Numerical evidence for a Bonderson-Slingerland non-abelian hierarchy state at $\nu = 12/5$

7th Symposium on Topological Quantum Computing
Paris March 30th – April 1st

Gunnar Möller
Cavendish Laboratory, University of Cambridge

Parsa Bonderson, Adrian Feiguin
Microsoft Station Q, Santa Barbara

Joost Slingerland
Dias, Dublin

[further contributions by **Arkadiusz Wójs**, Univ. Cambridge]

Overview

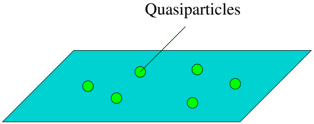
- Introduction & Motivation
  - The Bonderson-Slingerland hierarchy construction (a quick reminder – for details: see Parsa's talk on Monday)
  - Special case considered: $\nu = 12/5$
- Numerical verification of the BS state at $\nu = 12/5$
  - Search for an incompressible state at the shift of BS
  - Analysis of two-point correlation functions of BS
  - Overlaps of the BS and exact ground states
  - Competition between RR, HH, and BS states at $\nu = 12/5$
- Conclusions

Motivation
New trial states from the Bonderson-Slingerland hierarchy construction I

Extend Halperin-Haldane hierarchy construction to the 2nd LL

Hierarchy construction in LLL:

- Concept: ‘Condensation’ of quasiparticles above a mother QH state
- Statistics of qp’s determines the Laughlin-like wavefunctions suitable to describe correlations between quasiparticles



- Iterating condensation of qp’s on subsequent quantum liquids yields states of the HH-hierarchy

Haldane 1983, Halperin 1984

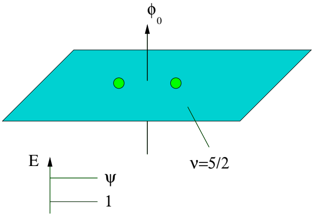
Motivation

New trial states from the Bonderson-Slingerland hierarchy construction II

Additional feature in 2nd LL: non-abelian statistics of qh in the mother-state!

energies for nearby quasiparticles will be split between fusion-channels

- Assume: all pairs of qh's prefer the vaccuum '1'-channel.
- Corresponding quasihole wavefunction is known for the Moore-Read Pfaffian state:



$$\Psi_0(\{w_\alpha\}) = \prod_{\alpha<\beta}(w_\alpha - w_\beta)^{\frac{1}{2}} \prod_{\alpha}^{M}\prod_{k}^{N}(z_k - w_\alpha)\prod_{i<j}(z_i - z_j)^2 \mathrm{Pf}\frac{1}{z_i - z_j}$$

P. Bonderson and J. K. Slingerland, Phys. Rev. B **78**, 067836 (2008).

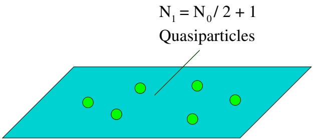
Motivation

New trial states from the Bonderson-Slingerland hierarchy construction III

Specialize to case of $\nu = 12/5$:

- semionic Laughlin qp's of Pfaffian may form liquid state with

$$\Phi_1(\{u_\alpha\}) = \prod_{\alpha<\beta} (u_\alpha - u_\beta)^{\frac{5}{2}}$$



This yields the hierarchy state

$$\Psi_{\nu=\frac{12}{5}}(\{z_i\}) = \int du_1 \dots du_{N_1} \Phi_1^*(u_\alpha)\Psi_0(z_i; u_\alpha)$$
$$\simeq \Psi^{(MR)}_{\nu=1} \times \Psi^{(CF)}_{\nu=\frac{2}{3}}$$

P. Bonderson and J. K. Slingerland, Phys. Rev. B **78**, 067836 (2008).

Motivation

Properties of the Bonderson-Slingerland hierarchy states

- The physics predicted by the BS hierarchy is fundamentally different from that predicted by other models

- For condensation in charge sector, all states inherit the statistics of the underlying mother-state, i.e., they realize Majorana Fermions described by the Ising CFT.

In particular, for $\nu = 12/5$, this implies the competition of three states with different shift S on sphere $[N_\phi = \nu^{-1} N - S]$

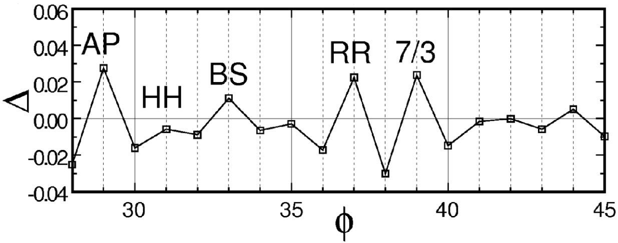
- the $\overline{RR}$ state, shift $S = -2$, $\Rightarrow$ parafermions
- the BS state, shift $S = +2$, $\Rightarrow$ Majorana fermions
- the HH/CF state, shift $S = +4$, $\Rightarrow$ abelian

$\Rightarrow$ Crucial to understand competition

Numerical search for a BS state at $\nu = 12/5$ - I

Charge gap as a function of the shift on sphere

Exact diagonalization / DMRG on sphere



[Data from DMRG for the Coulomb Hamiltonian in a thin layer, $N_e = 14$]

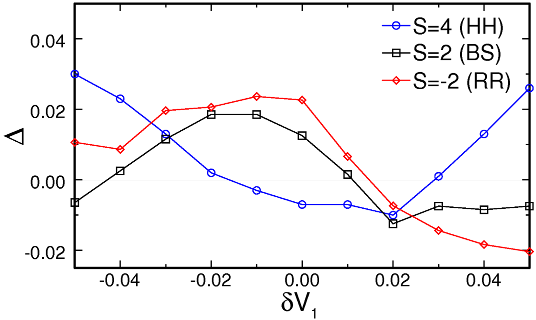
- clearly visible gap $\Delta(N_\phi) = E_{N_\phi+1} + E_{N_\phi-1} - 2E_{N_\phi}$ at the shift of the $\overline{RR}$ and BS states
- small local maximum for HH/CF state.

P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Numerical search for a BS state at $\nu = 12/5$ - II

Perturbation of the interaction around Coulomb

Simplest parametrization of interaction: $V_1^{\text{Coulomb}} \rightarrow V_1 + \delta V_1$



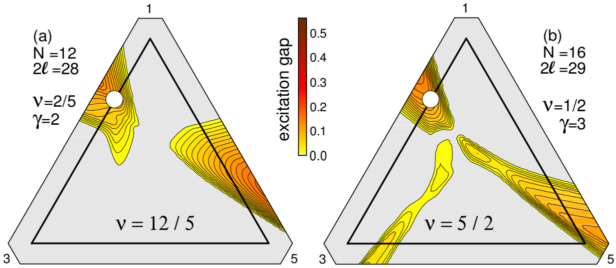
- Both BS and $\overline{RR}$ have a clear gap in region around Coulomb point, shown here for $N = 14$.

Bonderson, Feiguin, Möller and Slingerland, arXiv:0901.4965.

Numerical search for a BS state at $\nu = 12/5$ - III

Parametrization of general interactions

Neutral gap for general interactions U varying (V_1, V_3, V_5)



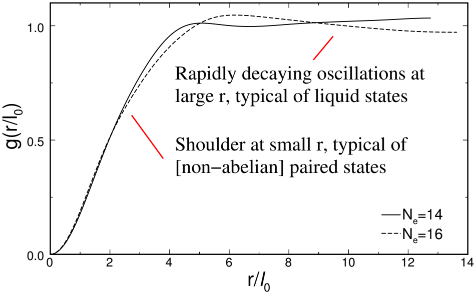
[courtesy: A. Wójs]

- Gap for general U reveals island of stability for the BS state very similar to that of its MR mother-state, and centered around the 2nd LL like potential.

A. Wójs, arXiv:0811.4072.

Numerical search for a BS state at $\nu = 12/5$ - IV
Correlations in the tentative BS state

Pair-correlation function $\langle \Psi^\dagger(\vec{R})\Psi(0)\rangle$ on the sphere



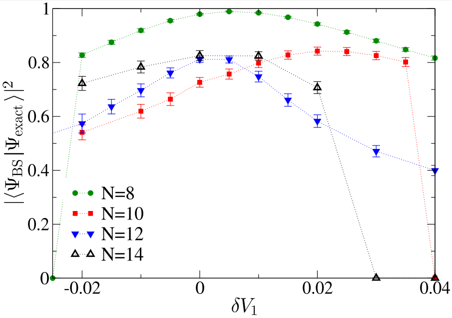
- Correlation function indicative of incompressible state with pairing nature
- Also, angular momentum $L^2 = 0$ for $N = 6, \ldots, 18$.

Bonderson, Feiguin, Möller and Slingerland, arXiv:0901.4965.

Numerical search for a BS state at $\nu = 12/5$ - V

Overlap of the BS state with the exact grondstate

Integrate $\mathcal{O} = \int d(z_1, \ldots, z_N)\Psi^*_{\mathrm{BS}}\Psi_{\mathrm{exact}}$ by Monte-Carlo sampling in position space



- Overlap large: up to 0.82 for $N = 14$ $[D_{L_z=0} \sim 1.9 \times 10^7]$.
- However, knowing that BS derives from the weak-pairing phase at $\nu = 5/2$, could this be improved?

Digression: weakly paired states

The Moore-Read state: one of many representatives in the weakly paired phase

- Moore-Read:
  $\Psi_{\mathrm{MR}} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] \prod_{i<j}(z_i - z_j)^2$
- want explicit expression for general paired state in same universality class!
  (see Read & Green, PRB 2000)
- start from BCS state: $|\mathrm{BCS}\rangle = \prod'_{\mathbf{k}}(u_k + v_k c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}})|0\rangle$
  [variational parameters $u_k, v_k \to g_k = v_k/u_k$]
- in position space: $\langle\{\mathbf{r}_i\}|\mathrm{BCS}\rangle = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_m)}\right]$

- Composite-fermionize BCS: $[\tilde\phi(z_i) = J_i^{-1}\mathcal{P}_{\mathrm{LLL}} J_i \phi(z_i)]$

$$\Psi^{\text{CF-BCS}} = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\, \tilde\phi_{\mathbf{k}}(z_i)\, \tilde\phi_{-\mathbf{k}}(z_j)\right] \prod_{i<j}(z_i - z_j)^2.$$

G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).

Digression: weakly paired states

The Moore-Read state: one of many representatives in the weakly paired phase

- Moore-Read:
  $\Psi_{\mathrm{MR}} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] \prod_{i<j}(z_i - z_j)^2$
- want explicit expression for general paired state in same universality class!
  (see Read & Green, PRB 2000)
  - start from BCS state: $|\mathrm{BCS}\rangle = \prod'_{\mathbf{k}}(u_k + v_k c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}})|0\rangle$
    [variational parameters u_k, $v_k \to g_k = v_k/u_k$]
  - in position space: $\langle\{\mathbf{r}_i\}|\mathrm{BCS}\rangle = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_m)}\right]$

- Composite-fermionize BCS: $[\tilde{\phi}(z_i) = J_i^{-1}\mathcal{P}_{\mathrm{LLL}} J_i \phi(z_i)]$

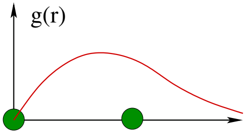
$$\Psi^{\text{CF-BCS}} = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\, \tilde{\phi}_{\mathbf{k}}(z_i)\, \tilde{\phi}_{-\mathbf{k}}(z_j)\right] \prod_{i<j}(z_i - z_j)^2.$$

G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).

Digression: weakly paired states

The Moore-Read state: one of many representatives in the weakly paired phase

- Moore-Read:
  $\Psi_{\mathrm{MR}} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] \prod_{i<j}(z_i - z_j)^2$



- want explicit expression for general paired state in same universality class!
  (see Read & Green, PRB 2000)

  - start from BCS state: $|\mathrm{BCS}\rangle = \prod'_{\mathbf{k}}(u_k + v_k c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}})|0\rangle$
    [variational parameters $u_k,\, v_k \rightarrow g_k = v_k/u_k$]
  - in position space: $\langle\{\mathbf{r}_i\}|\mathrm{BCS}\rangle = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_m)}\right]$

- Composite-fermionize BCS: $[\tilde{\phi}(z_i) = J_i^{-1}\mathcal{P}_{\mathrm{LLL}} J_i \phi(z_i)]$

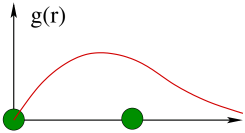
$$\Psi^{\text{CF-BCS}} = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\, \tilde{\phi}_{\mathbf{k}}(z_i)\, \tilde{\phi}_{-\mathbf{k}}(z_j)\right] \prod_{i<j}(z_i - z_j)^2.$$

G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).

Digression: weakly paired states

The Moore-Read state: one of many representatives in the weakly paired phase

- Moore-Read:
  $\Psi_{\mathrm{MR}} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] \prod_{i<j}(z_i - z_j)^2$
- want explicit expression for general paired state in same universality class!
  (see Read & Green, PRB 2000)



- start from BCS state: $|\mathrm{BCS}\rangle = \prod'_{\mathbf{k}}(u_k + v_k c^{\dagger}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}})|0\rangle$
  [variational parameters u_k, v_k $\rightarrow$ $g_k = v_k/u_k$]
- in position space: $\langle\{\mathbf{r}_i\}|\mathrm{BCS}\rangle = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_m)}\right]$

- Composite-fermionize BCS: $[\tilde{\phi}(z_i) = J_i^{-1}\mathcal{P}_{\mathrm{LLL}} J_i \phi(z_i)]$

$$\Psi^{\text{CF-BCS}} = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\, \tilde{\phi}_{\mathbf{k}}(z_i)\, \tilde{\phi}_{-\mathbf{k}}(z_j)\right] \prod_{i<j}(z_i - z_j)^2.$$

G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).

Digression: weakly paired states

Apply concept of general pair wavefunctions to BS wavefunction

- Bonderderson-Slingerland states derive from the weakly paired states at $\nu = 5/2 \Rightarrow$ make use of variational degrees of freedom in its pair wavefunction

previously: $\Psi_{\frac{2}{5}}^{(\mathrm{BS})} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] \prod_{i<j}(z_i - z_j)\Psi_{\frac{2}{3}}^{(\mathrm{CF})}$

with generalized pair wavefunction:

$\Rightarrow \Psi_{\frac{2}{5}}^{(\mathrm{BS})}[g_k] = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\, \tilde{\phi}_{\mathbf{k}}(z_i)\, \tilde{\phi}_{-\mathbf{k}}(z_j)\right] \prod_{i<j}(z_i - z_j)\Psi_{\frac{2}{3}}^{(\mathrm{CF})}$,

with the projected CF orbitals $\tilde{\phi}(z_i) = J_i^{-1}\mathcal{P}_{\mathrm{LLL}} J_i \phi(z_i)$,

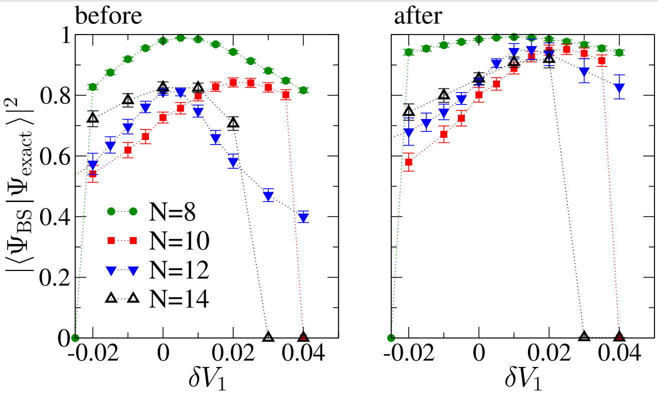
and with $\Psi_{\frac{2}{3}}^{(\mathrm{CF})}$ generated from CF in negative flux.

P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.
G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).
G. Möller and S. H. Simon, Phys. Rev. B **72**, 045344 (2005).

Numerical search for a BS state at $\nu = 12/5$ continued

More overlaps for the BS states with general pairing

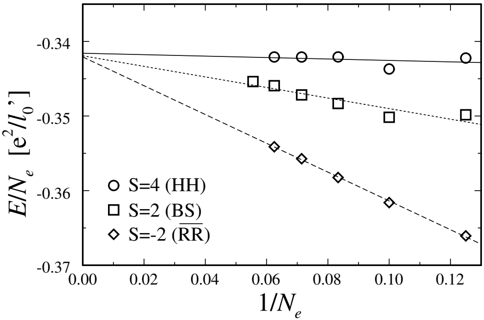
Overlaps in Monte-Carlo simulations, with optimization of $\{g_{\mathbf{k}}\}$



- Overlaps further increased: up to 0.92 for $N = 14$.
- Number of variational parameters on sphere small (≤ 5)

Competition of different trial states at $\nu = 12/5$

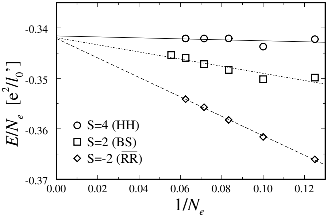
Having established the $\nu = 12/5$ state with shift $S = 2$ as a BS state: $\Rightarrow$ now study competition between different candidate states



- find $E/N_e = -0.3416(5)$, $-0.342(3)$, and $-0.3421(5)$ for $S = HH$, BS, and $\overline{RR}$ using $N \geq 12$.

$\Rightarrow$ very close competition, cannot confidently distinguish states

Competition of different trial states at $\nu = 12/5$ – discussion



Recapitulate

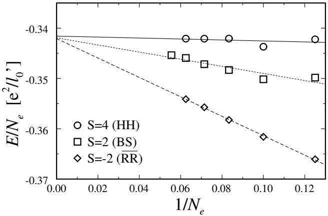
- $e_{HH} = -0.3416(5)$
- $e_{BS} = -0.342(3)$
- $e_{RR} = -0.3421(5)$

- Estimate of energies, including their order, susceptible to details of extrapolation (linear/quadratic, system sizes, etc.)
- Additional physical effects as Landau-level mixing and finite width likely to determine state that champions competition
- Torus data mostly supports $\overline{RR}$, but also indicates proximity of BS state

Both $\overline{RR}$ and BS can potentially be realized at $\nu = 12/5$, depending on details of sample geometry

P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Competition of different trial states at $\nu = 12/5$ – discussion



Recapitulate

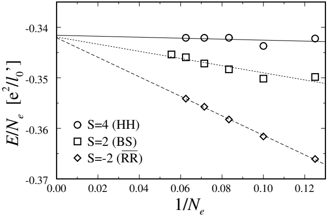
- $e_{HH} = -0.3416(5)$
- $e_{BS} = -0.342(3)$
- $e_{RR} = -0.3421(5)$

- Estimate of energies, including their order, susceptible to details of extrapolation (linear/quadratic, system sizes, etc.)
- Additional physical effects as Landau-level mixing and finite width likely to determine state that champions competition
- Torus data mostly supports $\overline{RR}$, but also indicates proximity of BS state

Both $\overline{RR}$ and BS can potentially be realized at $\nu = 12/5$, depending on details of sample geometry

P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Competition of different trial states at $\nu = 12/5$ – discussion



Recapitulate

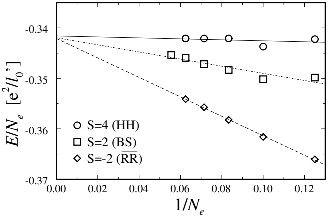
- $e_{HH} = -0.3416(5)$
- $e_{BS} = -0.342(3)$
- $e_{RR} = -0.3421(5)$

- Estimate of energies, including their order, susceptible to details of extrapolation (linear/quadratic, system sizes, etc.)
- Additional physical effects as Landau-level mixing and finite width likely to determine state that champions competition
- Torus data mostly supports $\overline{RR}$, but also indicates proximity of BS state

Both $\overline{RR}$ and BS can potentially be realized at $\nu = 12/5$, depending on details of sample geometry

P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Competition of different trial states at $\nu = 12/5$ – discussion



Recapitulate

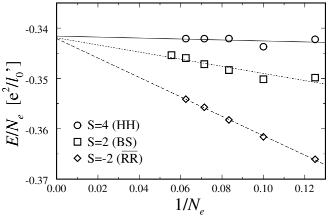
- $e_{\mathrm{HH}} = -0.3416(5)$
- $e_{\mathrm{BS}} = -0.342(3)$
- $e_{\mathrm{RR}} = -0.3421(5)$

- Estimate of energies, including their order, susceptible to details of extrapolation (linear/quadratic, system sizes, etc.)
- Additional physical effects as Landau-level mixing and finite width likely to determine state that champions competition
- Torus data mostly supports $\overline{RR}$, but also indicates proximity of BS state

Both $\overline{RR}$ and BS can potentially be realized at $\nu = 12/5$, depending on details of sample geometry

P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Competition of different trial states at $\nu = 12/5$ – discussion



Recapitulate

- $e_{HH} = -0.3416(5)$
- $e_{BS} = -0.342(3)$
- $e_{RR} = -0.3421(5)$

- Estimate of energies, including their order, susceptible to details of extrapolation (linear/quadratic, system sizes, etc.)
- Additional physical effects as Landau-level mixing and finite width likely to determine state that champions competition
- Torus data mostly supports $\overline{RR}$, but also indicates proximity of BS state

Both $\overline{RR}$ and BS can potentially be realized at $\nu = 12/5$, depending on details of sample geometry

P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Conclusions

- The Bonderson-Slingerland hierarchy construction predicts a $\nu = 12/5$ state with shift $S = +2$ on the sphere
- Multiple pieces of evidence establish this state as a robust incompressible state described by the BS wavefunction:
  - Clear neutral and charge gap for $N = 6, \ldots, 18$
  - 'Nice' correlations and $\langle L^2 \rangle = 0$ for all states of the series
  - Large overlap with the BS trial wavefunction
- Extent of the BS state in interaction space similar to that of the $\nu = 5/2$ state from which it derives.
- General pair-wavefunctions further increase overlaps
- Energetically competitive with $\overline{RR}$ (and HH), outcome may depend on finer experimental details:

P. Bonderson and J. K. Slingerland, Phys. Rev. B **78**, 067836 (2008).
P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Experimental signatures of competing trial states at $\nu = 12/5$

Summary of possible experimental probes:

	RR	BS	HH
qp charge	$\frac{e}{5}$	$\frac{e}{5}$	$\frac{2e}{5}$
weak tunnelling ($I \sim V^{4\Delta_{qp}-1}$)	$\Delta_{qp} = \frac{1}{5}$	$\Delta_{qp} = \frac{9}{80}$	$\Delta_{qp} = \frac{1}{5}$
strong tunnelling ($G \sim T^{4\Delta_e - 2}$)	$\Delta_e = 2$	$\Delta_e = \frac{3}{2}$	$\Delta_e = \frac{3}{2}$
braiding	Z_3 parafermions	Ising	abelian

- Distinguishing tunnelling exponents for edge states difficult
- Interferometry could clearly distinguish braiding statistics

W. Bishara, G. A. Fiete, C. Nayak, Phys. Rev. B **77**, 241306(R) (2008).
P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.

Experimental signatures of competing trial states at $\nu = 12/5$

Summary of possible experimental probes:

	RR	BS	HH
qp charge	$\frac{e}{5}$	$\frac{e}{5}$	$\frac{2e}{5}$
weak tunnelling $(I \sim V^{4\Delta_{qp}-1})$	$\Delta_{qp} = \frac{1}{5}$	$\Delta_{qp} = \frac{9}{80}$	$\Delta_{qp} = \frac{1}{5}$
strong tunnelling $(G \sim T^{4\Delta_e-2})$	$\Delta_e = 2$	$\Delta_e = \frac{3}{2}$	$\Delta_e = \frac{3}{2}$
braiding	Z_3 parafermions	Ising	abelian

- Distinguishing tunnelling exponents for edge states difficult
- Interferometry could clearly distinguish braiding statistics

W. Bishara, G. A. Fiete, C. Nayak, Phys. Rev. B **77**, 241306(R) (2008).
P. Bonderson, A. Feiguin, G. Möller and J. Slingerland, arXiv:0901.4965.