UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

24 April 2013

THEORETICAL PHYSICS 2

Answer ***three*** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1 Consider a spin-1/2 in a time varying field, described by the Hamiltonian

$$H(t) = \Omega_0 S_z + \frac{\Omega_R}{2}\left(S_+ e^{-i\omega t} + S_- e^{i\omega t}\right),$$

where $S_\pm = S_x \pm iS_y$, and $S_i = \frac{\hbar}{2}\sigma_i$, $i = x, y, z$, with σ_i the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Write the state of the system $|\Psi(t)\rangle$ in the form

$$|\Psi(t)\rangle = \exp\left(-i\omega t S_z/\hbar\right)|\Psi_R(t)\rangle$$

and show that $|\Psi_R(t)\rangle$ obeys the equation

$$i\hbar\frac{d\,|\Psi_R(t)\rangle}{dt} = H_{\mathrm{Rabi}}\,|\Psi_R\rangle$$

where

$$H_{\mathrm{Rabi}} = (\Omega_0 - \omega)\,S_z + \Omega_R S_x.$$

[*You might find it useful to write the Hamiltonian as a* 2×2 *matrix.*] [7]

(b) Find the eigenvalues of H_{Rabi} and describe the *complete* time evolution of the corresponding eigenstates (you don't need to find the eigenstates explicitly). [8]

(c) Find the evolution of the phase of the eigenstates of H_{Rabi} after time $2\pi/\omega$. Considering the *adiabatic* limit $\omega \ll \sqrt{\Omega_0^2 + \Omega_R^2}$, interpret your result in terms of Berry's phase. [9]

(d) Explain how your answers to (a), (b), and (c) would be modified if we have spin-s, rather than spin-1/2. [9]

Solution (a) is straightforward, the only hard part being to transform the rotating parts of the Hamiltonian

$$e^{i\omega S_z t/\hbar} S_+ e^{-i\omega t} e^{-i\omega S_z t/\hbar} = S_+,$$

which can be done just with 2×2 matrices if need be, though for part (d) they would be better off recognizing that it only depends on the angular momentum algebra. (b) eigenvalues are

$$E_\pm = \pm\frac{\hbar}{2}\sqrt{(\Omega_0 - \omega)^2 + \Omega_R^2},$$

and the time evolution of the eigenstate is

$$e^{-iE_\pm t/\hbar}\begin{pmatrix} e^{-i\omega t/2}\psi_\uparrow^{(\pm)} \\ e^{i\omega t/2}\psi_\downarrow^{(\pm)} \end{pmatrix}.$$

(c) In time $2\pi/\omega$ the phase evolves by an amount

$$\pi\left(1 \pm \frac{1}{\omega}\sqrt{(\Omega_0-\omega)^2+\Omega_\mathrm{R}^2}\right) \underset{\omega\to 0}{\longrightarrow} \pi\left(\pm\frac{1}{\omega}\sqrt{\Omega_0^2+\Omega_\mathrm{R}^2}+1\mp\frac{\Omega_0}{\sqrt{\Omega_0^2+\Omega_\mathrm{R}^2}}\right).$$

The first part is the dynamical phase, while the ω-independent part is the Berry phase, which may be written

$$\pi\left(1 \mp \cos\theta\right)$$

where $\cos\theta = \frac{\Omega_0}{\sqrt{\Omega_0^2+\Omega_\mathrm{R}}}$. $2\pi(1-\cos\theta)$ is the solid angle of a circular cap on the unit sphere subtending an angle 2θ, while $2\pi(1+\cos\theta)$ is minus this (mod 4π). (d) Nothing changes in (a), because the transformation depends only on the angular momentum relations. In (b) we have $2s+1$ levels with energies $m\hbar\sqrt{(\Omega_0-\omega)^2+\Omega_\mathrm{R}^2}$, $m=-s,\ldots s$. In (c) we likewise have m multiplying the solid angle rather 1/2.

2 In one dimension, the propagator $K(x,t|x',t')$ is the solution of the equation

$$\left[i\hbar\frac{\partial}{\partial t} - H\right] K(x,t|x',t') = i\hbar\delta(x-x')\delta(t-t') \text{ and}$$
$$K(x,t|x',t') = 0 \text{ for } t < t',$$

where H is the Hamiltonian. The momentum space propagator is defined by

$$K(x,t|x',t') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp dp'\, K(p,t|p',t') \exp\left(ipx/\hbar - ip'x'/\hbar\right).$$

(a) Show that the propagator for a free particle is

$$K_{\text{free}}(x,t|x',t') = \theta(t-t')\left(\frac{m}{2i\pi\hbar(t-t')}\right)^{1/2} \exp\left[-\frac{m(x-x')^2}{2i\hbar(t-t')}\right]$$

[7]

(b) Now consider a particle moving in a linear potential, described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \alpha x.$$

Find the equation satisfied by $K(p,t|p',t')$ and verify that the solution is

$$K(p,t|p',t') = \theta(t-t')\delta(p-p'+\alpha[t-t'])\exp\left(\frac{i\left(p^3-p'^3\right)}{6\alpha m\hbar}\right).$$

[8]

(c) Use the result of part (b) to obtain $K(x,t|x',t')$ for a particle moving in a linear potential. [9]

(d) By computing the classical action for a trajectory $(x',t') \to (x,t)$, show that the same result follows from the path integral. [9]

Solution (a) Bookwork. They could continue from the fundamental solution of the heat equation, or compute the Fourier transform of the momentum space propagator. (b) Equation is

$$\left[i\hbar\partial_t - i\hbar\alpha\partial_p - \frac{p^2}{2m}\right] K(p,t|p',t') = i\hbar\delta(t-t')\delta(p-p').$$

Verify solution by substitution. (c) This is a matter of computing the Fourier transform

$$\begin{aligned}
K(x,t|x',t') &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp dp'\, K(p,t|p',t') \exp\left(ipx/\hbar - ip'x'/\hbar\right) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp \exp\left(\frac{i(p^3 - [p+\alpha(t-t')]^3)}{6\alpha m\hbar} + ipx/\hbar - i(p+\alpha[t-t'])x'/\hbar\right) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp \exp\left(-\frac{i(t-t')}{2m\hbar}\left(p + \alpha(t-t')/2 - \frac{(x-x')m}{t-t'}\right)^2\right. \\
&\qquad \left. - \frac{i\alpha^2(t-t')^3}{24m\hbar} + \frac{i(x-x')^2 m}{2\hbar(t-t')} - \frac{i\alpha(x+x')(t-t')}{2\hbar}\right).
\end{aligned}$$

Doing the Gaussian integral gives the final form

$$K(x,t|x',t') = \left(\frac{m}{2i\pi\hbar(t-t')}\right)^{1/2} \exp\left(-\frac{i\alpha^2(t-t')^3}{24m\hbar} + \frac{i(x-x')^2 m}{2\hbar(t-t')} - \frac{i\alpha(x+x')(t-t')}{2\hbar}\right)$$

(d) For the path integral solution we need to find the classical action for the trajectory

$$X(t) = x' - \frac{\alpha t^2}{m} + (x-x')\frac{t}{T}$$

(I've set $T = t - t'$ for convenience). Then we have

$$S_{\text{cl}} = \int_0^T dt \left[\frac{1}{2}m\dot{X}^2 - \alpha X\right] = \frac{m(x-x')^2}{2T} - \frac{\alpha^2 T^3}{24m} - \frac{\alpha(x+x')T}{2},$$

which leads quickly to the result once you argue that the Gaussian path integral is unchanged from the free particle case.

3 A particle is initially in a plane wave state $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ and interacts with a scattering potential $V(\boldsymbol{r})$. Its wavefunction is a solution of the Lippmann–Schwinger equation

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}'\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi_{\boldsymbol{k}}(\boldsymbol{r}').$$

(a) Show that the differential cross-section in the (first) Born approximation is

$$\frac{d\sigma(\theta,\phi)}{d\Omega} = \left|\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}V(\boldsymbol{r})\right|^2$$

where $\boldsymbol{q} = \boldsymbol{k}_f - \boldsymbol{k}$ is the momentum transfer, and $\boldsymbol{k}_f$ is the final momentum, pointing in a direction specified by spherical coordinates (θ,ϕ). [7]

(b) Use this result to show that the total cross-section in the Born approximation is

$$\sigma_{\text{total}} = \frac{m^2}{\pi\hbar^4}\int d^3\boldsymbol{r}\, d^3\boldsymbol{r}'\, V(\boldsymbol{r})V(\boldsymbol{r}')\frac{\sin k\,|\boldsymbol{r}-\boldsymbol{r}'|}{k\,|\boldsymbol{r}-\boldsymbol{r}'|}e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

[8]

(c) By iterating the integral equation a second time find the *second* Born approximation to the scattering amplitude. [9]

(d) Show that the results of parts (b) and (c) are consistent with the optical theorem

$$\operatorname{Im} f(\theta=0) = \frac{k\sigma_{\text{total}}}{4\pi},$$

where $f(\theta,\phi)$ is the scattering amplitude. [9]

Solution (a) Bookwork based on approximating the exponent in the Lippmann–Schwinger equation by

$$k|\boldsymbol{r}-\boldsymbol{r}'| \sim kr - k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'$$

(b) Write the answer to (a) as a double integral

$$\sigma_{\text{total}} = \left(\frac{m}{2\pi\hbar^2}\right)^2\int d\Omega_{\boldsymbol{k}_f}\int d^3\boldsymbol{r}d^3\boldsymbol{r}'\, e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}')}V(\boldsymbol{r})V(\boldsymbol{r}').$$

Doing the integral over the solid angle gives the result. (c) Second Born approximation to the scattered wave is

$$\psi^{(2)}_{\boldsymbol{k}}(\boldsymbol{r}) = \left(\frac{m}{2\pi\hbar^2}\right)^2\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2\,\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}_1|}}{|\boldsymbol{r}-\boldsymbol{r}_1|}V(\boldsymbol{r}_1)\frac{e^{ik|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}V(\boldsymbol{r}_2)e^{i\boldsymbol{k}\cdot\boldsymbol{r}_2},$$

which gives the contribution to the scattering amplitude

$$f^{(2)} = \left(\frac{m}{2\pi\hbar^2}\right)^2\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2\, e^{-i\boldsymbol{k}_f\cdot\boldsymbol{r}_1}V(\boldsymbol{r}_1)\frac{e^{ik|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}V(\boldsymbol{r}_2)e^{i\boldsymbol{k}\cdot\boldsymbol{r}_2}.$$

(d) Evaluating at $\boldsymbol{k} = \boldsymbol{k}_f$ gives the forward scattering amplitude, which checks with the optical theorem.

4 A product state of a system of identical bosons or fermions is described in terms of single particle states $\varphi_\alpha(\boldsymbol{r})$, occupation numbers N_α, and creation and annihilation operators $a_\alpha^\dagger$, a_α. The field operator has the form

$$\psi(\boldsymbol{r}) \equiv \sum_\beta \varphi_\beta(\boldsymbol{r}) a_\beta,$$

while the density operator is

$$\hat{\rho}(\boldsymbol{x}) \equiv \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}).$$

(a) Show that in a product state the density-density correlation function has the form (with the plus sign for bosons and minus sign for fermions)

$$C_\rho(\boldsymbol{r},\boldsymbol{r}') \equiv \langle :\rho(\boldsymbol{r})\,\rho(\boldsymbol{r}'):\rangle = \langle\rho(\boldsymbol{r})\rangle\,\langle\rho(\boldsymbol{r}')\rangle \pm g(\boldsymbol{r},\boldsymbol{r}')g(\boldsymbol{r}',\boldsymbol{r}).$$

Here $:\cdots:$ denotes normal ordering, $\langle\rho(\boldsymbol{r})\rangle$ is the expectation value of the density, and

$$g(\boldsymbol{r},\boldsymbol{r}') = \sum_\alpha N_\alpha \varphi_\alpha^*(\boldsymbol{r})\varphi_\alpha(\boldsymbol{r}')$$

is the single particle density matrix. The term with both creation and both annihilation operators corresponding to the same state may be neglected in the limit of a large system. [10]

(b) Assuming that $g(\boldsymbol{r},\boldsymbol{r}') \to 0$ as $|\boldsymbol{r}-\boldsymbol{r}'| \to \infty$, find the ratio

$$\frac{C_\rho(\boldsymbol{r},\boldsymbol{r})}{\lim_{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} C_\rho(\boldsymbol{r},\boldsymbol{r}')}$$

for both bosons and fermions. [4]

(c) Now find the form of the three point correlation function

$$C_\rho^{(3)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3) \equiv \langle :\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)\rho(\boldsymbol{r}_3):\rangle,$$

expressing your answer in terms of $\langle\rho(\boldsymbol{r})\rangle$ and $g(\boldsymbol{r},\boldsymbol{r}')$.

[15]

(d) Find the ratio

$$\frac{C_\rho^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})}{\lim_{\substack{|\boldsymbol{r}_1-\boldsymbol{r}_2|\to\infty \\ |\boldsymbol{r}_1-\boldsymbol{r}_3|\to\infty \\ |\boldsymbol{r}_2-\boldsymbol{r}_3|\to\infty}} C_\rho^{(3)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)}$$

for both bosons and fermions. [4]

Solution(a) From the definition

$$
\begin{aligned}
C_\rho(\boldsymbol{r},\boldsymbol{r}') &= \pm \sum_{\alpha\beta\gamma\delta} \varphi^*_\alpha(\boldsymbol{r})\varphi_\beta(\boldsymbol{r})\varphi^*_\gamma(\boldsymbol{r}')\varphi_\delta(\boldsymbol{r}')\langle a^\dagger_\alpha a^\dagger_\gamma a_\beta a_\delta\rangle \\
&= \sum_{\alpha\gamma} |\varphi_\alpha(\boldsymbol{r})|^2|\varphi_\gamma(\boldsymbol{r}')|^2 N_\alpha N_\gamma \pm \sum_{\alpha\gamma} \varphi^*_\alpha(\boldsymbol{r})\varphi_\alpha(\boldsymbol{r}')\varphi^*_\gamma(\boldsymbol{r}')\varphi_\gamma(\boldsymbol{r}) N_\alpha N_\gamma \\
&= \langle\rho(\boldsymbol{r})\rangle\,\langle\rho(\boldsymbol{r}')\rangle \pm g(\boldsymbol{r},\boldsymbol{r}')g(\boldsymbol{r}',\boldsymbol{r}).
\end{aligned}
$$

Notice that the case $\alpha = \beta = \gamma = \delta$ is not handled correctly by this formula, but provides a negligible contribution as the size of the system goes to infinity. (b) The ratio is 2 for bosons and 0 for fermions. (c) The three point function is a straightforward generalisation of the calculation for the two point case. There are six different ways to pair the indices of the three creation operators with those of the three annihilation operators. Any such pairing results in an occupation number factor.

$$
\begin{aligned}
C^{(3)}_\rho(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3) &= \pm \sum_{abcdef} \varphi^*_a(\boldsymbol{r}_1)\varphi_b(\boldsymbol{r}_1)\varphi^*_c(\boldsymbol{r}_2)\varphi_d(\boldsymbol{r}_2)\varphi^*_e(\boldsymbol{r}_3)\varphi_f(\boldsymbol{r}_3)\langle a^\dagger_a a^\dagger_c a^\dagger_e a_b a_d a_f\rangle \\
&= \langle\rho(\boldsymbol{r}_1)\rangle\langle\rho(\boldsymbol{r}_2)\rangle\langle\rho(\boldsymbol{r}_3)\rangle \pm g(\boldsymbol{r}_1,\boldsymbol{r}_2)g(\boldsymbol{r}_2,\boldsymbol{r}_1)\langle\rho(\boldsymbol{r}_3)\rangle \\
&\quad \pm g(\boldsymbol{r}_1,\boldsymbol{r}_3)g(\boldsymbol{r}_3,\boldsymbol{r}_1)\langle\rho(\boldsymbol{r}_2)\rangle \pm g(\boldsymbol{r}_2,\boldsymbol{r}_3)g(\boldsymbol{r}_3,\boldsymbol{r}_2)\langle\rho(\boldsymbol{r}_1)\rangle \\
&\quad + g(\boldsymbol{r}_1,\boldsymbol{r}_2)g(\boldsymbol{r}_2,\boldsymbol{r}_3)g(\boldsymbol{r}_3,\boldsymbol{r}_1) + g(\boldsymbol{r}_1,\boldsymbol{r}_3)g(\boldsymbol{r}_3,\boldsymbol{r}_2)g(\boldsymbol{r}_2,\boldsymbol{r}_1)
\end{aligned}
$$

(d) The ratios are 6 and 0.

5 Consider a system of N bosons, each of which can occupy only two states: $|\uparrow\rangle$ and $|\downarrow\rangle$. We can introduce creation and annihilation operators $a_s^\dagger$, a_s, with $s = \uparrow, \downarrow$, to add and remove particles from the two states.

(a) Show that the operators

$$\begin{aligned} S_x &= \frac{\hbar}{2}\left(a_\uparrow^\dagger a_\downarrow + a_\downarrow^\dagger a_\uparrow\right) \\ S_y &= -i\frac{\hbar}{2}\left(a_\uparrow^\dagger a_\downarrow - a_\downarrow^\dagger a_\uparrow\right) \\ S_z &= \frac{\hbar}{2}\left(a_\uparrow^\dagger a_\uparrow - a_\downarrow^\dagger a_\downarrow\right) \end{aligned}$$

obey the angular momentum ($SU(2)$) commutation relations. [7]

(b) Show that $\boldsymbol{S}^2 \equiv S_x^2 + S_y^2 + S_z^2$ can be expressed in terms of N, the total number of bosons, and find the relationship between the spin quantum number s and N. [8]

(c) A totally symmetric wavefunction of N bosons can be written as $\Psi_{(s_1 s_2 \cdots s_N)}$, where the round brackets denote the operation of symmetrisation:

$$\Psi_{(s_1 s_2 \cdots s_N)} = \frac{1}{N!}\sum_P \Psi_{s_{P1} s_{P2} \cdots s_{PN}},$$

and the sum is over all permutations of N objects. How many independent components are needed to describe $\Psi_{(s_1 s_2 \cdots s_N)}$? Interpret this result in terms of angular momentum states. [5]

(d) What are the defining properties of the Lie group $SU(2)$? [3]

(e) Under an element U of $SU(2)$, the components ψ_s of a one boson state transform as $\psi \to \mathsf{U}\psi$. If ϕ_s and χ_s are the components of two one boson states, show that the quantity $\phi_\uparrow \chi_\downarrow - \phi_\downarrow \chi_\uparrow$ is invariant under $SU(2)$ transformations.

[10]

Solution (a) Straightforward. (b) We first find

$$\begin{aligned} S_x^2 + S_y^2 &= \frac{\hbar^2}{4}\left(\left[a_\uparrow^\dagger a_\downarrow + a_\downarrow^\dagger a_\uparrow\right]^2 - \left[a_\uparrow^\dagger a_\downarrow - a_\downarrow^\dagger a_\uparrow\right]^2\right) = \frac{\hbar^2}{2}\left(a_\uparrow^\dagger a_\downarrow a_\downarrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\uparrow a_\uparrow^\dagger a_\downarrow\right) \\ &= \frac{\hbar^2}{2}\left(N_\uparrow\left[N_\downarrow + 1\right) + N_\downarrow\left[N_\uparrow + 1\right)\right] \end{aligned}$$

Together with $S_z^2 = \frac{\hbar^2}{4}\left(N_\uparrow - N_\downarrow\right)^2$ we get $\boldsymbol{S}^2 = \hbar^2 s(s+1)$ where $s = (N_\uparrow + N_\downarrow)/2$. (c) $N + 1 = 2s = 1$ components required. This is just the number of independent angular momentum states of spin s. (d) 2×2 unitary matrices of determinant one. (e) For a general 2×2 matrix

$$\mathsf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we find that

$$\phi_\uparrow \chi_\downarrow - \phi_\downarrow \chi_\uparrow \to (ad - bc)\,(\phi_\uparrow \chi_\downarrow - \phi_\downarrow \chi_\uparrow)\ ,$$

so as long as the determinant is one this quantity is invariant (so being unitary is actually a red herring).

6 The one dimensional Klein–Gordon equation has the form

$$\left[\left(i\hbar\frac{\partial}{\partial t} - V(x)\right)^2 + c^2\hbar^2\frac{\partial^2}{\partial x^2} - m^2c^4\right]\psi(x,t) = 0,$$

where $V(x)$ is an external potential.

(a) When $V(x) = 0$, a general solution of the Klein–Gordon equation can be written

$$\psi(x,t) = \sum_k \sqrt{\frac{1}{2\omega_k}}\left[a_k \exp\left(i\left[kx - \omega_k t\right]\right) + b_k^\dagger \exp\left(-i\left[kx - \omega_k t\right]\right)\right],$$

where $\omega_k = \sqrt{k^2c^2 + m^2c^4/\hbar^2}$. How is this solution modified when $V(x) = V_0$? [6]

(b) Consider the potential step

$$V(x) = \begin{cases} V_0 & x > 0 \\ 0 & x < 0 \end{cases}.$$

Find the transmission and reflection amplitudes for an incoming plane wave of energy $E > mc^2$ incident from $x < 0$. [9]

(c) Paying careful attention to the analytic structure of the reflection and transmission amplitudes, describe their behaviour in the three regimes

$$\begin{aligned} I &: E > V_0 + mc^2 \\ II &: V_0 - mc^2 < E < V_0 + mc^2 \\ III &: E < V_0 - mc^2, \end{aligned}$$

assuming $V_0 > 2mc^2$. [12]

(d) What is the physical interpretation of the behaviour in regime III? [6]

Solution (a)

$$\psi(x,t) = \sum_k \sqrt{\frac{1}{2\omega_k}}\left[a_k \exp\left(i\left[kx - (\omega_k + V_0/\hbar)\,t\right]\right) + b_k^\dagger \exp\left(-i\left[kx - (\omega_k - V_0/\hbar)\,t\right]\right)\right],$$

(b) A plane wave incident from $x < 0$ has wavevector $k = \frac{1}{\hbar c}\sqrt{E^2 - m^2c^4}$ and gives rise to a wave on $x > 0$ with $q = \frac{1}{\hbar c}\sqrt{(E - V_0)^2 - m^2c^4}$, as well as reflected wave. Writing

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & x < 0 \\ te^{iqx} & x > 0 \end{cases}$$

we find that continuity of the wavefuntion and its first derivative yield

$$t = \frac{2}{1 + q/k}, \qquad r = \frac{1 - q/k}{1 + q/k}.$$

It would also be acceptable to normalise these to the flux of the particles by including a factor $\frac{q}{k}$ in t.
(c) In regime I, k and q are real and positive, and the reflection amplitude is decreasing to zero with increasing energy, while the transmission amplitude tends to 1. In regime II q is imaginary, so that r is a complex number of unit modulus. Although t is finite it is the amplitude of an evanescent wave. In regime III both k and q are real once more, but q is *negative* (that is, if we are taking E to have a positive imaginary infinitesimal, q is positive in regime I – as is physically necessary – and negative in regime III). Thus both t and r are becoming large.
(d) In regime III the potential step is able to create particle-antiparticle pairs, with the particle going to $-\infty$ and the antiparticle to $+\infty$. Ideally, reference should be made to part (a), noting that in this regime the dispersion of the particles on $x < 0$ overlaps with the dispersion of antiparticles on $x > 0$. Those especially on the ball will note that negative q antiparticles have a positive group velocity (i.e. moving away from $x = 0$).

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