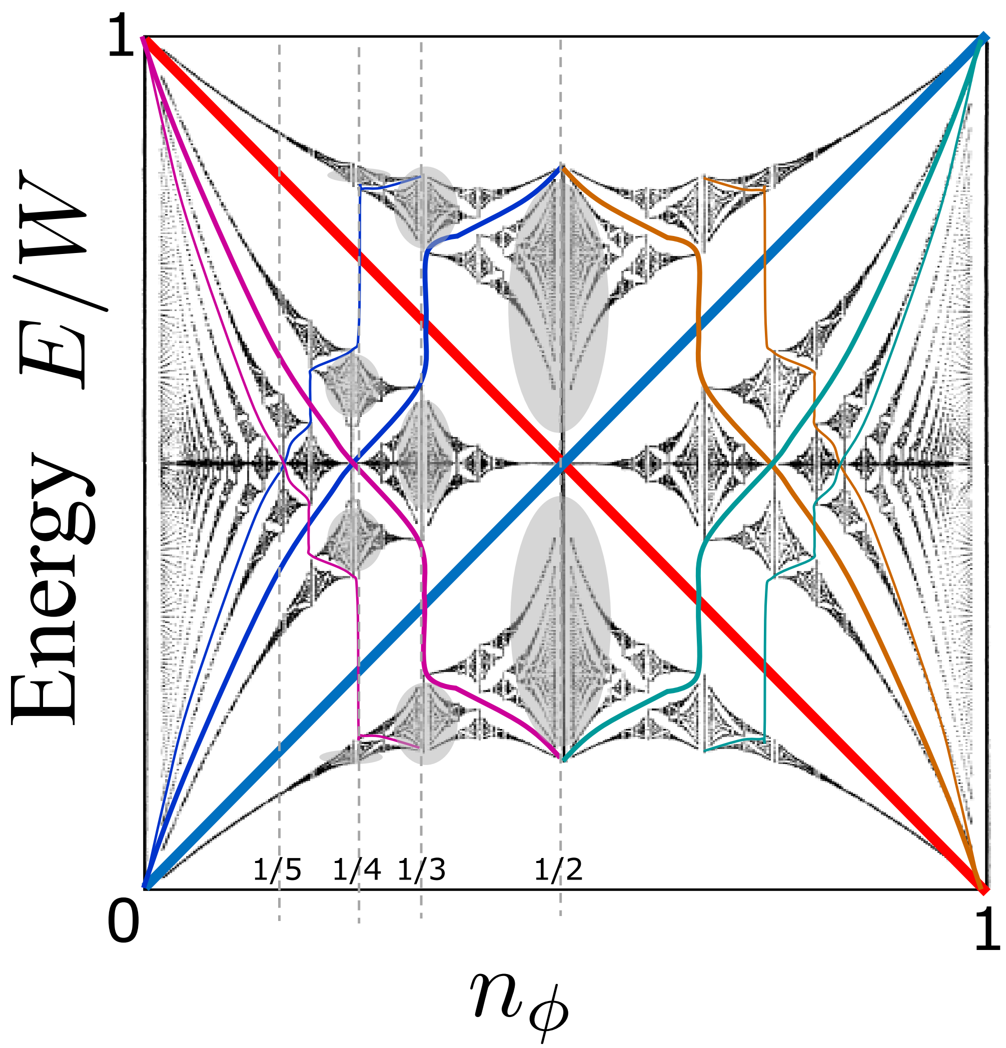


- higher Chern number bands yield new series of Abelian quantum Hall states!

first numerical evidence: GM & N.Cooper Phys. Rev. Lett. **103**, 105303 (2009)

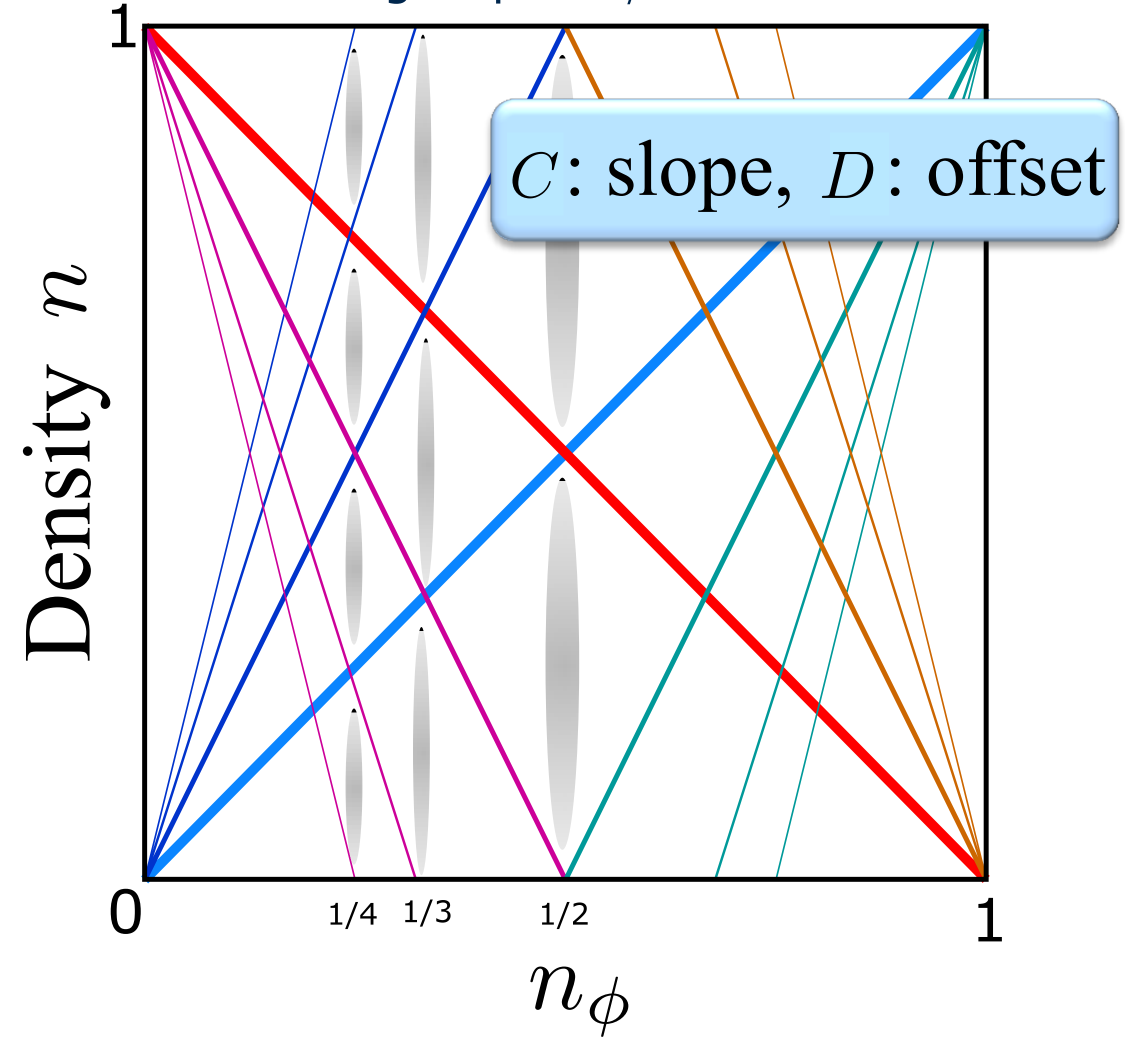
Density of filled bands in the Butterfly: Wannier Diagram

Hofstadter's Energy Spectrum



Wannier, *Phys. Status Solidi*. **88**, 757 (1978)

Tracing Gaps in ϕ and *n*



Density of filled bands n

Flux density n_ϕ

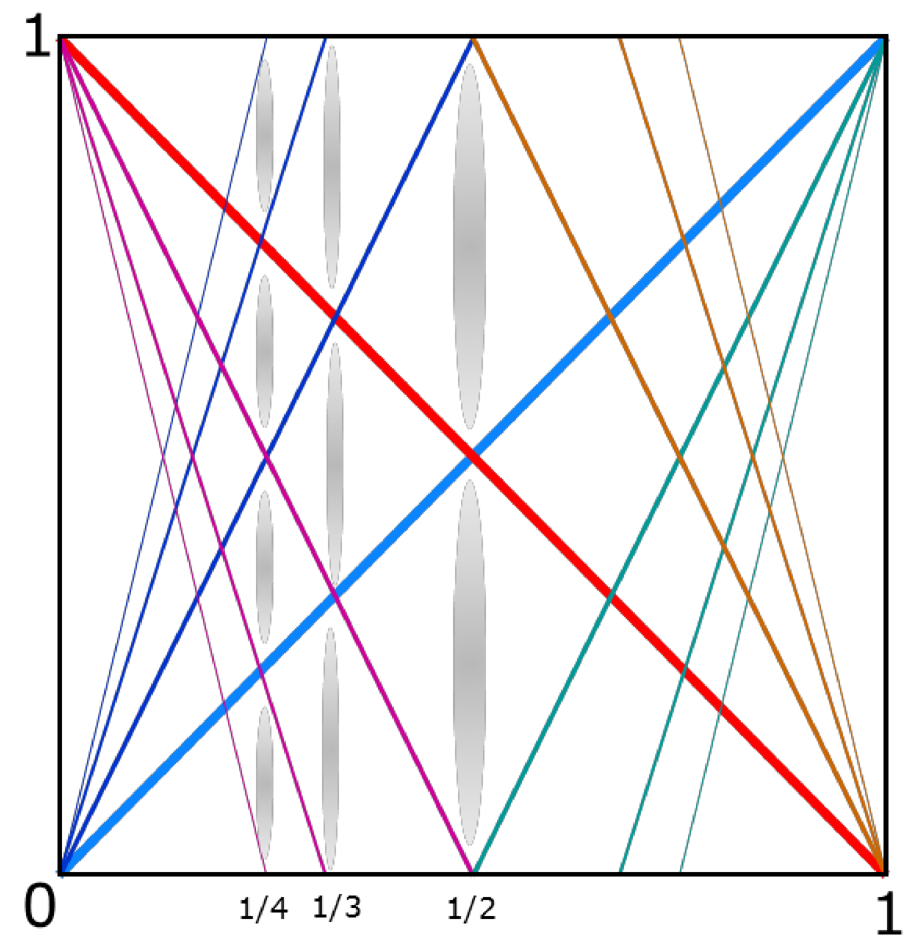
Diophantine equation for gaps

$$n = Cn_\phi + D, \quad C, D \in \mathbb{Z}$$

Streda & Thouless: Quantization of Hall conductivity

Wannier:

$$n = C n_\phi + D, \quad C, D \in \mathbb{Z}$$



Streda:

$$\sigma_{xy} = \frac{e}{\Phi_0}\frac{\partial n}{\partial n_\phi} = C\frac{e^2}{h}.$$

Thouless:

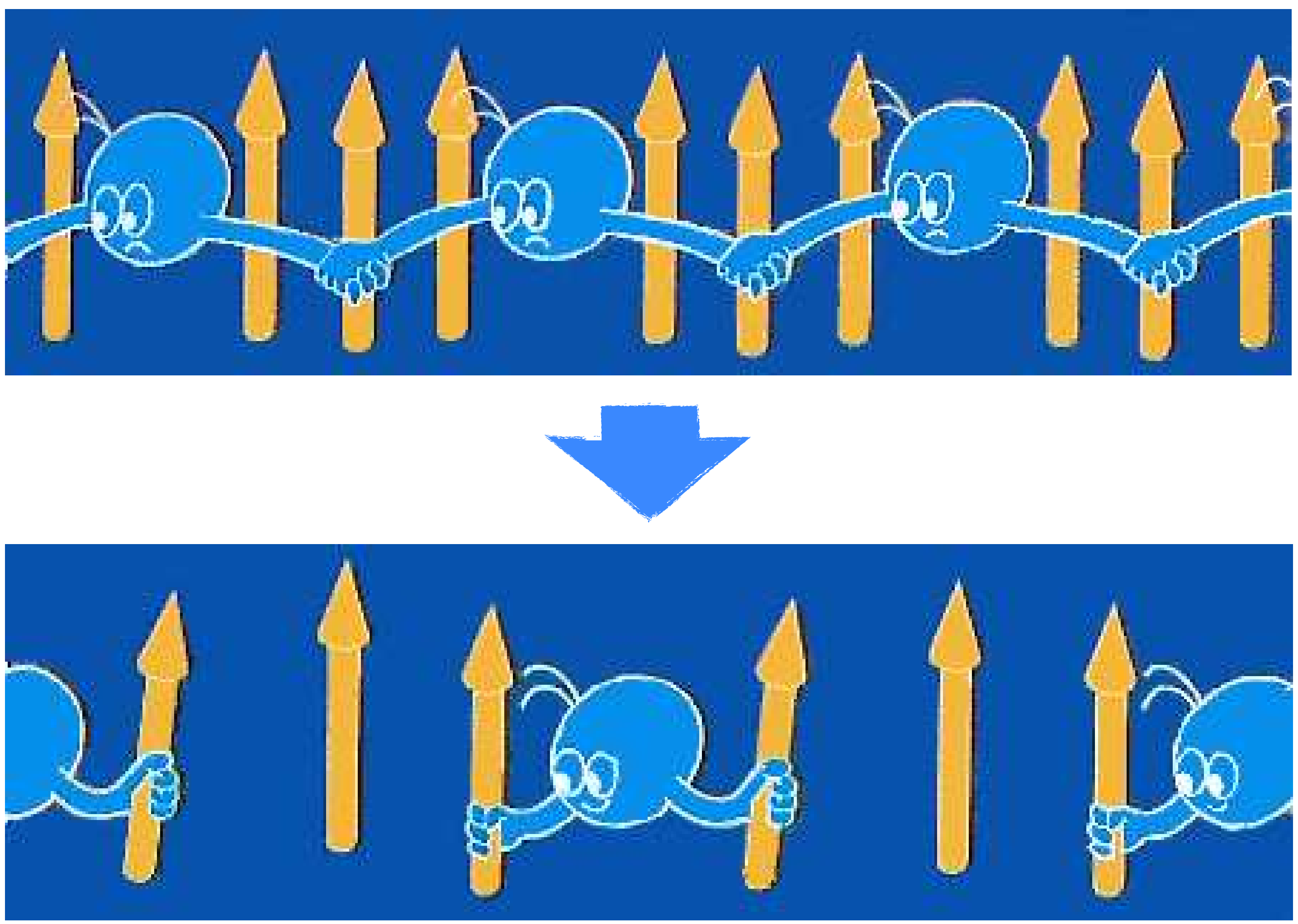
$$\sigma_{xy} = \frac{e^2}{2\pi h}\sum_{\text{filled bands } n}\int d^2k\mathcal{F}_{12}(k)$$

$$\Rightarrow \quad C = \sum_{\text{filled bands}} C_n$$

The Composite Fermion Approach

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}_\alpha^\dagger \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

Account for repulsive interactions $U > 0$ by "flux-attachment" (Fradkin 1988, Jain 1989)



drawings: K. Park

Continuum Landau-level for fermions at filling 1/3: three flux per particle

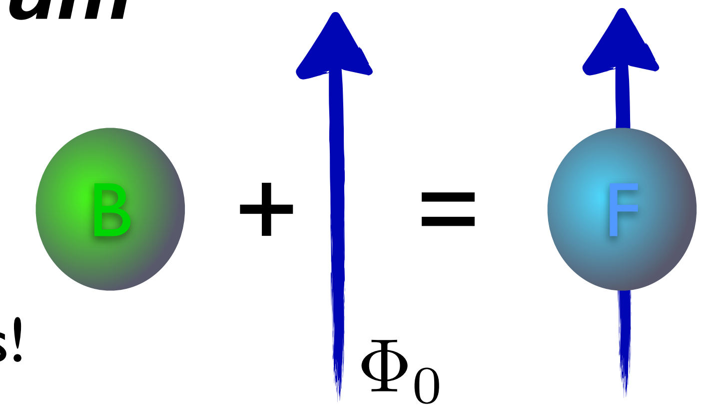
Composite fermions = electron + 2 flux quanta

$$\Psi \propto \prod_{i<j} (z_i - z_j)^2 \Psi_{\mathrm{CF}}$$

Bosons:
1 flux per composite particle

Composite Fermions in the Hofstadter Spectrum

1. Flux attachment for bosonic atoms: $n_\phi^* = n_\phi \mp n$



$$\Psi_B \propto \prod_{i<j} (z_i - z_j)\Psi_{\rm CF}$$ ⇒ transformation of statistics!

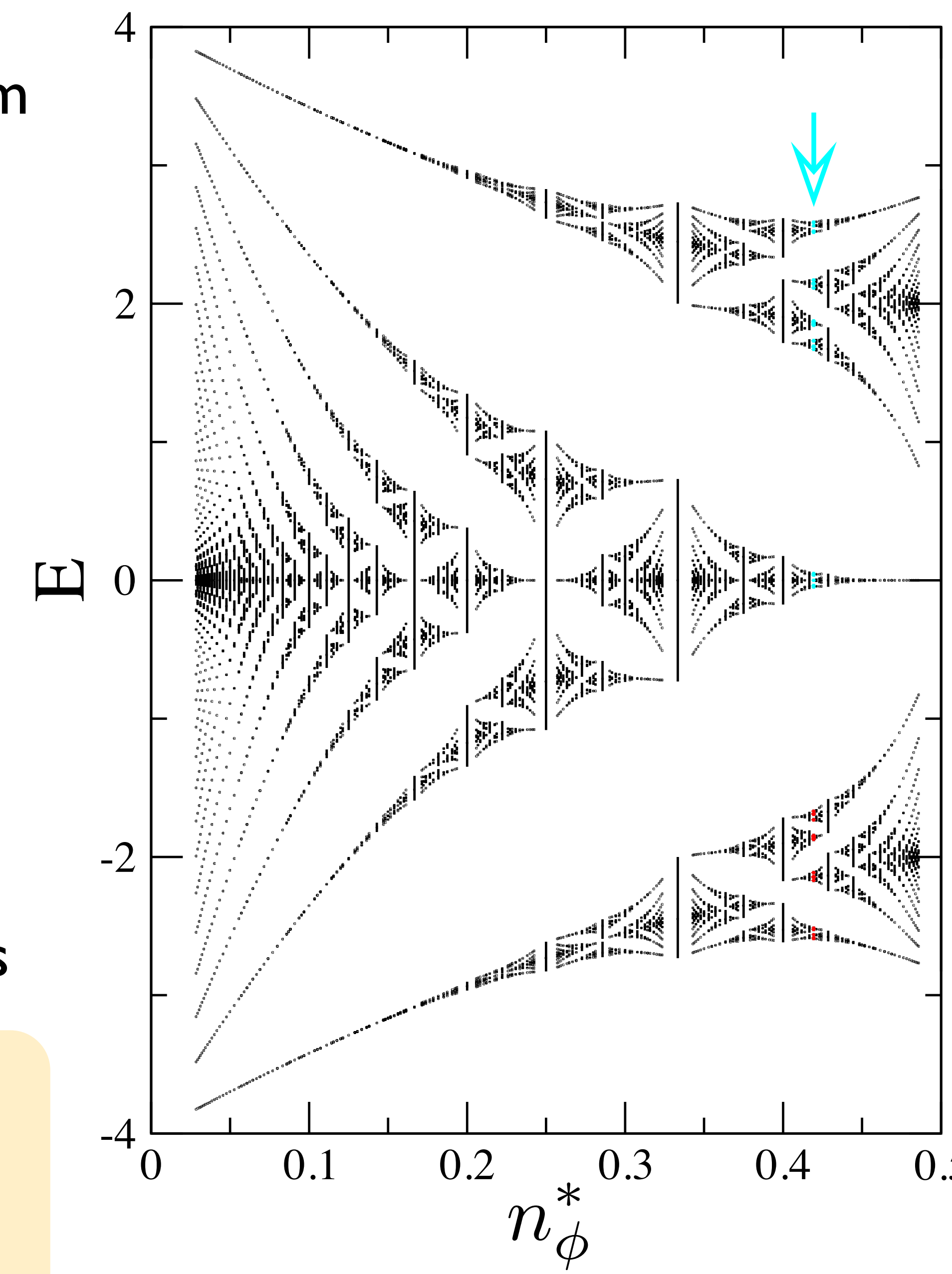
2. Effective spectrum at flux n_ϕ^* is again a Hofstadter problem

⇒ weakly interacting CF will fill bands, so obtain density n

by counting bands using fractal structure

⇒ linear relation of flux and density for bands under a gap

$$n = Cn_\phi + D, \quad C, D \in \mathbb{Z}$$



GM & N. R. Cooper, PRL 2009

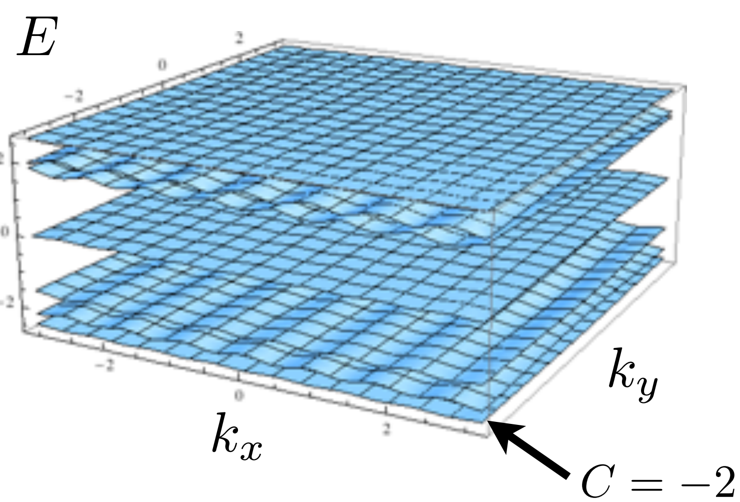
3. Construct Composite Fermion wavefunction

continuum: $$\Psi_{\rm B}(\{\mathbf{r}_i\}) \propto \mathcal{P}_{LLL} \underbrace{\prod_{i<j}(z_i - z_j)}\ \psi_{\rm CF}(\{\mathbf{r}_i\})$$

Vandermonde / Slater determinant of LLL states

lattice: $$\Psi_{\rm B}(\{\mathbf{r}_i\}) \propto \underbrace{\psi_{\rm J}^{(\phi_x,\phi_y)}(\{\mathbf{r}_i\})}\ \psi_{\rm CF}^{(-\phi_x,-\phi_y)}(\{\mathbf{r}_i\})$$

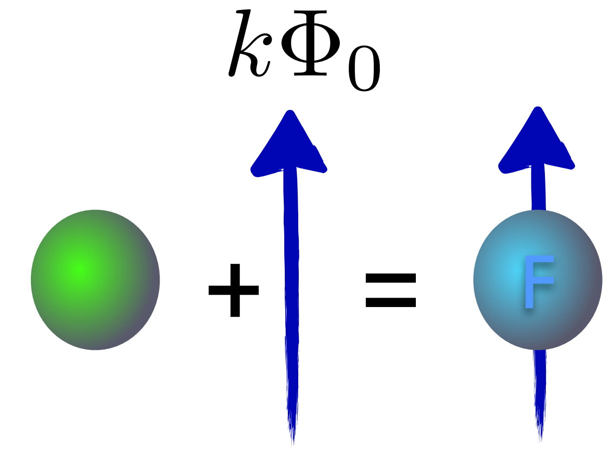
Slater determinant of Hofstadter
orbitals at flux density $n_\phi^0 = n$

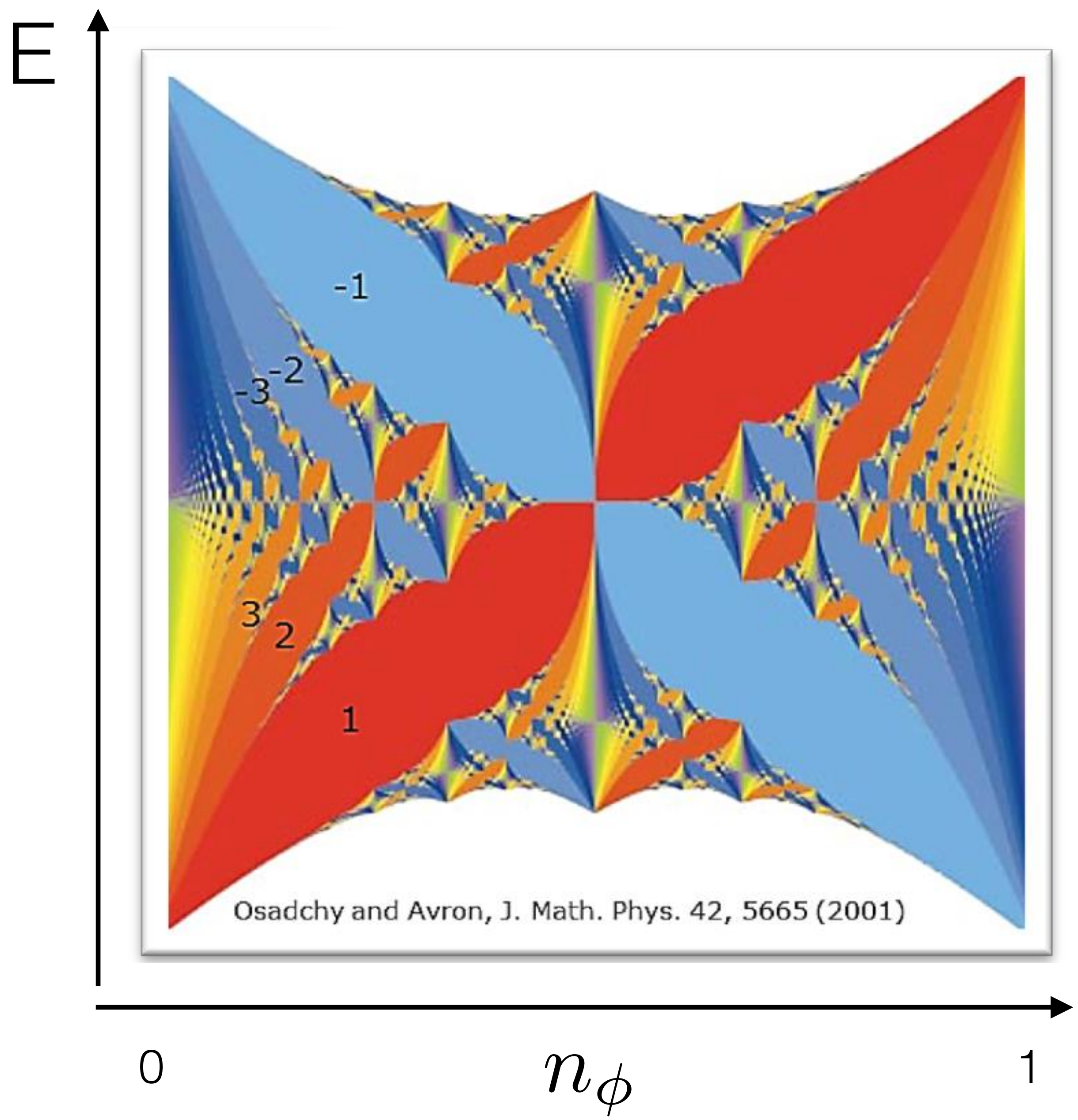


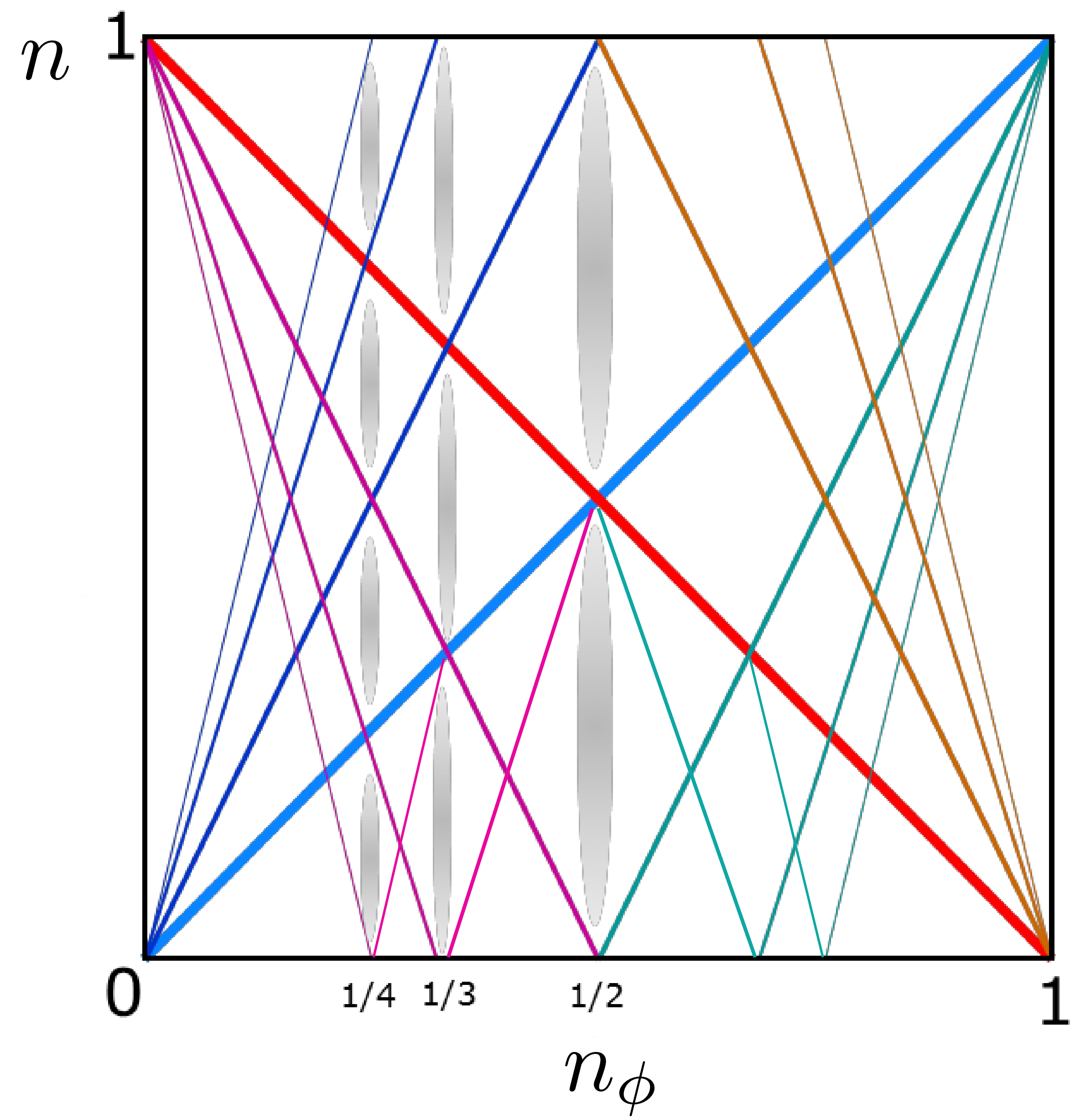
Simple Heuristics: Composite Fermions

$$n_\phi = kn + n_\phi^*$$

[k odd (even) for bosons (fermions)]







on the blackboard...

$$n_\phi = kn + n_\phi^*$$

useful to replace flux density by number of states in relevant low-energy manifold

$$n_s = C n_\phi + D$$

Composite fermions filling integer # bands, so can use the Diophantine equation for the CF gap:

$$n = n_s^* = C^* n_\phi^* + D^*$$

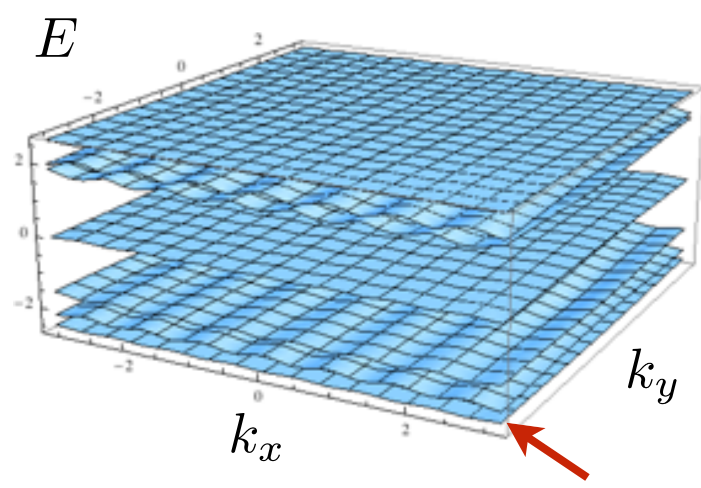
$$\Rightarrow \quad \frac{n_s}{C} - \frac{D}{C} = n\left(\frac{kC^* + 1}{C^*}\right) - \frac{D^*}{C^*}$$

Hence, a constant filling factor is defined only if $\frac{D}{C} = \frac{D^*}{C^*}$ – but that is indeed a representative case: as n small, the CF band structure looks similar to the original one, but CF may fill r bands. Then, we have $C^* = rC$ and the filling factors are

$$\nu = \frac{r}{kCr + 1}, \quad r \in \mathbb{Z}$$

Exact Diagonalization vs Theory Predictions:

Bosons with contact interactions, in lowest band



$$\mathcal{H} = \mathcal{P}_{\rm LB} \sum_i \hat{n}_i(\hat{n}_i - 1)\mathcal{P}_{\rm LB}$$

Check predictions for incompressible states:

filling:

$$\nu = \frac{r}{r|Ck| + 1}$$

GS degeneracy:

$$d_{\rm GS} = |rCk| + \mathrm{sgn}(r)$$

Chern number of GS's:

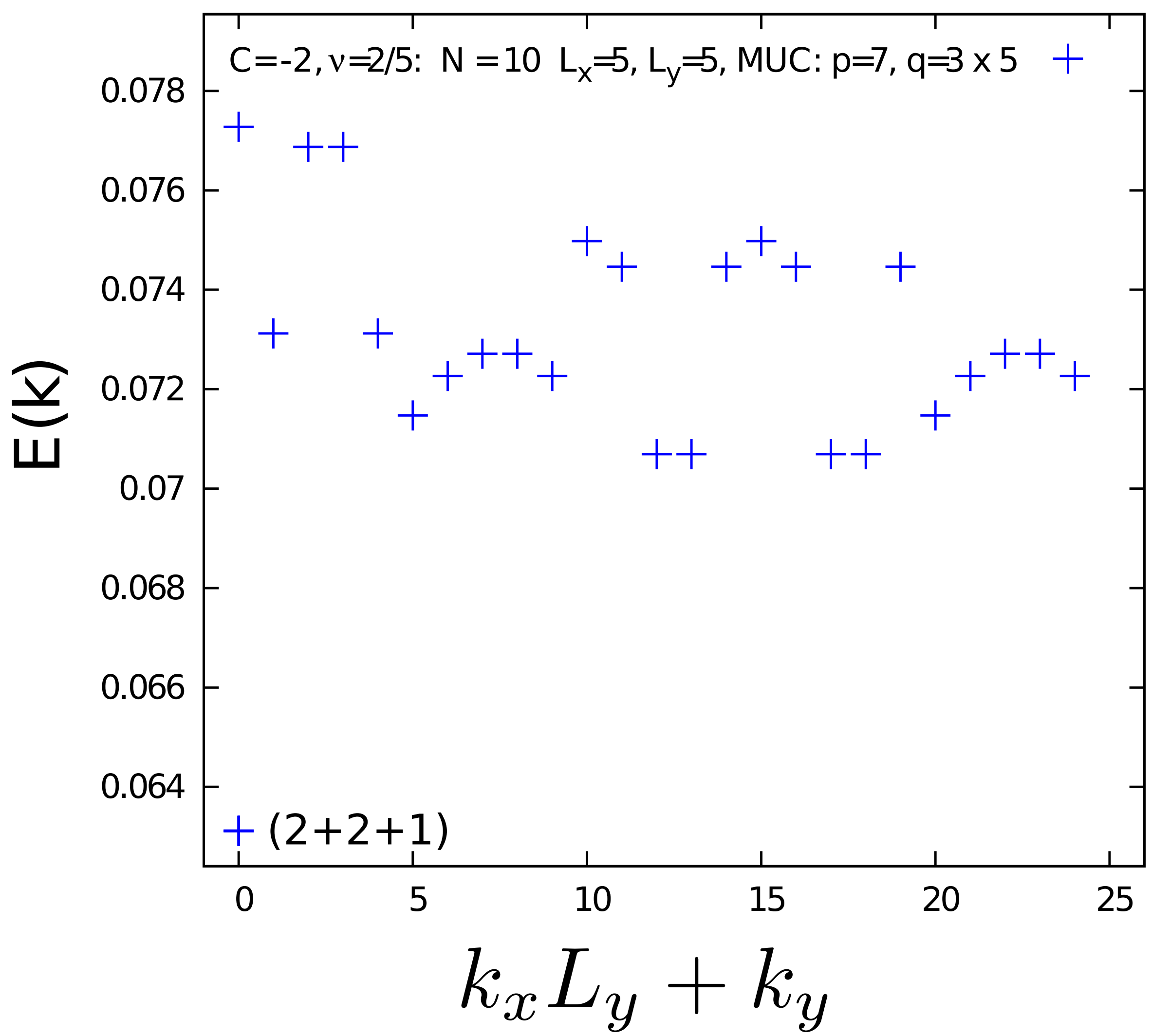
$$C_{\rm MB} = C^* = rC$$

GM & NR Cooper, PRL (2015), arXiv:1504.06623

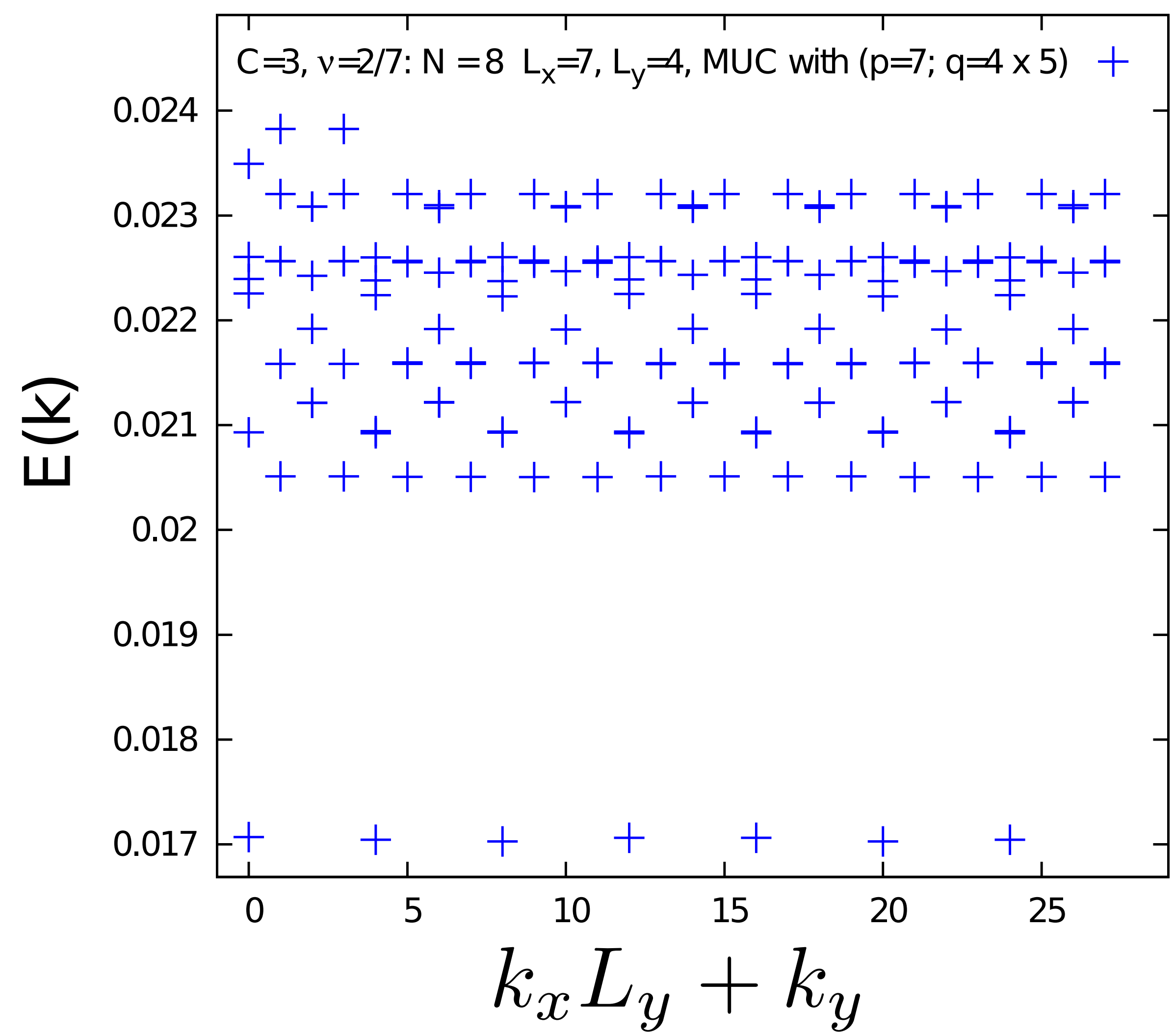
Exact Diagonalization: Spectra for new candidates

Example spectra for states with ‘positive flux attachment’: r=2

$C = 2$



$C = 3$

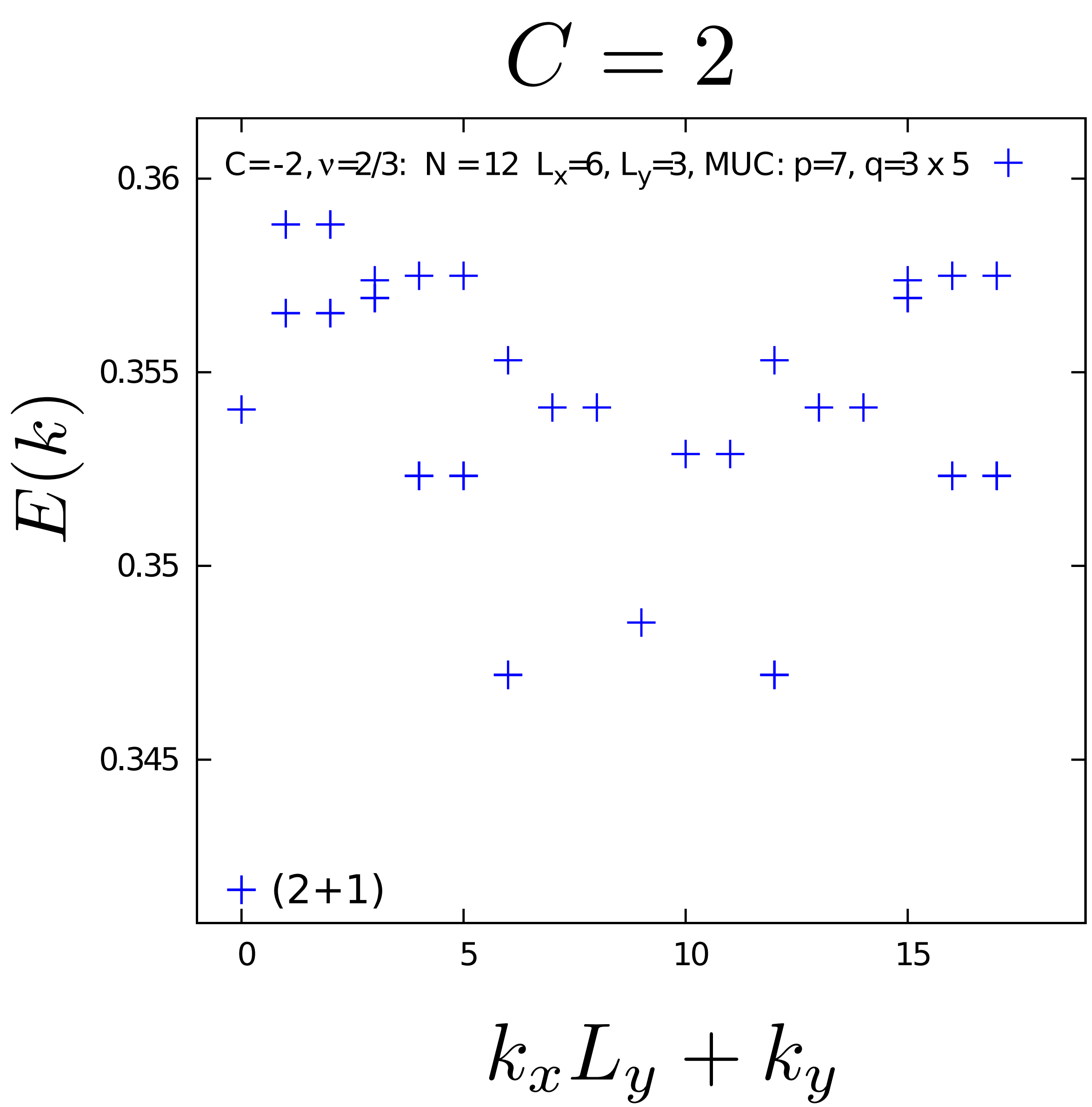


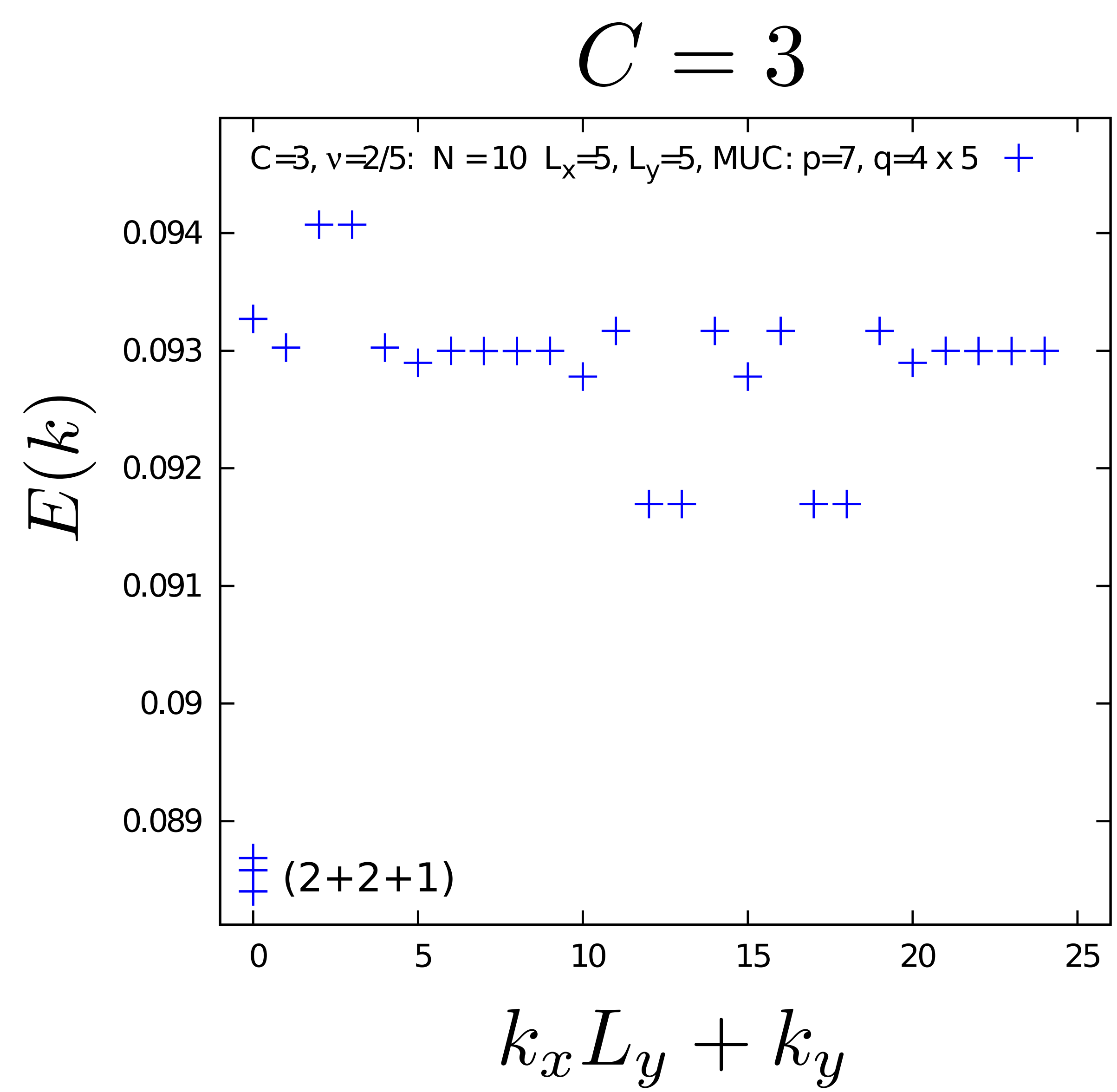
$$\nu = \frac{r}{r|Ck| + 1}$$

GM & NR Cooper, arXiv:1504.06623

Exact Diagonalization: Spectra for new candidates

Example spectra for states with 'negative flux attachment': r=-2



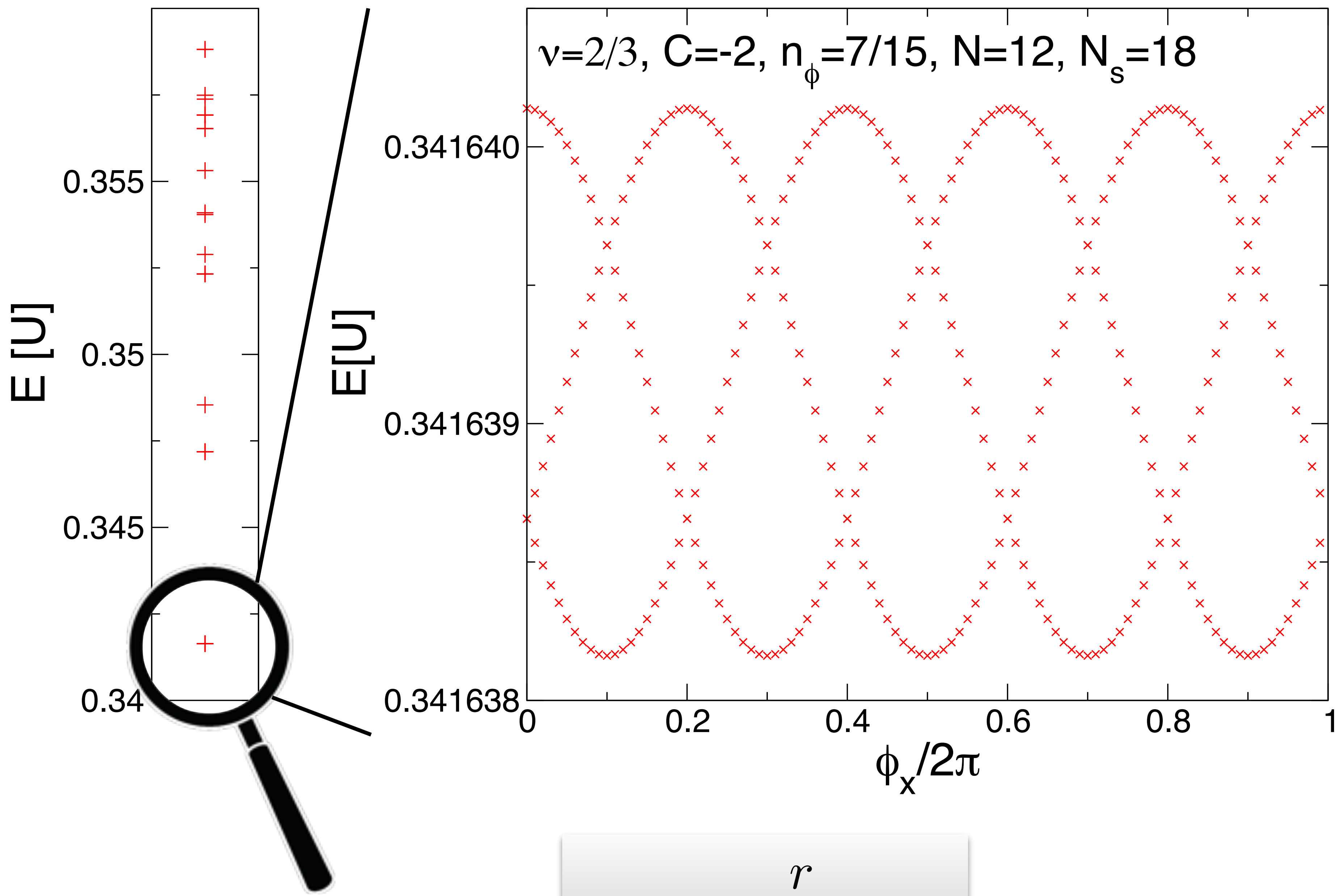


$$\nu = \frac{r}{r|Ck| + 1}$$

GM & NR Cooper, arXiv:1504.06623

Exact Diagonalization: Spectral flow

Evolution of the ground states under “threading flux”



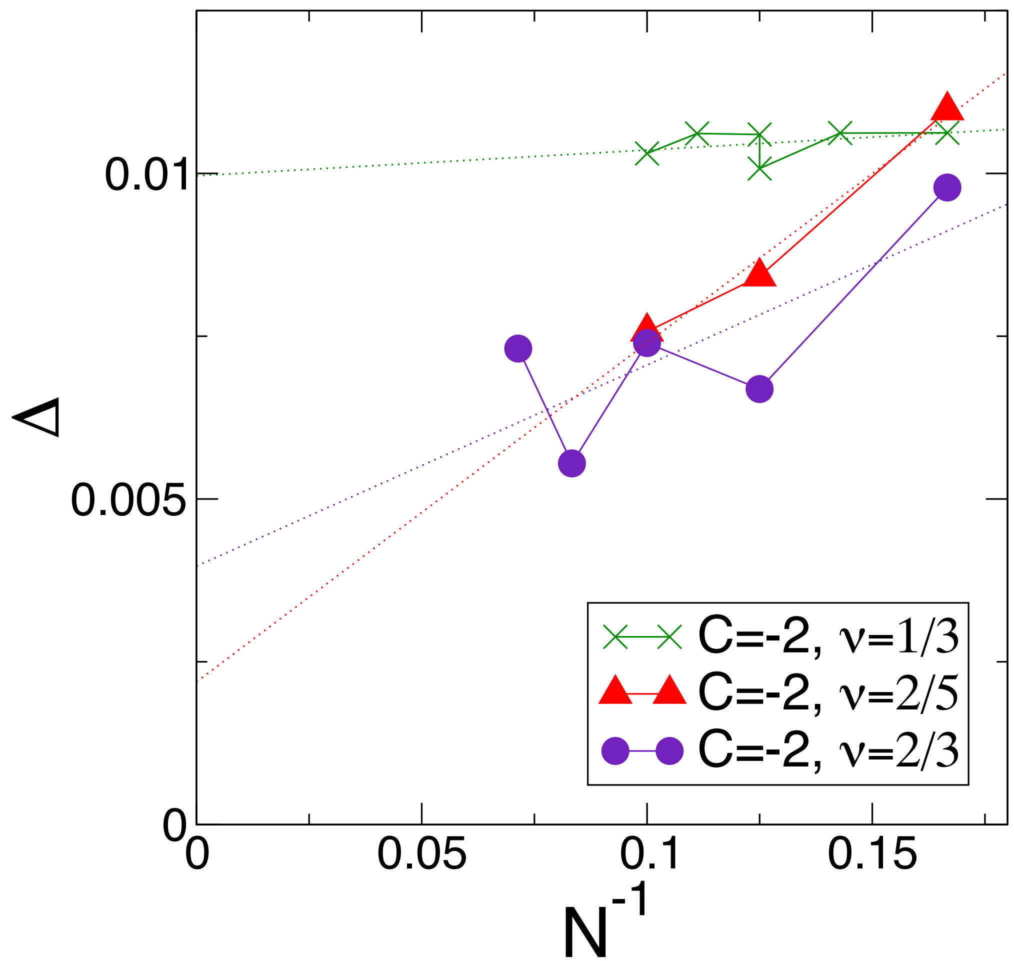
$$\nu = \frac{r}{r|Ck|+1}$$

GM & NR Cooper, arXiv:1504.06623

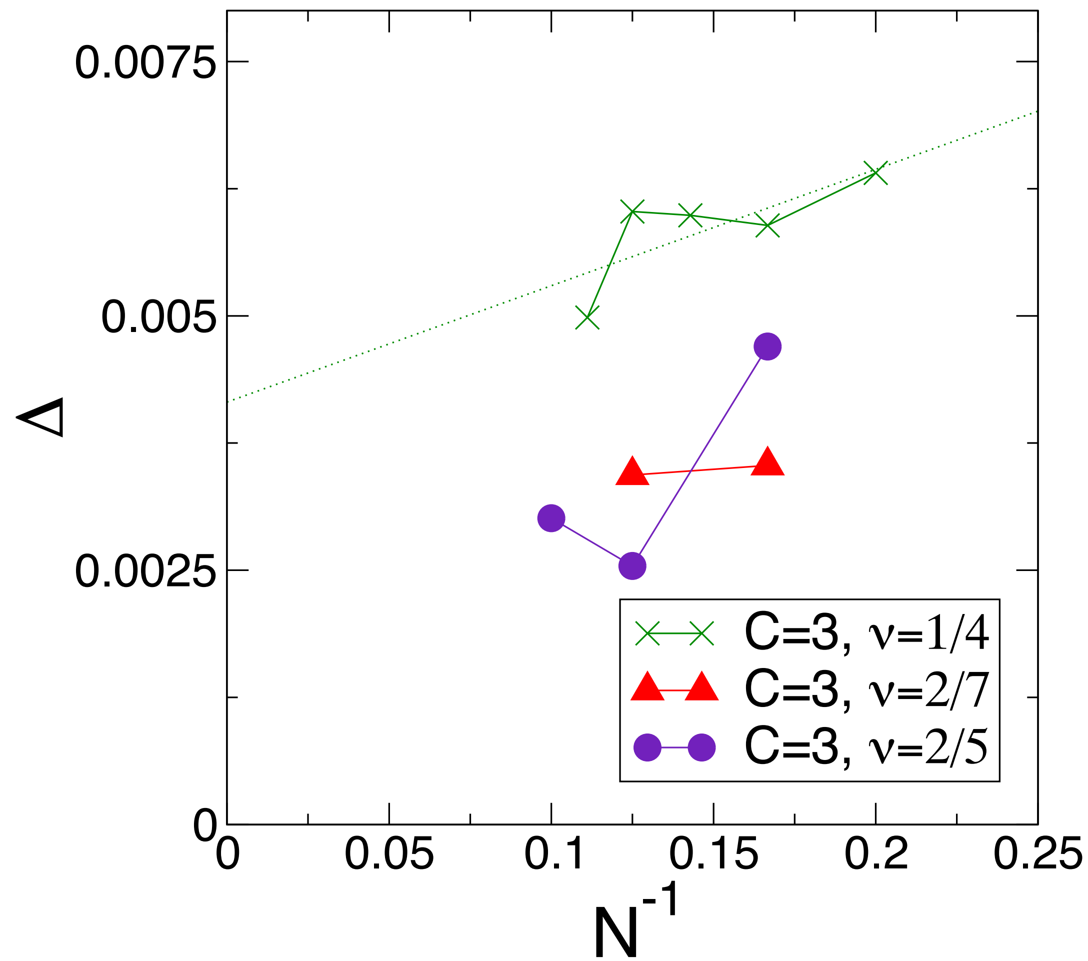
Exact Diagonalization: Finite Size Scaling of Gaps

Ascertain that GS degeneracy with finite gap is found consistently for different N_s

$C = 2$



$C = 3$



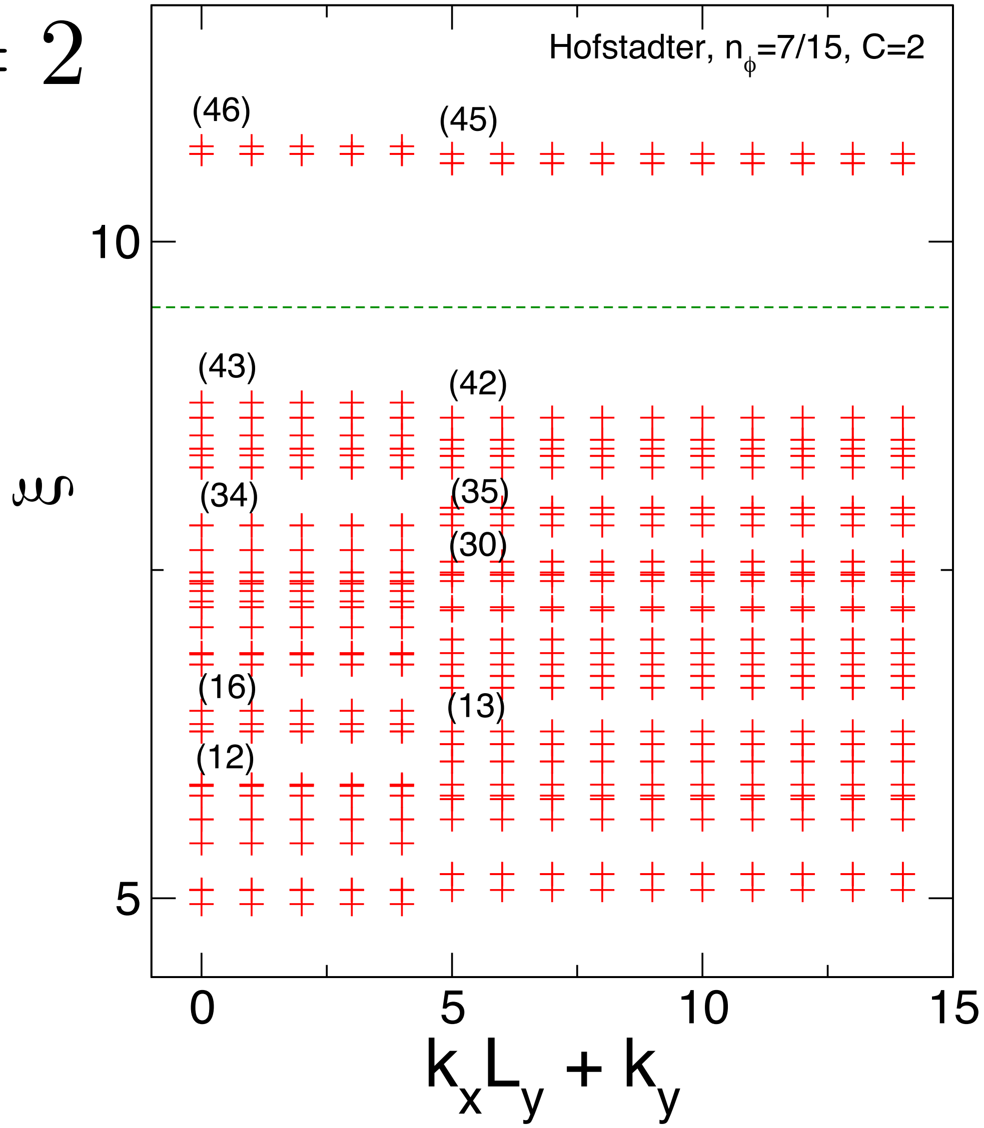
⇒ data suggests the composite fermion states are incompressible in the thermodynamic limit

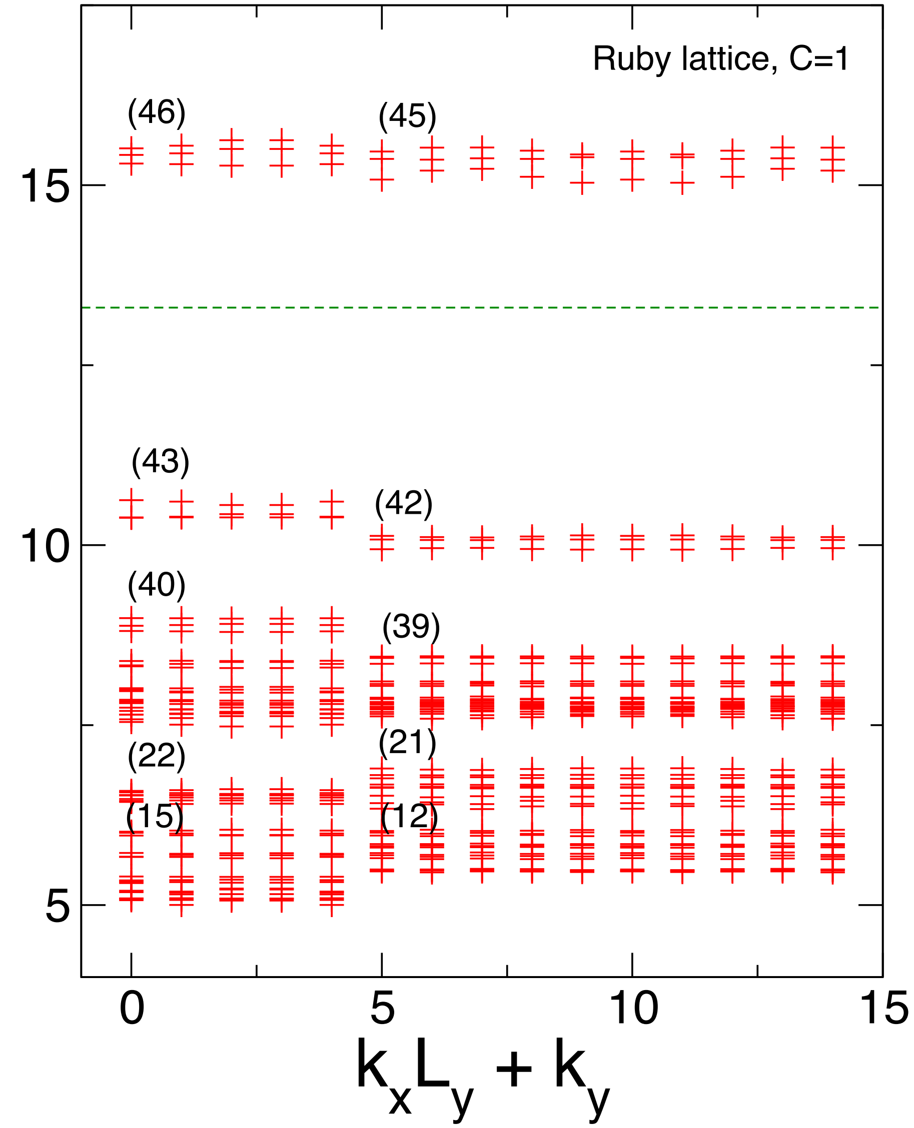
GM & NR Cooper, PRL (2015), arXiv:1504.06623

Exact Diagonalization: Particle Entanglement Spectra

Compare PES of a C=2 system to a known C=1 spectrum: ν=2/3

$C = 2$





$C = 1$

⇒ smaller entanglement energies; differences in detail

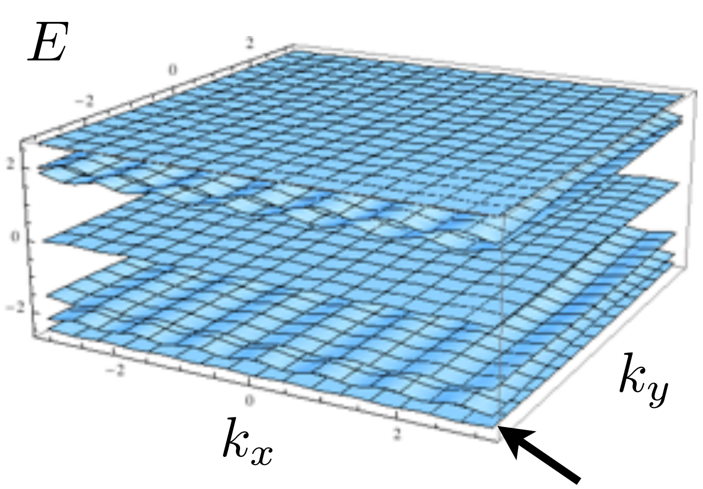
⇒ overall features similar

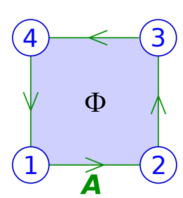
GM & NR Cooper, PRL (2015), arXiv:1504.06623

Universality of Predictions

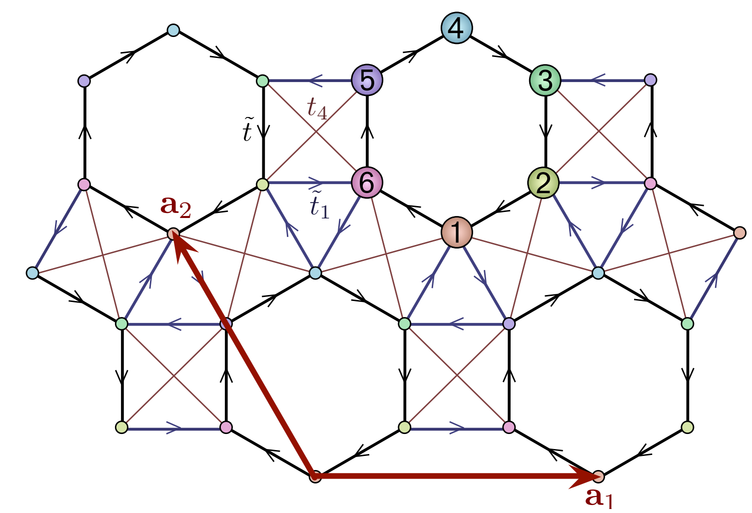
- Again, argue with adiabatic deformations:

- Hofstadter generates bands of any Chern #





- can deform to any other model…



- adiabatic connection for single bands as long as

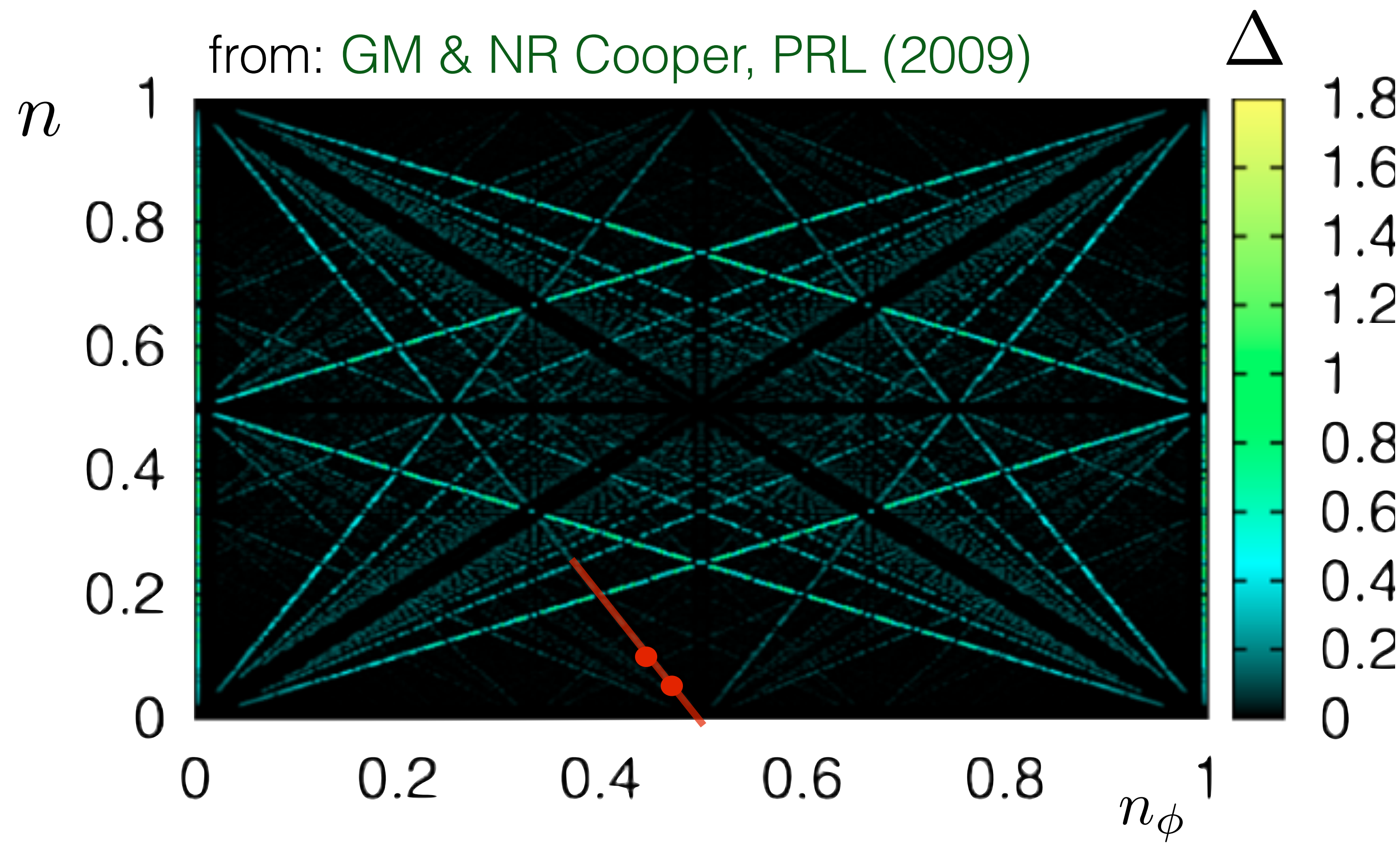
$$C_1 = C_2$$

⇒ flux attachment provides candidate states for all Chern bands $\nu = \dfrac{r}{r|Ck|+1}$

A Special case - Bosonic IQHE in C=2 bands

Bosons in a C=2 band with negative flux attachment (r=-1) ⇒ ν=1

from: GM & NR Cooper, PRL (2009)



alternative realisations:

- Quantum Hall Bilayers [Regnault & Senthil 2013]
- Honeycomb with correlated hopping [He et al. 2015]
- Optical Flux Lattices [Sterdyniak et al 2015]

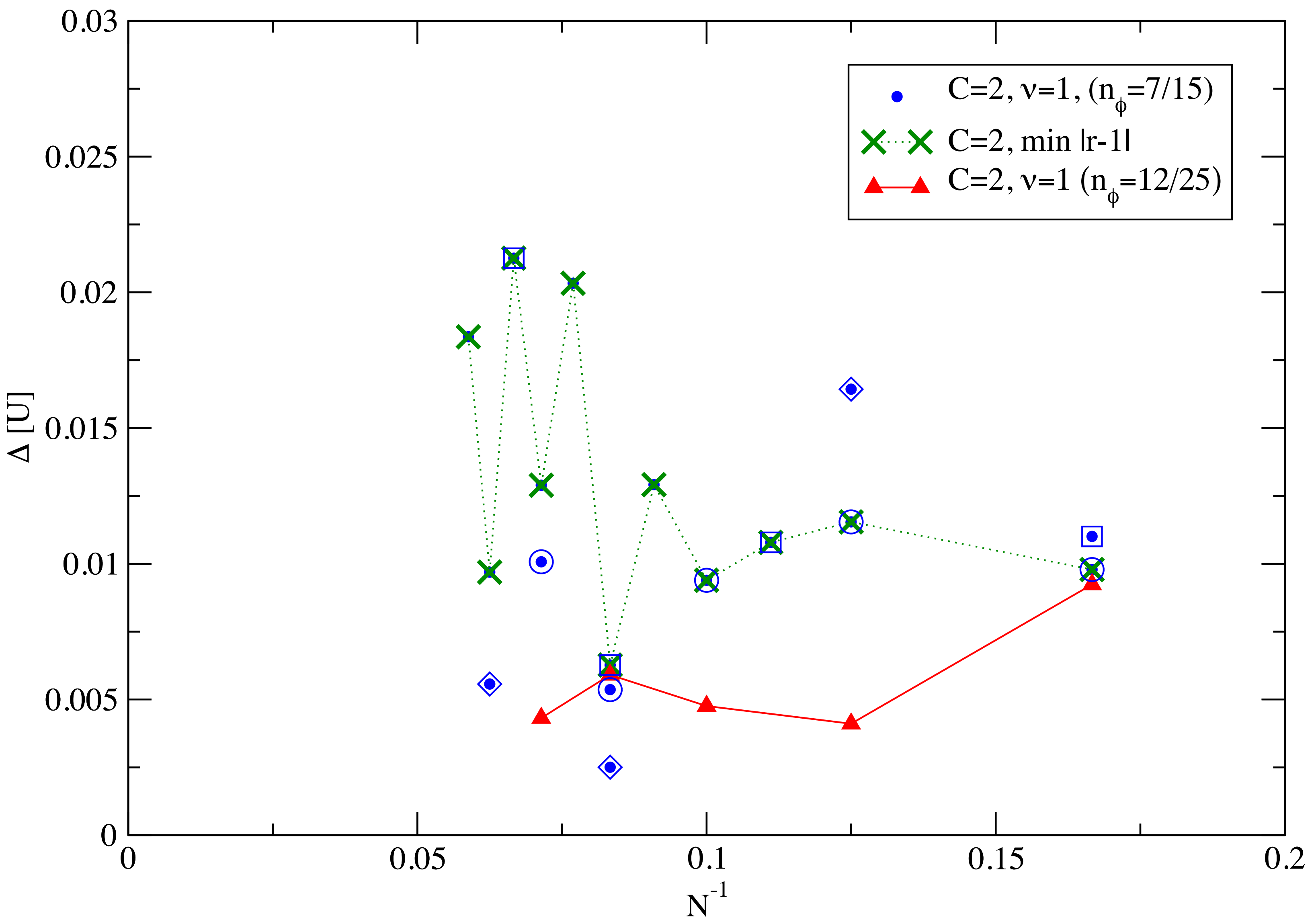
- first evidence in Hofstadter model GM & NR Cooper, PRL (2009)
- quasiparticles are fermions - not fractionalized

⇒ only symmetry protected topological phase [Senthil & Levin, PRL (2013)]

GM & NR Cooper, PRL (2009) & PRL (2015), arXiv:1504.06623; Hormozi et al. PRL 2012

A Special case - Bosonic IQHE in C=2 bands

Many-body gap: finite-size scaling at fixed flux density



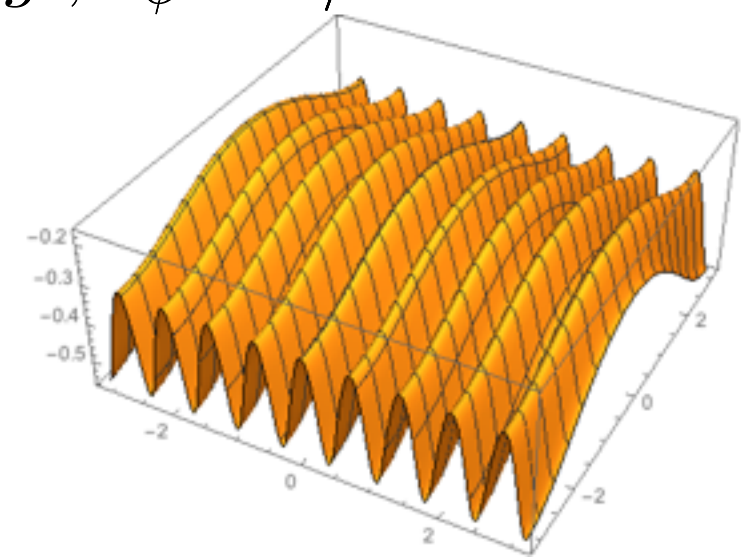
- significant geometry-dependency - but less so for flatter bands.

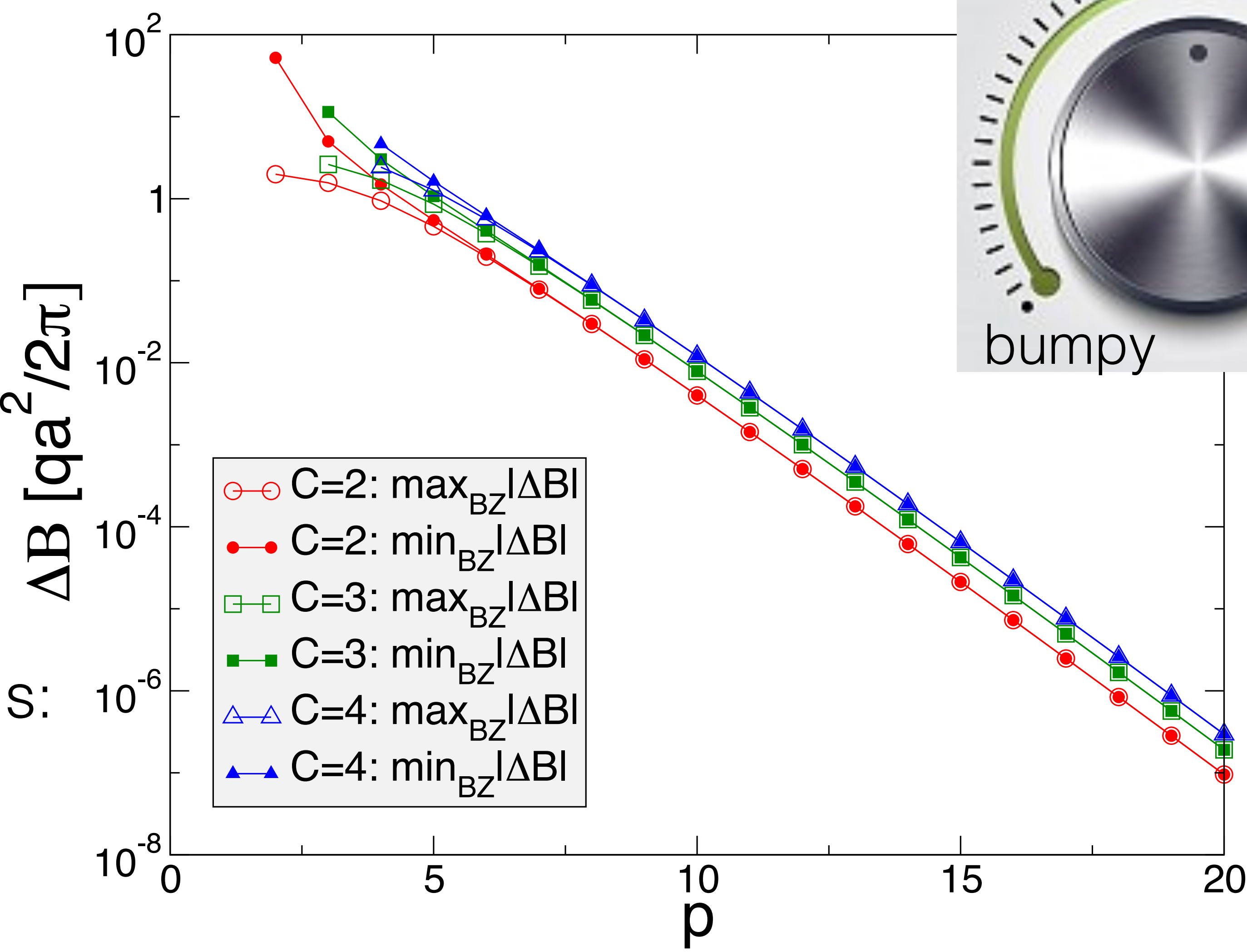
GM & NR Cooper, PRL (2009) & arXiv:1504.06623; Hormozi et al. PRL 2012

Tuning band flatness in the Hofstadter spectrum

Berry curvature exponentially flat in proximity to $n_\phi = 1/|C|$

$e.g., n_\phi = 4/9$





general case for single bands:

$$n_\phi = \frac{p}{|C|p - \mathrm{sgn}(C)}, \quad p \in \mathbb{N}$$

- can tune flatness of band geometry while keeping same physics

GM & NR Cooper, arXiv:1504.06623; Bauer, Jackson & Roy arxiv:1504.07185

Part II: Band Geometry & Stability of Fractional Chern Insulators

How to decide which lattice models have stable fractional Chern Insulators?

- single-particle dispersion - want flat bands

  many groups

  finite size matter a lot - success by iDMRG A. Grushin et al.

- shape of interactions - clear hierarchy of two-body energies desirable "Pseudopotentials"

  Läuchli, Liu, Bergholtz, Moessner + other proposals

- band geometry - ideally want even Berry curvature

  Regnault, Bernevig; Dobardzic, Milovanovic, …

  systematic study of geometric measures beyond Berry curvature

This Talk!

- Full story: all three aspects contribute

Which Berry Curvature?

Gauge invariance of the Bloch functions: one arbitrary U(1) phase for each **k**-point

$$|u^\alpha_{\mathbf{k}}\rangle \to e^{i\phi_\alpha(\mathbf{k})}|u^\alpha_{\mathbf{k}}\rangle$$

The above manifestly leaves H invariant:

$$H_{bc}(\mathbf{k}) = \sum_{\alpha=1}^{\mathcal{N}} E_\alpha(\mathbf{k}) u_b^{\alpha *}(\mathbf{k}) u_c^\alpha(\mathbf{k})$$

However, sublattice dependent phases are *not gauges*:

$$u_a^\alpha(\mathbf{k}) \to \widetilde{u}_b^\alpha(\mathbf{k}) = e^{i\mathbf{r}_b\cdot\mathbf{k}} u_b^\alpha(\mathbf{k})$$

as this substitution yields a modified Berry curvature:

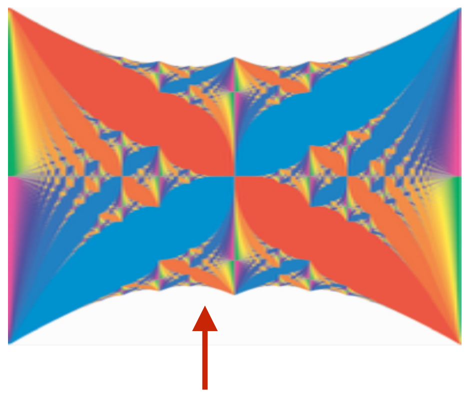
$$\widetilde{B}_\alpha(\mathbf{k}) - B_\alpha(\mathbf{k}) = \sum_{b=1}^{\mathcal{N}} r_{b,y}\frac{\partial}{\partial k_x}|u_b^\alpha(\mathbf{k})|^2 - r_{b,x}\frac{\partial}{\partial k_y}|u_b^\alpha(\mathbf{k})|^2$$

There is a unique choice such that the polarisation reduces to the correct semi-classical expression

and canonical position operator $\hat{R}_\mu \to -i\dfrac{\partial}{\partial k_\mu}$

see, e.g. Zak PRL (1989)

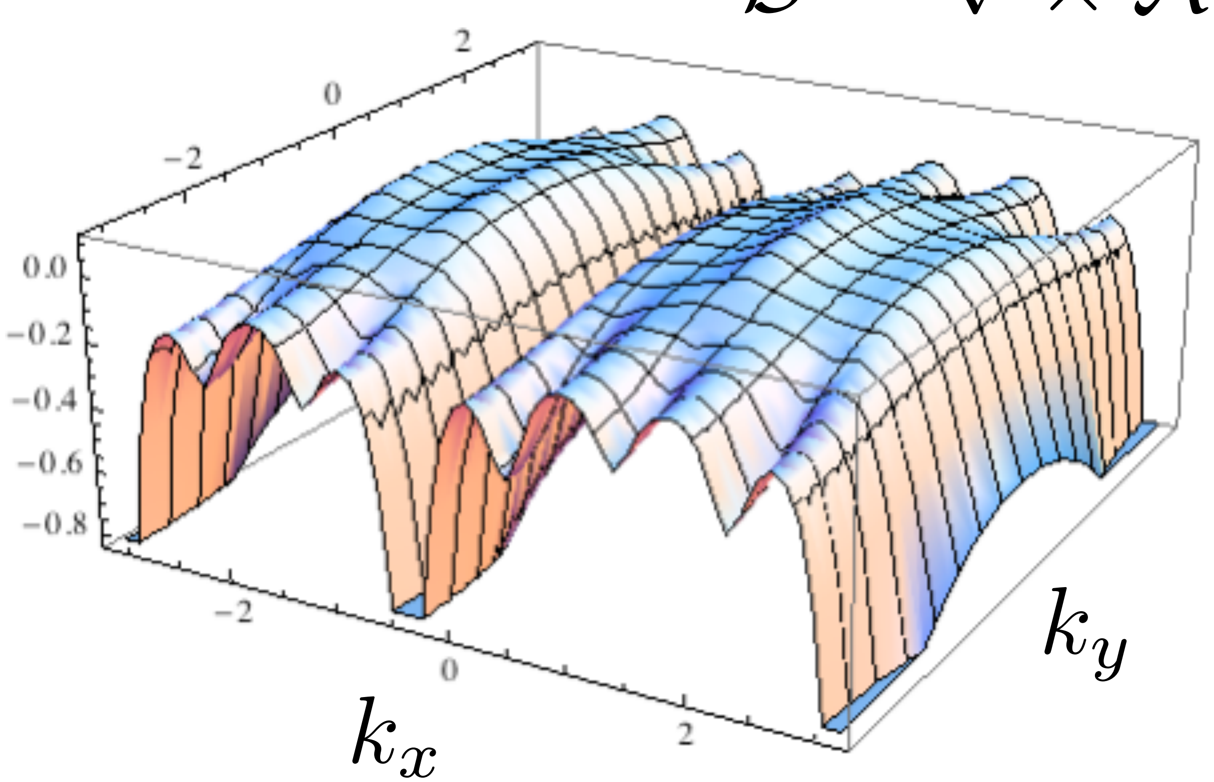
Example single particle properties



an example: Hofstadter spectrum in magnetic unit cell of 7x1, $n_\phi = 3/7$

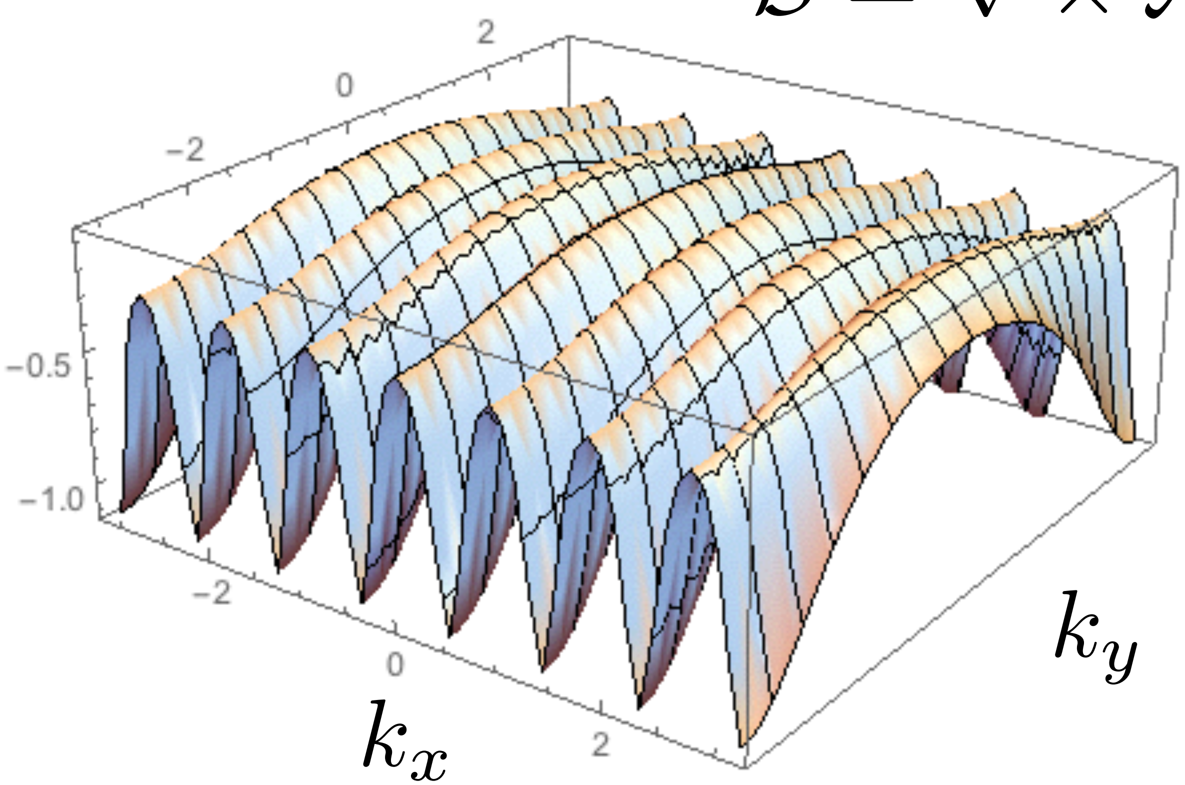
Curvature for Fourier transform with respect to unit cell position

$$\tilde{\mathcal{B}} = \nabla \times \tilde{\mathcal{A}}$$



Curvature for canonical Fourier transform

$$\mathcal{B} = \nabla \times \mathcal{A}$$



Magnetic unit cell

$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$-\frac{6\Phi}{7}$

net flux defined only mod Φ_0

GMP Algebra: Generating low-lying excitations

Girvin, MacDonald and Platzman, PRB **33**, 2481 (1986).

- single mode approximation captures low-lying neutral excitations in quantum Hall systems:

$$|\Psi_{\mathbf{k}}^{\mathrm{SMA}}\rangle = \hat{\rho}_{\mathbf{k}}|\Psi_0\rangle$$

for sp density operators $\hat{\rho}_{\mathbf{k}} = \sum_q \hat{\gamma}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{\gamma}_{\mathbf{q}}$

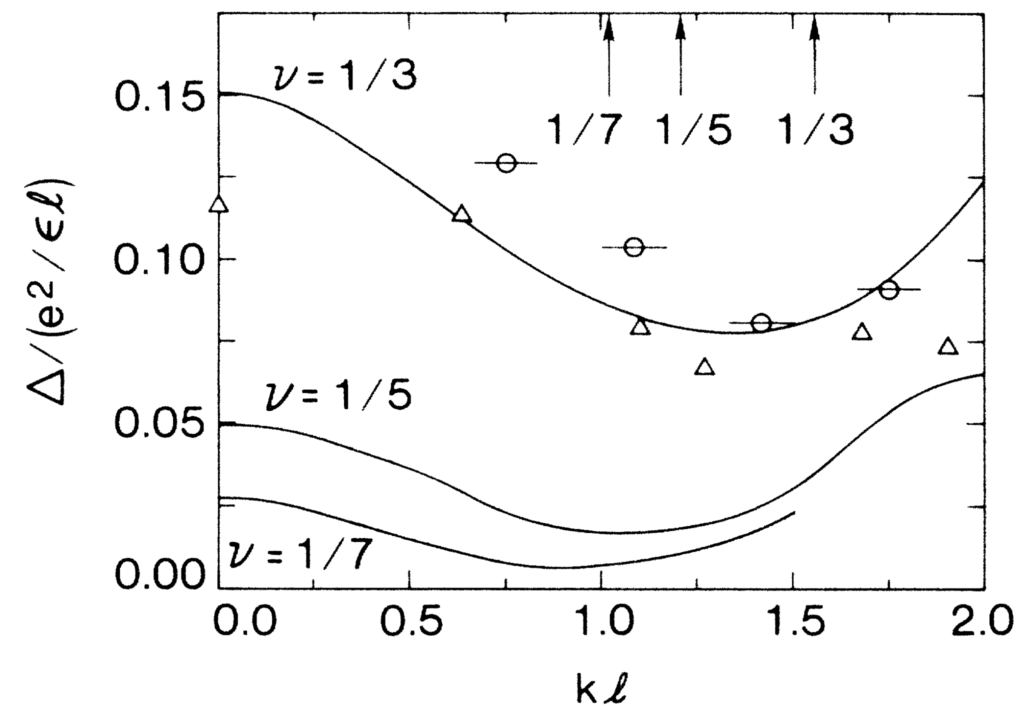


FIG. 4. Comparison of SMA prediction of collective mode energy for $\nu=\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ with numerical results of Haldane and Rezayi (Ref. 20) for $\nu=\frac{1}{3}$. Circles are from a seven-particle spherical system. Horizontal error bars indicate the uncertainty

GMP algebra (w/LLL form factor):

$$[\rho_{\mathrm{LLL}}(\mathbf{q}), \rho_{\mathrm{LLL}}(\mathbf{q}')] = 2i\sin\left(\tfrac{1}{2}\mathbf{q}\wedge\mathbf{q}'\ell_B^2\right)\exp\left(\tfrac{1}{2}\mathbf{q}\cdot\mathbf{q}'\ell_B^2\right)\rho_{\mathrm{LLL}}(\mathbf{q}+\mathbf{q}')$$

SMA carries over to Chern bands: Repellin, Neupert, Papić, Regnault, Phys. Rev. B 90 (2014)

Chern bands: generalised GMP algebra

- consider band-projected density operators for general Chern bands:

$$\widetilde{\rho}_{\mathbf{q}} \equiv P_\alpha e^{i\mathbf{q}\cdot\widehat{\mathbf{r}}} P_\alpha = \sum_{\mathbf{k}} \sum_{b=1}^{\mathcal{N}} u_b^{\alpha *}(\mathbf{k}+\mathbf{q}/2) u_b^{\alpha}(\mathbf{k}-\mathbf{q}/2) \gamma^{\alpha\dagger}_{\mathbf{k}+\mathbf{q}/2} \gamma^{\alpha}_{\mathbf{k}-\mathbf{q}/2}$$

- in general, the algebra of density operators does not close, i.e.

$$[\widetilde{\rho}_{\mathbf{q}}, \widetilde{\rho}_{\mathbf{k}}] \neq F(\mathbf{k},\mathbf{q}) \widetilde{\rho}_{\mathbf{k}+\mathbf{q}}$$

- intuitive consequences for FQH states:
  - no finite, closed set of low-energy excitations corresponding to the GMP single mode states
  - $\widetilde{\rho}_{\mathbf{q}}$ can generate many distinct eigenstates
  - strong violation of the algebra should signal an unstable, gapless phase

Conditions for closure of the generalised GMP algebra I

- conditions for closure can be derived in long-wavelength expansion

i) $\mathcal{O}(k^2)$:

$$\sigma_c \equiv \sqrt{\frac{A_{BZ}^2}{4\pi^2}\langle B^2\rangle - c_1^2}$$

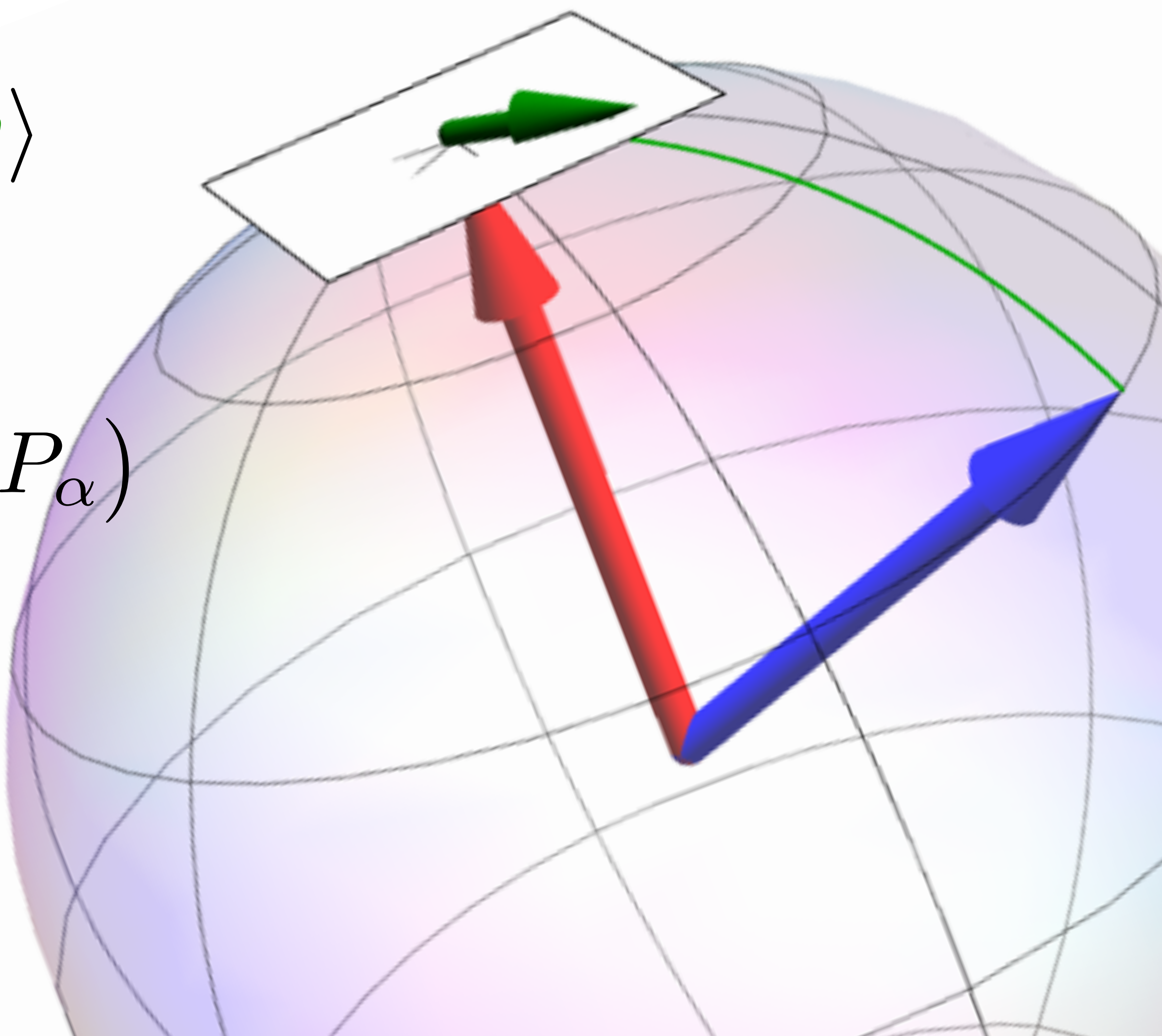
flatness of Berry curvature

ii) $\mathcal{O}(k^3)$: Pullback of Hilbert space metric constant over BZ

$$ds^2 = \langle\delta\psi|\delta\psi\rangle - \langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle$$

$$g_{\mu\nu} + \frac{i}{2}F_{\mu\nu}$$

$$= \sum_{\alpha\in\text{occ}} \text{tr}\,\left(\frac{\partial}{\partial k_\mu}P_\alpha\right)(1-P_\alpha)\left(\frac{\partial}{\partial k_\nu}P_\alpha\right)$$



deviations $\sigma_g \equiv \sqrt{\frac{1}{2}\sum_{\mu,\nu}\langle g_{\mu\nu}g_{\nu\mu}\rangle - \langle g_{\mu\nu}\rangle\langle g_{\nu\mu}\rangle}$

Conditions for closure of the generalised GMP algebra II

- single condition in terms of metric g:

$$T(\mathbf{k}) \equiv \mathrm{tr}\ g^{\alpha}(\mathbf{k}) - |B_{\alpha}(\mathbf{k})| = 0$$

- algebra of projected density operators reduces exactly to the GMP algebra

- Now: test how violations of the closure constraints correlate with gap

R. Roy, arxiv:1208.2055 (PRB 2014); Parameswaran, Roy, Sondhi C. R. Physique (2013)

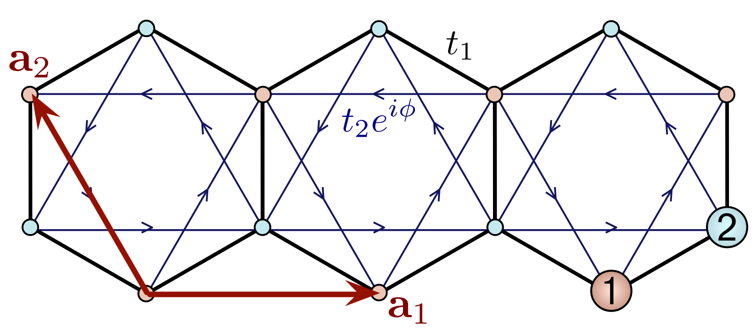
Target models to examine

- Hamiltonian: bosonic states with on-site interactions — defined independent of specific lattice

2-body contact	3-body contact
↓	↓
Laughlin $\nu = \frac{1}{2}$	Moore-Read $\nu = 1$

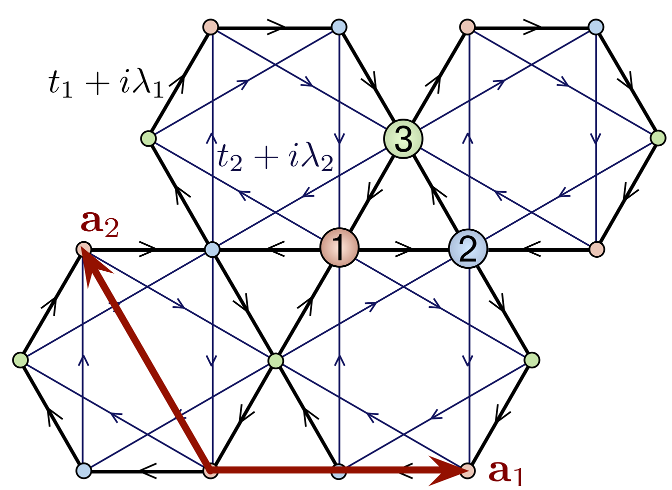
- lattice geometries:

Haldane model



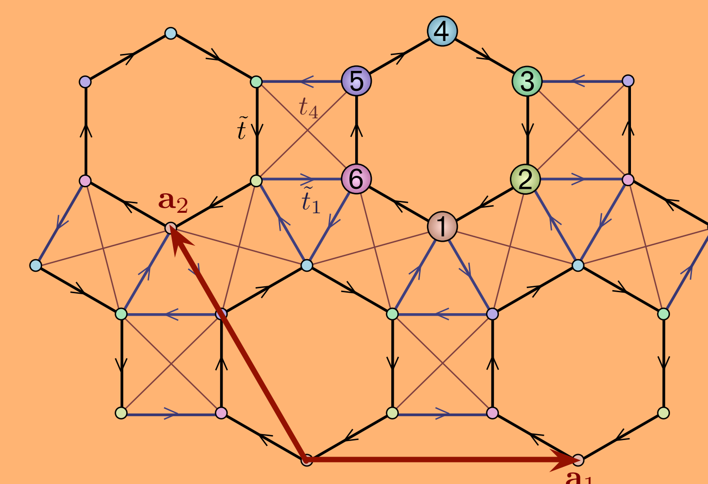
$\mathcal{N} = 2$

Kagomé model



$\mathcal{N} = 3$

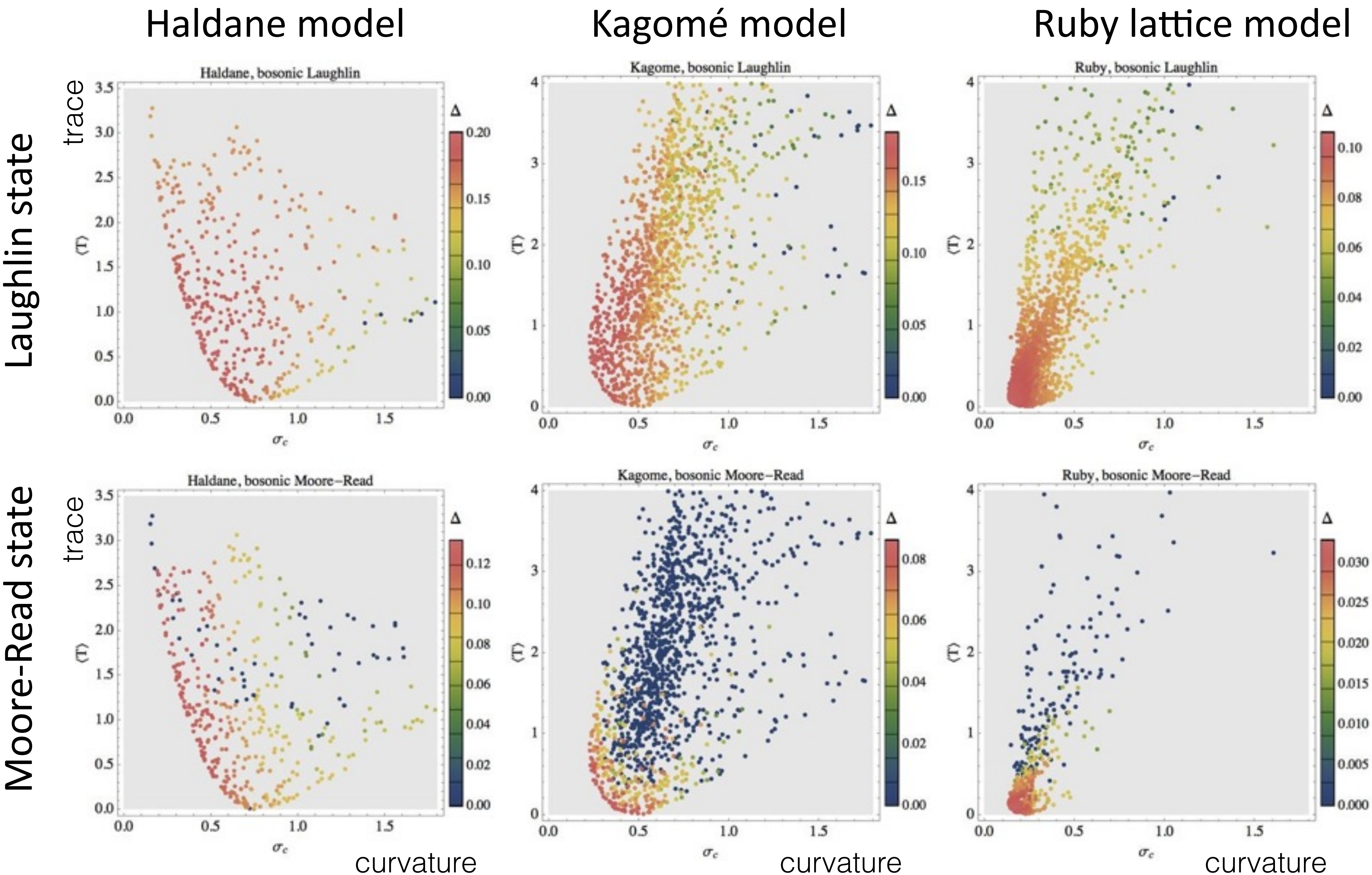
Ruby lattice model



$\mathcal{N} = 6$

other models, see: T. Jackson, GM, R. Roy, Nature Comm. (2015); arxiv:1408.0843

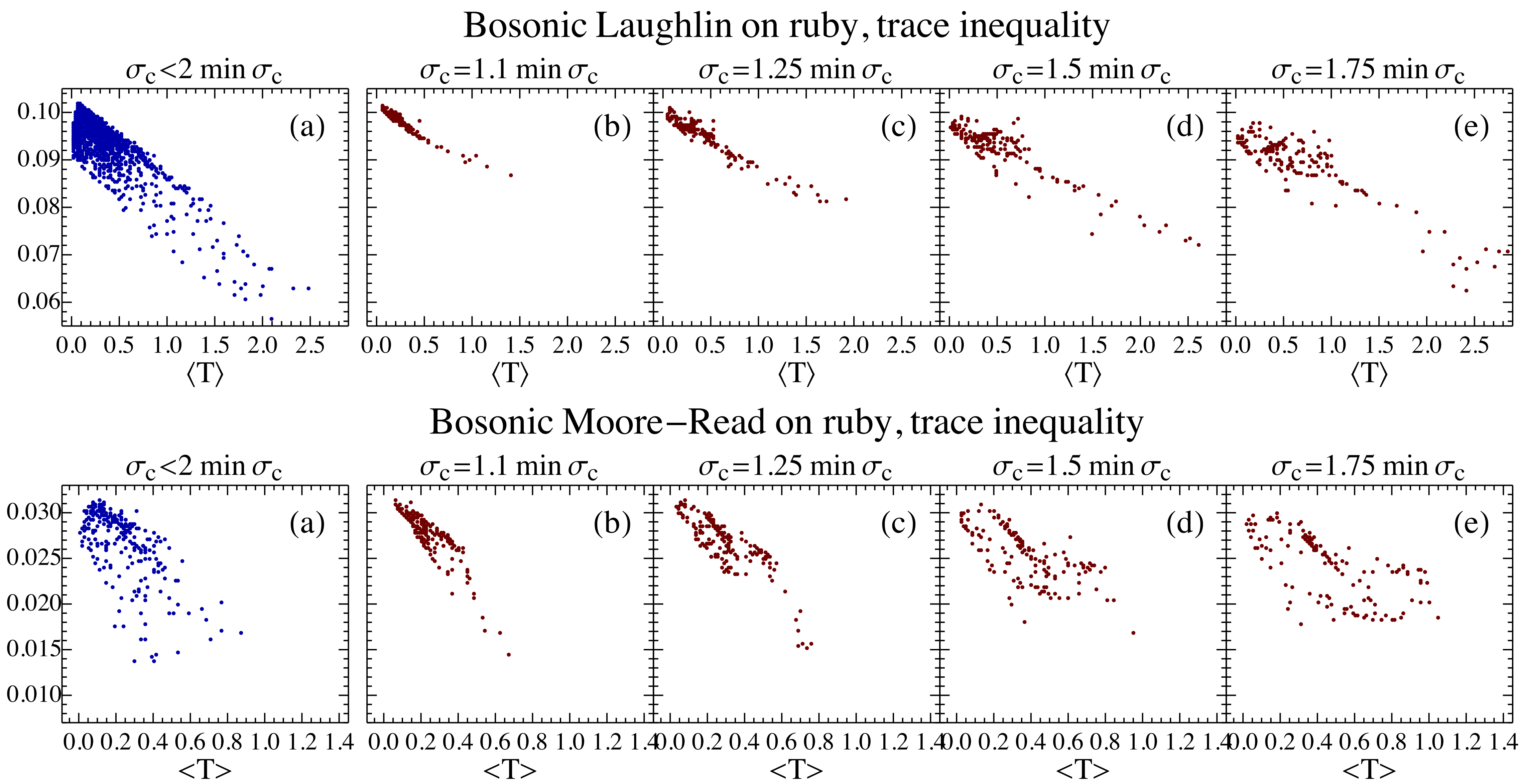
Model Comparison: Gaps vs. RMS B and trace inequality



- Parameters yielding max gap are always in lower-left corner
- Demonstrates relevance of both band-geometric quantities

An Advert: Role of band geometry beyond Berry curvature

Influence of metric tensor g via “trace”: $T(\mathbf{k}) \equiv \mathrm{tr}\ g^{\alpha}(\mathbf{k}) - |B_{\alpha}(\mathbf{k})|$



- Systematic correlation of many-body gap and trace of metric tensor

T. Jackson, GM, R. Roy, Nature Comm. (2015); arxiv:1408.0843

Conclusions

- Composite fermion theory predicts filling factors of stable incompressible phases in general Chern bands at

$$\nu = \frac{r}{r|Ck|+1}$$

[k odd (even) for bosons (fermions)]

- Series includes a Bosonic Integer QHE in C=2 bands

- Numerical evidence matches the predictions (bosons, contact int.):
  - correct GS degeneracy
  - robust gap

- Outlook: expect interesting physics for CF Fermi liquids at |C|>1

$$\lim_{r\to\infty} \nu^{C^*=rC} = \frac{1}{kC}$$

- Though sub-leading to Berry curvature, flatness of Fubini Study metric correlates with magnitude of many-body gap of FCI

GM & NR Cooper, PRL, arXiv:1504.06623; T. Jackson, GM, R. Roy Nat. Comm, arxiv:1408.0843