Tale 64
The Envelope.

1. The boundary of the targer area. Consider a simple mechanical problem: a gun shuts a bullet with initial velocity v under angle ϕ to horizon. The trajectory of the bullet $\{x(t), y(t)\}$, parametrized by time t, is given by the following equations:

$$x(t) = vt\cos\phi; \qquad y(t) = vt\sin\phi - \frac{gt^2}{2}. \tag{1}$$

The maximal hight $h = v^2/2g$ corresponds to initial angle $\phi = \pi/2$, while the maximal horizontal length $L = 2h$ corresponds to initial angle $\phi = \pi/4$. Excluding time t from Eqs (1), obtain the shape of the bullet's trajectory

$$y = x\ \theta - \frac{4\ x^2}{h}\ (1+\theta^2), \qquad \theta = \tan\phi, \tag{2}$$

corresponding to different initial angles $\pi/2 \geq \phi \geq \pi/4$. All these trajectories cover the part of $(x,\ y)$-plane in which the target can be hit by the bullet. The problem is to find out the boundary of this area (shown by the dashed line in Fig 1). At each its point $(X,\ Y)$, this boundary meets

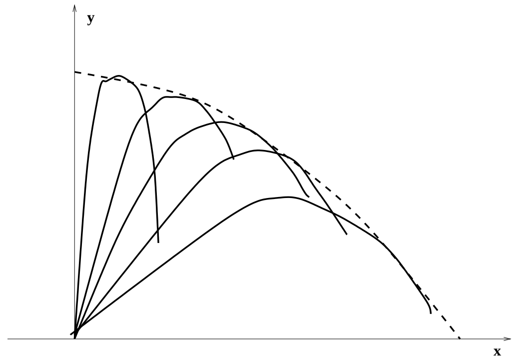

Figure 1: Envelope of the family of trajectories

tangentially one of trajectories $y(x,\ \theta)$. This is why the dashed line $Y(X)$ is called "the envelope of the family of lines". At each point, the envelope corresponds to maximal hight y at given abscissa X. Therefore,

$$\frac{\partial y(X,\ \theta)}{\partial \theta} = 0. \tag{3}$$

Using Eq (2), this gives the condition (3) in the following form:

$$0 = X - \frac{X^2}{4h}\,\theta; \tag{4}$$

which shows that the envelope

$$Y = h - \frac{X^2}{4h} \tag{5}$$

to the family of parabolic trajectories (2) is a parabola itself[1].

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[1] The receipt:

$$Y = y(X, \theta), \qquad \left(\frac{\partial y}{\partial \theta}\right)_X = 0$$

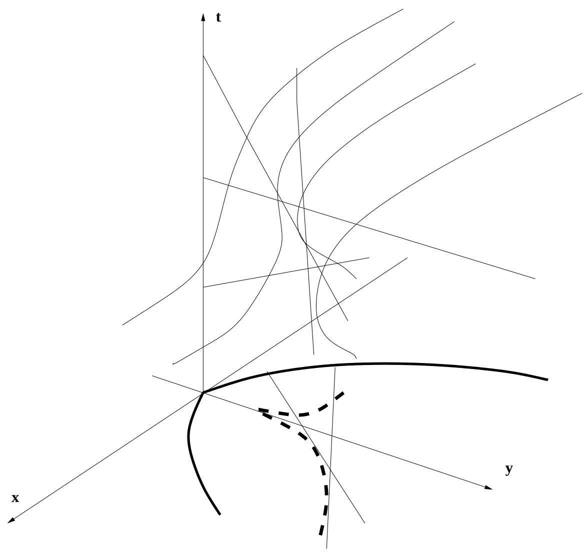

Figure 2: Enveloping curve and enveloping surface of the family of straigth lines - normals to a parabola

2. Enveloping surface and its singularity. Consider now a more complicated case. Let we have the parabola, which we present in the form:

$$x = t, \qquad y = t^2. \tag{8}$$

At any point of parabola, characterized by parameter t, let

can be rewritten if the family is given the form of equation $F(x, y, \theta) = 0$. Then it reads

$$F(X, Y, \theta) = 0, \quad \left(\frac{\partial F}{\partial \theta}\right)_{X,Y} = 0 \tag{6}$$

For the case of the family is given in the parametric form $x(t, \theta),\ y(t, \theta)$, the envelope may be given in the form:

$$X(t) = x(t, \theta),\ Y(t) = y(t, \theta),\ \dot{x}\left(\frac{\partial y}{\partial \theta}\right)_t + \dot{y}\left(\frac{\partial x}{\partial \theta}\right)_t = 0. \tag{7}$$

us draw a normal to it:

$$y - t^2 = \frac{x - t}{-1/2t} \to 2t^3 + t(1 - 2y) - x = 0. \qquad (9)$$

The family of straight lines given by Eq (9) can be drawn either on the plane (x, y) or in the space (x, y, t). In both cases, the family has an evelope: either an enveloping line $Y(X)$ or an enveloping surface $t(x, y)$. Looking at this surface (Fig 2), we can see that it has a fold. This means that, for any given x and y, Eq (9) has either one or three solutions. The projection of the edges of this fold to the plane (x, y) is precisely the envelope $Y(X)$, which should be found from the system of equations:

$$\left.\begin{array}{l} 2t^3 + t(1 - 2y) - x = 0 \\ 3t^2 - y = 0 \end{array}\right\} \to 27(y - 1/2)^3 - 8x^2 = 0 \quad (10)$$

and, as it is seen from Eq (10), has the shape of the half-cubic parabola. The cusp of the enveloping curve at the point $(0, .5)$ corresponds to the void of the fold of the enveloping surface.

3. Evolute and envelope. Consider a pendulum, restricted from above by a rigid body. The contour of this body is shown in Fig 3 by curve **C**. Pendulum suspension point let be A, and the length of the string let be equal to the length of the arch of the curve **C** between points A and B. So, let us keep the heavy body at the point B and, therefore, the string goes along the curve **C**. After we release the heavy body at the end of the string, the pendulum swings, drawing the curve **C'**, called *the evolute* of curve **C**. The property of the evolute is that the normal straight line

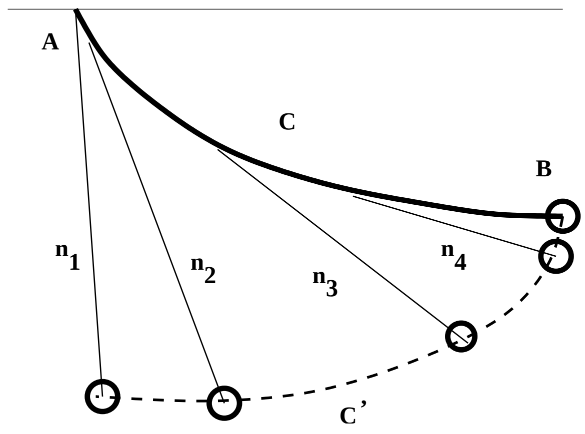

Figure 3: Evolute Pendulum.

n_i at any of its point is touching tangentially the dual curve **C**. Therefore, the dual curve **C** is an envelope of the family of the normals n_i of its evolute **C'**.

EXAMPLE. Show that is the curve **C** is a cycloid, then its evolute is identical cycloid, shifted by half a period to the left (right) and by its hight down. Find the length of the total arch of a cycloid.