UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 24 April 2024, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer ***all four*** *questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains 4 sides, excluding this one, and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof. (The booklet is available for separate download.)

V7.1

1 (a) Consider a Hermitian eigensystem $H(\mathbf{g})|u_n(\mathbf{g})\rangle = E_n(\mathbf{g})|u_n(\mathbf{g})\rangle$, which depends parametrically on g^μ, $\mu = 1, ..., N$. Starting from the overlap of two infinitessimally close states in parameter space,

$$\langle u_n(\mathbf{g})|u_n(\mathbf{g}+\delta\mathbf{g})\rangle = |\langle u_n(\mathbf{g})|u_n(\mathbf{g}+\delta\mathbf{g})\rangle| \cdot e^{i\boldsymbol{\mathcal{A}}_n(\mathbf{g})\cdot\delta\mathbf{g}},$$

show that, upon leading order in $\delta\mathbf{g}$, $\mathcal{A}_n^a = \frac{\langle u_n(\mathbf{g})|\nabla_g^a u_n(\mathbf{g})\rangle - \langle\nabla_g^a u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}{2i\langle u_n(\mathbf{g})|u_n(\mathbf{g}\rangle}$. Assuming that the states are normalised, show this gives the Berry potential,

$$\mathcal{A}_n^a = -i\langle u_n(\mathbf{g})|\nabla_g^a u_n(\mathbf{g})\rangle. \qquad (1)$$

[6]

(b) We now take a specific class of Hamiltonians of the above form,

$$H(\mathbf{q}) = v(q_x\sigma_x + q_y\sigma_y + q_z\sigma_z),$$

where $\mathbf{q}$ is assumed to be an effective momentum, σ_i are the Pauli matrices and v is a velocity. What happens at $\mathbf{q} = 0$? Show that $H(\mathbf{q})$ has two eigenenergies $E_\pm = \pm v|\mathbf{q}|$. [3]

(c) Show that the eigensstate $|+\rangle$, with $H|+\rangle = E_+|+\rangle$, is given by

$$|+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{pmatrix},$$

where we rewrote $(q_x, q_y, q_z) = (|\mathbf{q}|\sin\theta\cos\varphi, |\mathbf{q}|\sin\theta\sin\varphi, |\mathbf{q}|\cos\theta)$. Give also an expression for $|-\rangle$, where $H|-\rangle = E_-|-\rangle$. [6]

(d) Show that the Berry potential $\mathcal{A}_+^a = -i\langle +|\nabla^a +\rangle = \frac{\sin^2(\theta/2)\hat{\boldsymbol{\varphi}}}{|\mathbf{q}|\sin\theta}$ and that the Berry curvature, $\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}_+$, equates to $\frac{\hat{\boldsymbol{q}}}{2|\mathbf{q}|^2}$. [5]

(e) What is the result when we integrate the Berry curvature over a sphere of constant $\mathbf{q}$? Give an interpretation. How does the result change when we would instead consider the Hamiltonian $\tilde{H}(\mathbf{q}) = v(q_x\sigma_x + q_y\sigma_y - q_z\sigma_z)$? Motivate your answer [a calculation is not directly needed]. How do the results change when we make a gauge transformation, $|+\rangle \rightarrow e^{i\beta(\mathbf{q})}|+\rangle$? [5]

Solution 1. (a) *[Derivation addressed in Lecture.]*

We have

$$\frac{\langle u_n(\mathbf{g})|u_n(\mathbf{g}+\delta\mathbf{g})\rangle}{|\langle u_n(\mathbf{g})|u_n(\mathbf{g}+\delta\mathbf{g})\rangle|} = e^{\boldsymbol{\mathcal{A}}_n(\mathbf{g})\cdot\delta\mathbf{g}}.$$

We expand in g to get $|u_n(\mathbf{g}+\delta\mathbf{g})\rangle = |u_n(\mathbf{g}\rangle + \delta\mathbf{g}\cdot\nabla|u_n(\mathbf{g}\rangle + \dots$, which we write as $|u_n(\mathbf{g}\rangle + \delta|u_n(\mathbf{g}\rangle + \dots$

(1) Expanding the right hand side gives $1 + i\boldsymbol{\mathcal{A}}_n(\mathbf{g})\cdot\delta\mathbf{g} + \cdots$

(2) Expanding the left hand side gives

$$(\langle u_n(\mathbf{g})|u_n(\mathbf{g}\rangle + \langle u_n(\mathbf{g})|\delta u_n(\mathbf{g}\rangle + \dots)(\frac{1}{\sqrt{\langle u_n(\mathbf{g})|u_n(\mathbf{g}) + \delta u_n(\mathbf{g}) + \dots\rangle\cdot\langle u_n(\mathbf{g}) + \delta u_n(\mathbf{g}) + \dots|u_n(\mathbf{g})\rangle}}.$$

Writing the left bracket of the above expression as

$$\frac{1}{\sqrt{(\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle)^2}\cdot\sqrt{1+\frac{\langle u_n(\mathbf{g})|\delta u_n(\mathbf{g})\rangle}{\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}+\frac{\langle \delta u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}{\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}+\dots}}$$

we obtain

$$(1+\frac{\langle u_n(\mathbf{g})|\delta u_n(\mathbf{g})\rangle}{\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}+\dots)\cdot\frac{1}{\sqrt{1+\frac{\langle u_n(\mathbf{g})|\delta u_n(\mathbf{g})\rangle}{\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}+\frac{\langle \delta u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}{\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}+\dots}}.$$

Writing $x = \frac{\langle u_n(\mathbf{g})|\delta u_n(\mathbf{g})\rangle}{\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}+\frac{\langle \delta u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}{\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}$ and using that $\frac{1}{\sqrt{1+x}} = 1 - x/2 + \dots$ and comparing order of the left and right hand side of the top equation we get the desired result for the first order. Using $\delta\langle u_n|u_n\rangle = 0 = \langle\delta u_n|u_n\rangle + \langle u_n|\delta u_n\rangle$, we can see the result reduces to the Berry potential.

(b) *[Unseen but very similar to derivation addressed in lecture and notes.]* This is Hamiltonian for a Weyl node at $\mathbf{q} = 0$. There is a singularity and the energies are gapped around $\mathbf{q} = 0$. Given the structure of the Pauli matrices one can square both sides to get the eigenenergies.

(c) *[Very similar to derivation addressed in lecture and notes.]* We rewrite the Hamiltonian as

$$H(\mathbf{q}) = \begin{pmatrix} q_z & q_x - iq_y \\ q_x + iq_y & -q_z \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}.$$

Using the double angle formula it is easily checked that $H|+\rangle = E_+|+\rangle$, an expression for $|-\rangle$ is

$$|-\rangle = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\varphi}\cos(\theta/2) \end{pmatrix}.$$

(d) *[Very similar to derivation addressed in lecture and notes.]* Using $\nabla = \hat{\boldsymbol{r}}\partial_r + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta + \frac{\hat{\boldsymbol{\varphi}}}{\sin\theta}\partial_\varphi$ and using the expression for $|+\rangle$ we directly obtain the result. As the Berry potential only depends on θ and only has a $\hat{\boldsymbol{\varphi}}$ component the curvature has one term,

$$\begin{aligned}\nabla \times \boldsymbol{\mathcal{A}}_{+} &= \frac{1}{|\mathbf{q}|\sin(\theta)}\partial_{\theta}[\sin(\theta)\frac{\sin^2\theta/2}{|\mathbf{q}|\sin\theta}]\hat{\mathbf{r}} \\ &= \frac{\hat{\mathbf{q}}}{\mathbf{q}^2}.\end{aligned}$$

(e) *[Very similar to derivation addressed in lecture and notes.]* There is a single monopole as seen in the lecture, hence it gives 1 in units of 2π. For the other Hamiltonian the result is the opposite as the chirality is reversed. Under gauge transformations, by analogy to $U(1)$ electromagnetism, the Berry curvature is unchanged but the potential transforms as $\boldsymbol{\mathcal{A}}_{\pm} \rightarrow \boldsymbol{\mathcal{A}}_{\pm} + \boldsymbol{\nabla}\beta\mathbf{q}$.

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2 Consider a very thin and long wire of length L and width W, smoothly connected to a reservoir of electrons on both ends. The reservoirs at the left and right end are kept at different electrostatic potentials $V = V_R - V_L$.

(a) Write down the energy eigenvalues and eigenstates of the electron wave-functions in the wire.

[*hint: You can assume a rectangular shaped wire and combine the quantum particle in an infinite well problem for the transverse direction y with a free-particle in the longitudinal direction x.*] [4]

(b) For each eigenstate calculate the electrical current density $j_{n,k}(x,y)$ and current $\mathcal{I}_{n,k}$. Here n, k are the eigenstate labels. The total current through the wire is

$$I = 2\frac{L}{2\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} dk \mathcal{I}_{n,k}[f(\epsilon_{n,k} + eV_L)\theta(k) + f(\epsilon_{n,k} + eV_R)\theta(-k)], \tag{2}$$

where $f(x)$ is the Fermi distribution function and $\theta(x)$ is the Heaviside step function. On physical grounds, justify the presence of these two functions and the first factor of 2 on the RHS of Eq. (2) . [6]

(c) Now take the zero temperature limit, and show that the conductance of the wire is given by $G = 2Ne^2/h$, where N is the number of occupied energy levels n (also referred to as open channels). Why is conductivity not infinite? Where is the associated energy loss happening?

[*hint: For the first part, it is convenient to perform integral over energy instead of k. For this try to express $\mathcal{I}_{n,k}$ as a derivative of energy*] [6]

(d) Now consider an impurity in the middle of the wire through which an electron in state n is reflected by probability r and transmitted by a probability t. Moreover, assume that the reflection and transmission is diagonal in n, *i.e.* the quantum number n remains unchanged before and after scattering. Obtain the modified expression for total current and show that the conductance is modified to $G = 2N|t|^2e^2/h$. [6]

(e) Now assume that the transmission and reflection probabilities are $t_{n,n'}$ and $r_{n,n'}$ for scattering between two channels with quantum number n and n', write down the generalized expression for conductance. You will get full marks even if you just guess the correct answer from what you obtained in (d) without any derivation. [3]

(a) In the longitudinal direction the wavefunction will look like a plane wave with quantum numbers labeled by momenta k. In the transverse direction due to confinement in an infinite potential well of width w the wavefunctions will be sinusoidal with quantum number n, which take integer values. Combining these, we have

$$\psi_{n,k}(x,y) = \frac{1}{\sqrt{L}}\sqrt{\frac{2}{W}}e^{ikx}\sin\left(\frac{n\pi y}{W}\right)$$

for the eigenstates. Here $1/\sqrt{L}$ factor comes from normalisation along the length and $\sqrt{2/W}$ factor comes from normalisation along the width. The corresponding energy eigenvalues are

$$\epsilon_{n,k} = \frac{\hbar^2 k^2}{2m} + \frac{n^2\hbar^2\pi^2}{2mW^2}.$$

(b) The electric current density should be given by $-ej_p$, where $-e$ is the charge of an electron and j_p is the probability current density. We can then obtain the electric current density for each state as

$$j_{n,k}(x,y) = \frac{ie\hbar}{2m}[\psi^*_{n,k}\partial_x\psi_{n,k} - \psi_{n,k}\partial_x\psi^*_{n,k}]$$

After evaluating the above expression for the above wavefunction, we get

$$j_{n,k}(x,y) = -\frac{2e\hbar k}{mLW}\sin^2\left(\frac{n\pi y}{W}\right)$$

We can obtain the current per state by integrating the current density along the width W and obtain

$$\mathcal{I}_{n,k} = -\frac{2e\hbar k}{mL}$$

To obtain the total current, we need to add the current contribution from all the states. However, we also need to take into account the occupation of each of these states. This is done by taking into account the Fermi statistics that the electrons follow while occupying their quantum states. The Heaviside step function takes into account the fact that right-moving electrons from the left reservoir and left-moving electrons from the right reservoir contribute to the total current through the wire, since these are the only states that will enter the wire from reservoirs. Finally the factor of 2 in front takes into account two spins of each electron.

(c) First note that

$$\mathcal{I}_{n,k} = -\frac{2e}{L\hbar}\frac{\partial\epsilon_{n,k}}{\partial k}$$

Putting this in the total current expression, we obtain

$$\begin{aligned}
I &= -\frac{2e}{h}\sum_{n=1}^{\infty}\int_0^{\infty} dk \frac{\epsilon_{n,k}}{\partial k}[f(\epsilon_{n,k}+eV_L) - f(\epsilon_{n,k}+eV_R)] \\
&= -\frac{2e}{h}\sum_{n=1}^{\infty}\int_0^{\infty} d\epsilon_{n,k}[f(\epsilon_{n,k}+eV_L) - f(\epsilon_{n,k}+eV_R)] \\
&= -\frac{2e}{h}\sum_{n=1}^{\infty}\int_{\gamma n^2}^{\infty} d\epsilon[f(\epsilon+eV_L) - f(\epsilon+eV_R)] \\
&\sim -\frac{2e^2}{h}(V_L - V_R)\sum_{n=1}^{\infty}\theta(-eV_R - \gamma n^2) = \frac{2e^2}{h}VN.
\end{aligned}$$

Here we have used the shorthand notation $\gamma = (\hbar\pi)^2/(2mW^2)$ From the total current, we can obtain the conductance

$$I = GV \implies G = 2N\frac{e^2}{h}.$$

We see that the conductance is quantized. Given that within the wire there is no backscattering of the electron or sources of energy loss, naively one would expect no resistance for the electrons to go through the wire, so infinite conductance. However, the quantum mechanics is preventing that. To see that imagine the wire has infinite conductance, that means (i) there is flow of current even when both reservoirs are at same potential, *i. e.* $V = 0$, (ii) or for finite potential difference, arbitrarily large current can flow through the wire.

The first condition is not possible because if the two reservoirs are at equal potential the left moving and right moving electrons in the wire must be equal to conserve total charge. Therefore, the net current must be zero in this situation. The second condition of arbitrarily large current is forbidden due to Fermi statistics that electrons have to follow. Taking two spins into account, not more than two electrons can occupy each quantum state in the wire. Because the energy needs to be conserved, the electrons coming from the reservoirs cannot just take arbitrarily high energy state in the wire to transfer current from one end to the other. Therefore it puts a limitation on how many electrons can transfer through the wire at a time, putting a limit on current. Imagine that the wire is a highway for electrons and each energy level is a lane and electrons have to follow the traffic laws to move.

Since there is no source of energy loss in the wire, it has to happen at the reservoirs. As electrons move from one reservoir to the other, they equilibrate in the new environment by some inelastic collisions that leads to energy loss, which is eventually the source of finite resistance.

(d) Restricting to the left side of the scattering impurity, it has current contribution from electrons coming from the left reservoir and the right reservoir. The current from left reservoir should come with a probability factor $1 - |r|^2$, taking into account the part that is reflected back from the scattering due to impurity. The current contribution from the right reservoir comes with a probability factor $|t|^2$, that takes into account the fact that only part from the right reservoir that reaches the left side of the impurity is the one that is transmitted. Therefore, the total current expression must modify as

$$I = 2\frac{L}{2\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} dk\mathcal{I}_{n,k}[(1-|r|^2)(\epsilon_{n,k}+eV_L)\theta(k)+|t|^2 f(\epsilon_{n,k}+eV_R)\theta(-k)],$$

You can also derive this expression step-by-step by write down normalised wavefunction on left and right, writing down probability current from the wavefunction, and using the continuity conditions. However, if you just give the above physical reasoning and just write down the expression, you will get full marks. Now following the steps in (c), we obtain

$$\begin{aligned} I &= -\frac{2e}{h}\sum_{n=1}^{\infty}\int_{\gamma n^2}^{\infty} d\epsilon[(1-|r|^2)f(\epsilon+eV_L)-|t|^2 f(\epsilon+eV_R)] \\ &\sim \frac{2e^2}{h}VN|t|^2. \end{aligned}$$

And the corresponding conductance becomes

$$G = 2N\frac{e^2}{h}|t|^2$$

(e) For the more general case, we should get

$$G = 2N\frac{e^2}{h}\sum_{n,n'}\frac{|t_{n,n'}|^2}{N}$$

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3 Consider spinless particles in a magnetic field described by the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}(\mathbf{r}))^2,$$

where we set the speed of light $c = 1$, $\mathbf{p}$ represents momentum and $\mathbf{A}(\mathbf{x})$ is the gauge potential.

(a) Give an expression for the velocity $\mathbf{v}$.
$\big[$*hint: Recall* $[AB, C] = A[B, C] + [A, C]B$ *and* $[A, BC] = B[A, C] + [A, B]C$.$\big]$ [5]

(b) We now take a specific gauge $\mathbf{A}(\mathbf{r}) = \frac{1}{2}(-By, Bx, 0)$, where B is the strength of the magnetic field. Setting $e = \hbar = B = m = 1$ we then get following Hamiltonian

$$H_{symm} = \frac{1}{2}[(-i\frac{\partial}{\partial x} + \frac{y}{2})^2 + (-i\frac{\partial}{\partial y} - \frac{x}{2})^2].$$

Show that H_{symm} can be written in quantised form as

$$H_{symm} = \hat{a}^\dagger \hat{a} + \frac{1}{2},$$

where $\hat{a} = \frac{1}{\sqrt{2}}[(\frac{x}{2} + \frac{\partial}{\partial x}) + i(\frac{y}{2} + \frac{\partial}{\partial y})]$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}[(\frac{x}{2} - \frac{\partial}{\partial x}) - i(\frac{y}{2} - \frac{\partial}{\partial y})]$. Also verify that $[\hat{a}, \hat{a}^\dagger] = 1$. What is the interpretation of acting with $\hat{a}^\dagger$ on the vacuum state? [6]

We now consider turning the magnetic field off and switching on interactions. In second quantised form the system is then described by a Hamiltonian that reads

$$H_{int} = \int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r})[\frac{-\hbar^2}{2m}\nabla^2]\hat{\psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r} \int d\mathbf{r}' \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')V(\mathbf{r}, \mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'),$$

where $\hat{\psi}^\dagger(\mathbf{r})$ are the field operators and $V(\mathbf{r}, \mathbf{r}')$ is a two-body interaction term.

(c) Write down the commutation/anti-commutation relations for the field operators $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ when the system is generally composed of Fermions or Bosons. [3]

(d) Define the total angular momentum J. Assuming that we have Fermions and a two-body interaction term of the form $V(\mathbf{r}, \mathbf{r}') = V(|\mathbf{r} - \mathbf{r}'|)$. Show that total angular momentum J is conserved. Here you may use that $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ vanish for large $\mathbf{r}$. You may also use that V derivatives thereof are symmetric in $\mathbf{r}$.
$\big[$*hint 1: Recall* $[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B$ *and* $\frac{d|\mathbf{r}|}{d\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$.$\big]$
$\big[$*hint 2: Also note that the boundary conditions allow for integration by parts.*$\big]$ [8]

(e) What does the result imply for the "Landau levels" obtained by acting with $\hat{a}^\dagger$ on the vacuum when such interaction terms are present? [3]

(a) *[Unseen, but very similar to derivation addressed in lecture and notes.]* We use the Heisenberg equation of motion and obtain

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{i}{\hbar}[H, \mathbf{r}] = \frac{1}{2m}\{\mathit{\Pi}[\mathit{\Pi}, \mathbf{r}] + [\mathit{\Pi}, \mathbf{r}]\mathit{\Pi}\},$$

where $\mathit{\Pi} = \mathbf{p} - e\mathbf{A}(\mathbf{x})$. Using that $[\mathit{\Pi}, \mathbf{r}] = [\mathbf{p} - e\mathbf{A}(\mathbf{x}), \mathbf{r}] = -[\mathbf{r}, \mathbf{p} - e\mathbf{A}(\mathbf{x})] = -i\hbar$. Hence, we obtain

$$\mathbf{v} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{x})}{m}.$$

(b) *[Unseen, but very similar to derivation addressed in lecture and notes.]* By direct evaluation we obtain

$$\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{1}{2}[(\frac{x}{2} - \frac{\partial}{\partial x}) - i(\frac{y}{2} - \frac{\partial}{\partial y})][(\frac{x}{2} + \frac{\partial}{\partial x}) + i(\frac{y}{2} + \frac{\partial}{\partial y})] \\
&= \frac{1}{2}[\frac{x^2}{4} - \frac{1}{2} - \partial_x^2 - i\frac{xy}{4} + i\frac{x}{2}\partial_y - i\frac{y}{2}\partial_x + i\partial_x\partial_y + i\frac{xy}{4} + i\frac{x}{2}\partial_y - i\frac{y}{2}\partial_x - i\partial_x\partial_y + \frac{y^2}{4} - \frac{1}{2} - \partial_y^2] \\
&= \frac{1}{2}[\frac{x^2}{4} - \partial_x^2 + ix\partial_y - iy\partial_x + \frac{y^2}{4} - \partial_y^2 - 1] \\
&= \frac{1}{2}[(-i\partial_x + \frac{y}{2})^2 + (-i\partial_y - \frac{x}{2})^2 - 1].
\end{aligned}$$

Where we used $\partial x_i = \frac{\partial}{\partial x_i}$, with $x_i = x, y$. Similarly, we get

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \frac{1}{2}\{[(\frac{x}{2} + \frac{\partial}{\partial x}), (\frac{x}{2} - \frac{\partial}{\partial x})] + [(\frac{y}{2} + \frac{\partial}{\partial y}), (\frac{y}{2} - \frac{\partial}{\partial y})]\} \\
&= \frac{1}{2}\{[\frac{x}{2}, -\frac{\partial}{\partial x}] + [\frac{\partial}{\partial x}, \frac{x}{2}] + [\frac{y}{2}, -\frac{\partial}{\partial y}] + [\frac{\partial}{\partial y}, \frac{y}{2}]\} \\
&= \frac{1}{2}\{[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}]\}.
\end{aligned}$$

We can thus interpret $\hat{a}^\dagger$ as a creation operator. In fact the Landau level is created by acting with $\hat{a}^\dagger$ on the vacuum as well [(*not*) part of the question] an operator $\hat{b}^\dagger = \frac{1}{\sqrt{2}}[(\frac{x}{2} - \frac{\partial}{\partial x}) + i(\frac{y}{2} - \frac{\partial}{\partial y})]$. Specifically, $|N, m_z\rangle = \frac{(\hat{b}^\dagger)^{N+m_z}}{\sqrt{m_z+N!}}\frac{(\hat{a}^\dagger)^N}{\sqrt{N!}}|vac\rangle$.

(c) *[Bookwork.]* In case of Bosons we have $[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')$, $[\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')] = [\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}') = 0$. The same for Fermions in terms of anti-commutators.

(d) *[Unseen, but similar to derivation addressed in lecture and notes.]* We define total angular momentum as $J = \int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r})[\mathbf{r} \times \nabla]\hat{\psi}(\mathbf{r})$

To check the conservation we use the Heisenberg equation of motion

$$\dot{J} = \frac{i}{\hbar}[H_{int}, J].$$

We first check the kinetic part. This part of the above commutator is

$$\int d\mathbf{r}d\mathbf{r}'\, [\hat{\psi}^\dagger(\mathbf{r})(\frac{-\hbar^2}{2m}\nabla^2)\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')(\mathbf{r}' \times \nabla)\hat{\psi}(\mathbf{r}')].$$

Using $[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B$ and the relations in (a). We see we get two terms.

$$\frac{-\hbar^2}{2m}\int d\mathbf{r}d\mathbf{r}'\, \hat{\psi}^\dagger(\mathbf{r})\{(\nabla^2_{\mathbf{r}})\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\}(\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}(\mathbf{r}') - \hat{\psi}^\dagger(\mathbf{r}')\{\hat{\psi}^\dagger(\mathbf{r}), (\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}(\mathbf{r}')\}(\nabla^2_{\mathbf{r}})\hat{\psi}(\mathbf{r}).$$

Given the boundary conditions we can integrate by parts twice and using that $\{\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}')$, we get

$$\frac{-\hbar^2}{2m}\int d\mathbf{r}d\mathbf{r}'\, \nabla^2_{\mathbf{r}}\hat{\psi}^\dagger(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')(\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}(\mathbf{r}') - \hat{\psi}^\dagger(\mathbf{r}')\{\hat{\psi}^\dagger(\mathbf{r}), (\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}(\mathbf{r}')\}(\nabla^2_{\mathbf{r}})\hat{\psi}(\mathbf{r}).$$

Integrating the second term by parts in the above expression in similar fashion we get

$$\frac{-\hbar^2}{2m}\int d\mathbf{r}d\mathbf{r}'\, \nabla^2_{\mathbf{r}}\hat{\psi}^\dagger(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')(\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}(\mathbf{r}') + \delta(\mathbf{r}-\mathbf{r}')(\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}^\dagger(\mathbf{r}')(\nabla^2_{\mathbf{r}})\hat{\psi}(\mathbf{r}).$$

Note that in the above, $\{\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\} = \{\hat{\psi}^\dagger(\mathbf{r}')\, \hat{\psi}(\mathbf{r})\}$. More importantly, we need to sate a word of caution. We note that we skipped a few steps and use the fact that under either $\int d\mathbf{r}$ or $\int d\mathbf{r}'$ the derivatives of the other coordinate act as a constant. This allows for the intergation by parts as peformed with some intermediate steps. Integrating the second term yet again and using the vanishing boundary condition, we see the terms are opposite and hence cancel.

Now for the potential part we have

$$\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\int d\mathbf{r}'' V(\mathbf{r},\mathbf{r}')[\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'), \hat{\psi}^\dagger(\mathbf{r}'')(\mathbf{r}''\times\nabla_{\mathbf{r}''})\hat{\psi}(\mathbf{r}'')].$$

We use $[A, BC] = [A, B]C + B[A, C]$ with $A = \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')$, $B = \hat{\psi}^\dagger(\mathbf{r}'')$ and $C = (\mathbf{r}''\times\nabla_{\mathbf{r}''})\hat{\psi}(\mathbf{r}'')$. Evaluating $[A, B]$ we note

$$\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r}'') - \hat{\psi}^\dagger(\mathbf{r}'')\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}') =$$
$$\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r}'')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}') - \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r}'')\hat{\psi} + \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\{\hat{\psi}(\mathbf{r})\delta(\mathbf{r}'-\mathbf{r}'') + \hat{\psi}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'')\}$$

For $B[A, C]$ or

$$\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\int d\mathbf{r}'' V(\mathbf{r},\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r}'')[\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'), (\mathbf{r}''\times\nabla_{\mathbf{r}''})\hat{\psi}(\mathbf{r}'')].$$

we partially integrate as above to get[1]

$$\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\int d\mathbf{r}'' V(\mathbf{r},\mathbf{r}')(\mathbf{r}''\times\nabla_{\mathbf{r}''})\hat{\psi}^\dagger(\mathbf{r}'')[\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'), \hat{\psi}(\mathbf{r}'')].$$

Then for

$$[\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'), \hat{\psi}(\mathbf{r}'')],$$

we see similar to above

$$[\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'), \hat{\psi}(\mathbf{r}'')] = (\hat{\psi}^\dagger(\mathbf{r})\delta(\mathbf{r}'-\mathbf{r}'') + \hat{\psi}^\dagger(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}''))\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}').$$

[1] again also involving more steps to get the final result

Hence, we obtain

$$\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\int d\mathbf{r}''V(\mathbf{r},\mathbf{r}')\{\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')(\hat{\psi}(\mathbf{r})\delta(\mathbf{r}'-\mathbf{r}'')+\hat{\psi}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}''))\}(\mathbf{r}''\times\nabla_{\mathbf{r}''})\hat{\psi}^\dagger(\mathbf{r}'')+$$
$$(\mathbf{r}''\times\nabla_{\mathbf{r}''})\hat{\psi}^\dagger(\mathbf{r}'')\{(\hat{\psi}^\dagger(\mathbf{r})\delta(\mathbf{r}'-\mathbf{r}'')+\hat{\psi}^\dagger(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}''))\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\}$$

or

$$\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'V(\mathbf{r},\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})(\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}^\dagger(\mathbf{r}')+V(\mathbf{r},\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}')(\mathbf{r}\times\nabla_{\mathbf{r}})+$$
$$V(\mathbf{r},\mathbf{r}')(\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}^\dagger(\mathbf{r}'')\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')+V(\mathbf{r},\mathbf{r}')(\mathbf{r}\times\nabla_{\mathbf{r}})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')$$

Now we can see that each term gives zero. This because of the form of the potential. Indeed, partially integrating the first term we have for example

$$\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'(\mathbf{r}'\times\nabla_{\mathbf{r}'})V(\mathbf{r},\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r})(\mathbf{r}'\times\nabla_{\mathbf{r}'})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')$$

Hence we all terms we get are of the form

$$\propto\int d\mathbf{r}\int d\mathbf{r}'\{\mathbf{r}'\times V'\}(|\mathbf{r}-\mathbf{r}'|)\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\cdots$$

Given the sign change of $\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$, the assumed symmetry of V and v' and properties of the other functions we see this equates to zero and hence angular momentum is conserved as $\dot{J}=\frac{i}{\hbar}[H_{int},J]=0$.

We note that this could have been anticipated from the structure of Hamiltonian and assumed form of the potential. Also the above guidance skips some steps but also allows for shortcuts, i.e. for the potential once can directly conclude that there are terms as in the the last above equations.

(e) *[Unseen, but direct interpretation.]* Under these interactions, assuming well defined angular momentum of Landau levels [as we can see (b)], it stays a proper constant of motion. Fractional states, incompressible Landau levels due to interactions, still can have well defined momentum.

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4 Consider the Hong-Ou-Mandel setup. Precisely, consider a 50:50 beam splitter (i.e. each incident photon has an equal probability of getting reflected or transmitted from the beam splitter), with two input modes and two output modes. Two identical photons are simultaneously incident in the two input modes (one in each mode).

(a) Draw the figure for all possible experimental outcomes. Taking into account the unitarity of the scattering matrix and assuming real-valued reflection and transmission amplitudes, assign appropriate overall signs for all possible outcomes. [4]

(b) Write down the two-photon states in the input and in the output modes. [*hint: The most convenient is to write Fock states in the mode basis*]. For the input and output states, write down the density matrices. Take the partial trace over one of the mode and obtain the reduced density matrices. Then explicitly calculate the entanglement entropies to show that the experiment generates entanglement in the output modes. [5]

(c) Now consider a three-qubit state known as the GHZ state

$$\psi_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Write down the one and two particle reduced density matrices by taking partial traces (You can choose to trace out particle 2 and 3 to calculate one particle reduced density matrix and trace out particle 1 to calculate two particle reduced density matrix). Calculate the entanglement entropies. Is the mutual entanglement equal? [5]

(d) By using the two particle reduced density matrix, show that indeed the GHZ state is highly entangled. [*hint: You can show that by considering a small perturbation to the GHZ state by mixing it with a three-qubit state of your choice and showing that entanglement entropy decreases as mixing is increased.*] [6]

(e) Now consider another highly entangled three qubit state, called the W state

$$\psi_W = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

Trace out one of the particle and calculate the entanglement entropy. Compare it with the result of the GHZ state. Which state is more entangled? [5]

(a) The scattering matrix will take the following form:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Let's label the two input modes as a, b and the two output modes as c, d, such that $\hat{a}^\dagger$ creates a photon in mode a. One can then relate the output and input modes with the scattering matrix as follows:

$$\begin{pmatrix} \hat{c}^\dagger \\ \hat{d}^\dagger \end{pmatrix} = S \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix}$$

$$\implies \hat{c}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{b}^\dagger), \quad \hat{d}^\dagger = \frac{1}{\sqrt{2}}(-\hat{a}^\dagger + \hat{b}^\dagger)$$

It may look a little strange to write this scattering matrix connection in terms of creation operators. The above equation physically just means that a photon in output mode c (or d) is created by an equal superposition of reflected mode a (or b) and transmitted mode b (or a). This is what we would expect physically.

(b) The input two photon state is $\hat{a}^\dagger\hat{b}^\dagger|0,0\rangle = |1,1\rangle$. The resultant output state is obtained as follows:

$$\begin{aligned} \hat{a}^\dagger\hat{b}^\dagger|0,0\rangle &= \frac{1}{2}(\hat{c}^\dagger - \hat{d}^\dagger)(\hat{c}^\dagger + \hat{d}^\dagger)|0,0\rangle \\ &= \frac{1}{2}((\hat{c}^\dagger)^2 - (\hat{d}^\dagger)^2)|0,0\rangle \\ &= \frac{1}{\sqrt{2}}(|2,\, 0\rangle - |0,\, 2\rangle) \end{aligned}$$

We have represented the input and output states as the Fock state in the beam splitters input and output modes as the basis respectively. The density matrices are

$$\rho_{input} = |1,1\rangle\langle 1,1|$$

and

$$\rho_{output} = \frac{1}{2}(|2,0\rangle\langle 2,0| + |0,2\rangle\langle 0,2| - |2,0\rangle\langle 0,2| - |0,2\rangle\langle 2,0|)$$

The reduced density matrices are obtained by taking a partial trace over mode b in input and mode d in output.

$$\rho^{red}_{input} = |1\rangle\langle 1|$$

and

$$\rho^{red}_{output} = \frac{1}{2}(|2\rangle\langle 2| + |0\rangle\langle 0|)$$

The entanglement entropies are respectively

$$S_{input} = -\log 1 = 0$$

and

$$S_{output} = -Tr(\rho_{output}^{red} \log \rho_{output}^{red}) = \log 2$$

Therefore, the output modes have become entangled.

(c) The reduced density matrix for particle 1 after tracing out particle 2 and 3 is

$$\rho_1^{red} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Similarly, after tracing out particle 1, the reduced density matrix is

$$\rho_{23}^{red} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|).$$

The entanglement entropies are respectively

$$S_1 = -Tr(\rho_1^{red} \log \rho_1^{red}) = \log 2$$
$$S_{23} = -Tr(\rho_{23}^{red} \log \rho_{23}^{red}) = \log 2,$$

which are equal. That simply shows that qubit 1 is entangled with qubit 2 and 3 with the same amount as qubit 2 and 3 are entangled with qubit 1.

(d) The simplest perturbation to GHZ, we can consider is

$$\psi = \lambda \psi_{GHZ} + \sqrt{1-\lambda^2}|111\rangle.$$

We can trace out qubit 2 and 3, to obtain a reduced density matrix

$$\rho_1^{red} = \left(1 - \frac{\lambda^2}{2}\right)|0\rangle\langle 0| + \left(\frac{\lambda^2}{2}\right)|1\rangle\langle 1|.$$

After some manipulations, we can express the entanglement entropy as

$$S_1 = -\frac{\lambda^2}{2}\log\left(\frac{\lambda^2}{2}\right) - \left(1 - \frac{\lambda^2}{2}\right)\log\left(1 - \frac{\lambda^2}{2}\right). \tag{3}$$

Now consider the difference between the entanglement entropy of the GHZ state and the current state

$$\Delta S = \log 2 + \frac{\lambda^2}{2}\log\left(\frac{\lambda^2}{2}\right) + \left(1 - \frac{\lambda^2}{2}\right)\log\left(1 - \frac{\lambda^2}{2}\right) > 0. \tag{4}$$

Therefore, the entanglement entropy decreases as the GHZ state is perturbed. This shows that GHZ state is highly entangled.

(e) If we trace out the first qubit, we obtain the reduced density matrix

$$\rho_{23}^{red} = \frac{1}{3}(|01\rangle\langle 01| + |10\rangle\langle 10| + |00\rangle\langle 00|)$$

Now, to calculate the entanglement entropy remember, we are taking log of a matrix. The operation can be most conveniently done by diagonalizing the matrix. Therefore, we obtain

$$S_{23} = -\sum_i \lambda_i \log \lambda_i, \tag{5}$$

where $\lambda_i = 0,\ 1/3,\ 2/3$ are the eignevalues of ρ_{23}. By summing over all the eigenvalues, we get

$$S_{23} = \frac{1}{3}\log(9/4), \tag{6}$$

which is smaller than $\log 2$. Therefore, from this measure, the GHZ state is more entangled.
