UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

23 April 2014

THEORETICAL PHYSICS 2

Answer ***three*** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1

(a) Explain what is meant by the *adiabatic approximation* in quantum mechanics, stating a general condition for its validity. [6]

(b) The state of a system with a time dependent Hamiltonian $H(t)$ may be written in terms of the instantaneous eigenstates $|\varphi_\alpha(t)\rangle$ of $H(t)$

$$|\Psi\rangle = \sum_\alpha a_\alpha(t) \exp\left(-\frac{i}{\hbar}\int^t E_\alpha(t')\,dt'\right)|\varphi_\alpha(t)\rangle\,.$$

Show that the time dependent Schrödinger equation implies that the amplitudes $\{a_\alpha(t)\}$ obey

$$\frac{da_\alpha}{dt} = -\sum_\beta \langle\varphi_\alpha|\left(\frac{d}{dt}|\varphi_\beta\rangle\right) a_\beta \exp\left(\frac{i}{\hbar}\int^t [E_\alpha(t') - E_\beta(t')]\,dt'\right)$$

[8]

(c) Consider a particle in the time-dependent infinite square well potential of width $L(t)$ at time t

$$V(x) = \begin{cases} 0 & 0 < x < L(t) \\ \infty & x < 0 \text{ or } x > L(t). \end{cases}$$

Write the wavefunction in terms of the instantaneous eigenstates

$$\Psi(x,t) = \sqrt{\frac{2}{L(t)}}\sum_{n=1}^{\infty} a_n(t)\exp\left(-\frac{i}{\hbar}\int^t E_n(t')\,dt'\right)\sin\left(\frac{n\pi x}{L(t)}\right),$$

where $E_n(t) = \frac{1}{2m}\left(\frac{\pi\hbar n}{L}\right)^2$. By finding $\langle\varphi_n|\left(\frac{d}{dt}|\varphi_p\rangle\right)$, show that $\{a_n(t)\}$ obey

$$\frac{da_n}{dt} = \frac{\dot{L}}{L}\sum_{p\neq n} a_p(-1)^{n+p}\frac{2np}{p^2-n^2}\exp\left(\frac{i}{\hbar}\int^t [E_n(t') - E_p(t')]\,dt'\right)$$

[10]

(d) By using

$$a_n = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

on the right hand side of these equations, and integrating, show that for $L(t) = vt$, the probability to make a transition from $n = 1$ at time $t = 0$ to $n = 2$ at time t is approximately

$$\frac{16}{9}\left|\int_0^t \frac{\exp(-i\alpha/t')}{t'}dt'\right|^2$$

where $\alpha = \frac{3\pi^2\hbar}{2mv^2}$. [9]

2

(a) Define the *propagator* and explain why

$$K(x,t|x',t') = \sum_{\alpha} \varphi_\alpha(x)\varphi^*_\alpha(x')e^{-iE_\alpha(t-t')/\hbar}, \qquad t > t'$$

where E_α and $\varphi_\alpha(x)$ are respectively the eigenvalues and eigenfunctions of a one dimensional Hamiltonian. [5]

(b) Using the above expression, find the propagator for a particle moving on a ring of radius R, with Hamiltonian

$$H = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2},$$

where θ is the angle. Leave your result expressed as a sum. [7]

(c) Find *all* the classical trajectories starting from (θ', t') and finishing at (θ, t). [7]

(d) Find the propagator from the path integral, expressing your result as a sum over classical paths. You may use the result for the propagator in one dimension

$$K(x,t|x',t') = \left(\frac{m}{2i\pi\hbar(t-t')}\right)^{1/2} \exp\left[-\frac{m(x-x')^2}{2i\hbar(t-t')}\right], \qquad t > t'.$$

[7]

(e) Use the identity

$$\sum_{p=-\infty}^{\infty} \exp\left[-\frac{\alpha}{2}(x+2\pi p)^2\right] = \sqrt{\frac{1}{2\pi\alpha}} \sum_{q=-\infty}^{\infty} \exp\left[-\frac{q^2}{2\alpha} - iqx\right]$$

to prove that the results of parts (b) and (d) are equal. [7]

3

(a) At low energies, scattering from a spherically symmetric potential $V(|\mathbf{r}|)$ is dominated by s-wave ($l = 0$) scattering. Explain why this implies that the wavefunction takes the form

$$\psi(\mathbf{r}) \sim 1 - \frac{a}{r}$$

outside the interaction region (i.e. the region where the scattering potential is non-zero), where a is the scattering length. [6]

(b) A pair of scatterers have scattering lengths a_1 and a_2 and are located at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. The low energy wavefunction outside the interaction region has the form

$$\psi(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}} + c_1\frac{e^{ik|\mathbf{r}-\mathbf{r}_1|}}{k|\mathbf{r}-\mathbf{r}_1|} + c_2\frac{e^{ik|\mathbf{r}-\mathbf{r}_2|}}{k|\mathbf{r}-\mathbf{r}_2|},$$

where $\mathbf{k}_i$ is the wavevector of the incoming particles, and $k = |\mathbf{k}_i|$. Find c_1 and c_2. [12]

(c) Far from the scatterers the wavefunction has the form

$$\psi(\mathbf{r}) \xrightarrow[\mathbf{r}\to\infty]{} e^{i\mathbf{k}_i\cdot\mathbf{r}} + f(\hat{\mathbf{r}})\frac{e^{ikr}}{r},$$

where $\hat{\mathbf{r}}$ is a unit vector parallel to $\mathbf{r}$. Find the scattering amplitude $f(\hat{\mathbf{r}})$ in terms of c_1 and c_2. [6]

(d) Find the total cross section in terms of c_1 and c_2. [9]

4

(a) Give the form of the density matrix in thermal equilibrium of a system with Hamiltonian H in the canonical ensemble at temperature T. [4]

(b) For a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

show that in thermal equilibrium the average kinetic energy and average potential energy are equal at all temperatures. You may use the standard result

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$
$$p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right)$$

with $[a, a^\dagger] = 1$. [8]

(c) Show that

$$\overline{\langle x^2 \rangle} = \frac{\hbar}{2m\omega}\coth\left(\frac{\hbar\omega}{2k_B T}\right)$$

where $\overline{\langle \cdots \rangle}$ denotes the quantum and thermal average. [10]

(d) Use the identity

$$\exp\left[\beta\left(A + B\right) + \beta^2\left[A, B\right]/2 + O(\beta^3)\right] = \exp(\beta A)\exp(\beta B)$$

to find the high temperature form of the density matrix $\langle x|\rho|x'\rangle$ in the position representation, when the β^2 and higher terms in the exponent on the left hand side are neglected. [11]

5 A system of bosons moving on a ring of radius R is described by the Hamiltonian $H = H_1 + H_2$, where

$$H_1 = \int_0^{2\pi} \left[\frac{\hbar^2}{2mR^2} \frac{d\psi^\dagger}{d\theta} \frac{d\psi}{d\theta} \right] d\theta$$

is the single particle Hamiltonian and

$$H_2 = \frac{U}{2} \int_0^{2\pi} \psi^\dagger \psi^\dagger \psi \psi \, d\theta$$

describes interactions between the particles. $\psi^\dagger(\theta)$ and $\psi(\theta)$ satisfy

$$\left[\psi(\theta), \psi^\dagger(\theta')\right] = \delta(\theta - \theta')$$
$$\left[\psi(\theta), \psi(\theta')\right] = \left[\psi^\dagger(\theta), \psi^\dagger(\theta')\right] = 0$$

(a) In the basis of angular momentum eigenstates

$$\varphi_l(\theta) = \frac{e^{il\theta}}{\sqrt{2\pi}}, \qquad l = 0, \pm 1, \pm 2, \ldots$$

$\psi(\theta)$ may be expressed

$$\psi(\theta) = \sum_{l=-\infty}^{\infty} \varphi_l(\theta) a_l,$$

where a_l annihilates a particle in state l. By considering *only* states $l = 0$ and 1, show that the Hamiltonian takes the form

$$H = \mathcal{E} a_1^\dagger a_1 + \frac{U}{4\pi} \left[a_0^\dagger a_0^\dagger a_0 a_0 + a_1^\dagger a_1^\dagger a_1 a_1 + 4 a_1^\dagger a_0^\dagger a_0 a_1 \right]$$

and identify $\mathcal{E}$. [10]

(b) Find the energy of the product state

$$|N_0, N_1\rangle = \frac{1}{\sqrt{N_0! N_1!}} \left(a_0^\dagger\right)^{N_0} \left(a_1^\dagger\right)^{N_1} |\mathrm{VAC}\rangle .$$

[6]

(c) Show that in the state

$$|\chi\rangle = \frac{1}{\sqrt{N!}} \left[\cos\frac{\chi}{2} a_0^\dagger + \sin\frac{\chi}{2} a_1^\dagger \right]^N |\mathrm{VAC}\rangle .$$

the occupation of the states $l = 0$ and 1 follows a binomial distribution, with average occupancies $\overline{N}_0 = N\cos^2(\chi/2)$ and $\overline{N}_1 = N\sin^2(\chi/2)$. [10]

(d) Show that for a large number of particles N, the expectation value of the energy *per particle* is approximately

$$E(\chi)/N = \mathcal{E}\sin^2(\chi/2) + \frac{nU}{2}\left[1 + \frac{1}{2}\sin^2\chi\right],$$

where $n = N/(2\pi)$. [7]

6 Consider 2×2 complex matrices of the form

$$\mathsf{M} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1$$

(a) Explain why these matrices form a Lie group. [6]

(b) By considering M close to the identity $\mathsf{M} = \mathbb{1} + \mathsf{m} + \cdots$, show that an element m of the Lie algebra of this group may be written

$$\mathsf{m} = \lambda \mathsf{m}_1 + \mu \mathsf{m}_2 + \nu \mathsf{m}_3$$

where λ, μ, and ν are real, and

$$\mathsf{m}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{m}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathsf{m}_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the commutation relations of the m_a. [11]

(c) A pair of bosons (or oscillator variables) $a^\dagger$, a and $b^\dagger$, b satisfying the usual commutation relations are used to define

$$K_1 = \frac{1}{2}\left[a^\dagger b^\dagger + ab\right], \qquad K_2 = -\frac{i}{2}\left[a^\dagger b^\dagger - ab\right], \qquad K_3 = \frac{1}{2}\left[a^\dagger a + b b^\dagger\right].$$

Show that iK_a for $a = 1, 2, 3$ have the same commutation relations as the m_a in the previous part. [8]

(d) Relate $C = K_3^2 - K_1^2 - K_2^2$ to the number of a and b quanta, and explain why it commutes with the K_a. [8]

END OF PAPER