Tale 47
π^2 in the exponent

This tale from my own experience is very instructive. As students we were taught to estimate solutions beforehand up to a numerical factor, but it turned out that the powers of π could (and should) also be estimated.

Ballistic point contact

Consider electron in a two-dimensional potential, which has the form of two large reservoirs with a narrow channel between them, as shown in Fig 1. For simplicity, let as consider the boundary a rectangular wall of infinite hight. If a voltage V is applied between the reservoirs, the current $I = GV$ (G is the conductance) will flow through the channel. Its width $w(x)$ as a function of coordinate x along the channel has the form

$$w(x) = w + \frac{x^2}{2R}, \tag{1}$$

where w is the width in the center of the constriction, and R is the curvature radius of the boundary at this center. If $w \ll R$, the width changes adiabatically with x. In another words, for each value of x, the channel has almost parallel boundaries at a local width $w(x)$. It is this picture we have in mind, when we apply the adiabatic approximation to our problem. In this approximation, the wave function $\Psi(x, y)$ can be factorized $\Psi(x, y) = \phi_x(y)\psi(x)$, and the functions $\phi_x(y)$ and $\psi(x)$ satisfy the equations:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_n}{dy^2} = \epsilon_n(x)\phi \quad \phi_n(0) = \phi_n(w(x)) = 0; \tag{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \epsilon_n(x)\psi = E\psi. \tag{3}$$

The solution of Eq (2) is

$$\phi_n(y) \propto \sin\left(\frac{\pi n}{w(x)}\, y\right); \qquad \epsilon_n(x) = \frac{\hbar^2\pi^2 n^2}{2mw^2(x)}. \tag{4}$$

Thus, for each value of x the wave function has $n-1$ nodes in the transverse direction, which gives rise to the effective potential $\epsilon_n(x)$ (see Fig 2), affecting the motion of the electron in the x-direction. If $w \ll R$ and adiabatic approximation is valid, the potential $\epsilon_n(x)$ is smooth and obeys the conditions of the semi-classical approximation. Therefore, the transmission coefficient T_n is equal 1, if the energy E is greater than the hight of n-th barrier, and equals zero in the opposite case. According to the Landauer formula, the conductance is related to T_n and, therefore,

$$G = \frac{e^2}{2\pi\hbar}\sum_n T_n = \left[\frac{kw}{\pi}\right], \quad k = \sqrt{\frac{2mE}{\hbar^2}}. \tag{5}$$

where $[A]$ denotes a the integer part of A. So, the conductance G as a function of w is quantized, i.e. the dependence $G(w)$ has steep steps and very flat plateaus, as shown in Fif 3. The value of conductance at the n-th plateau is $e^2 n/2\pi\hbar$.

Parabolic Barrier

This result is approximative, of course, approximate and valid if $w < R$. The largest deviation corresponds to that value of energy E, which coincides with top of of the n-th barrier. At this energy T_n is neither 1, nor 0. Fortunately, any smooth barrier we can approximate near its top by a parabola

$$U(x) = -\frac{m\omega^2 x^2}{2},$$

so the transmition coefficient could be found in the most general form (see Appendix 3 to Tale 6):

$$T = \left\{1 + \exp\left[-\frac{2\pi\epsilon}{\hbar\omega}\right]\right\}^{-1}, \tag{6}$$

where the energy ϵ is counted from the top of the barrier. As a result, shape of step in conductance could be presented as

$$\delta G = \frac{e^2}{2\pi\hbar} \cdot \left\{1 + \exp\left[-\frac{z}{\delta z}\right]\right\}^{-1}; \tag{7}$$

$$z = \frac{kw}{\pi} - \left[\frac{kw}{\pi}\right], \qquad \delta z = \frac{1}{\pi^2}\sqrt{\frac{w}{2R}}. \tag{8}$$

If to use dimensionless variable z, the length of the plateau is 1, while the width of the step is δz. Eq (8) shows that if $w \ll R$ $\delta z \ll 1$. But even if $w \approx R$, the step remains very steep because of the factor π^2. This gives for precision of ballistic quantization at $w \approx R$

$$\frac{2\pi\hbar}{e^2} \cdot \delta G \sim e^{-\pi^2} \approx 10^{-4}.$$

This amazing phenomenon that a dimensionless number of the order of unity turns out to be 10^{-4} is a very curious lesson. It must be most seriously taken by those, who disbelieve in miracles.