Tale 46
Green Function and Resistance Networks

Consider a square network formed by identical resistors of unit length and of resistance R_0. What is the resistance between neighboring sites 1 and 2 (see Fig 1.)? In order to find an answer

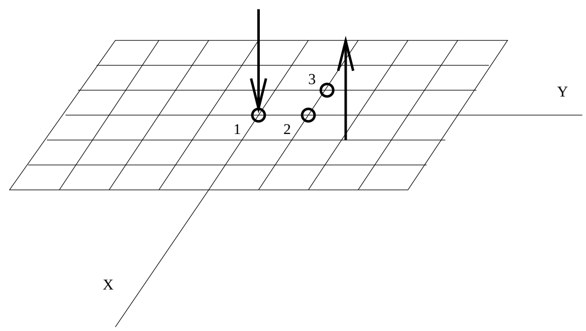

Figure 1: Resistors network with two leads, through which the current is entering and leaving the circuit.

to this, let us assume that the electrostatic potential $\phi(\mathbf{r})$ at infinity is zero ($\phi(\infty) = 0$). Let us consider a source of current I at site 1 and a drain of the same strength I at site 2. The resulting current distribution is the superposition of two current distribitions: the current flowing from point 1 to infinity and that flowing from infinity to point 2. Since the corrent flowing along the bond $1 - 2$ in both these distruibutions is $I/4$, the resulting current is $I/2$ and, therefore, the resistance $R_{12} = R_0/2$.

Now, what is the resistance between next nearest neighboring sites 1 and 3 along a diagonal [1]

[1] L.S.Levitov says that this question is popular at the entrance exams in Ecole Normale

Assume again that there are a source and drain at sites current 1 and 3 respectively. To find the voltage drop V_{13}, we will consider the problem in a general form.

According to Ohm's law the voltage drop $V_{\mathbf{r}-\mathbf{r}'} = \phi(\mathbf{r}) - \phi(\mathbf{r}')$ along any resistor in the network is

$$R_0 I(\mathbf{r},\mathbf{r}') = \phi(\mathbf{r}) - \phi(\mathbf{r}'). \tag{1}$$

The current conservation leads to the following equation

$$\begin{aligned} &I(x+1,y;x,y) + I(x-1,y;x,y) + I(x,y+1;x,y) + \\ &+ I(x,y-1;x,y) = -I\delta(\mathbf{r}-\mathbf{r}_s) + I\delta(\mathbf{r}-\mathbf{r}_d), \end{aligned} \tag{2}$$

where $\mathbf{r}_s$ and $\mathbf{r}_d$ are coordinates of the source and drain respectively. Therefore, the electrostatic potential $\phi(\mathbf{r})$ obeys the following equation in finite differences:

$$\begin{aligned} &\phi(x+1,y) + \phi(x-1,y) + \phi(x,y+1) + \phi(x,y-1) - \\ &-4\,\phi(x,y) = IR_0\left[\delta(\mathbf{r}-\mathbf{r}_s) + I\delta(\mathbf{r}-\mathbf{r}_d)\right]. \end{aligned} \tag{3}$$

The boundary condition as $\mathbf{r} \to \infty$ is $\phi(\infty) = 0$. If the Green's function $G(x,y;x',y')$ is determined by the equation

$$\begin{aligned} &G(x+1,y;x',y') + G(x-1,y;x',y') + G(x,y+1;x',y') + \\ &+ G(x,y-1;x',y') - 4\,G(x,y;x',y') = \delta(\mathbf{r}-\mathbf{r}'), \end{aligned} \tag{4}$$

then the potential drop $V_{\mathbf{r},\mathbf{r}'}$ between $\mathbf{r}$ and $\mathbf{r}'$ is

$$V_{\mathbf{r}-\mathbf{r}'} = 2IR_0[G(0) - G(|\mathbf{r}-\mathbf{r}'|)] \tag{5}$$

The Green's function $G(\mathbf{r},\mathbf{r}')$ can be expressed through the complete set of the wave functions $\psi_\lambda(\mathbf{r})$ and the eigenvalues λ of

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the discrete Laplace operator in the left-hand side of Eq (4):

$$G(\mathbf{r},\mathbf{r}') = \sum_{\lambda} \frac{\psi_\lambda(\mathbf{r})\psi^*_\lambda(\mathbf{r}')}{\lambda} \tag{6}$$

The complete set of the wave functions on a square lattice is given by the functions:

$$\psi(\mathbf{r},\mathbf{k}) = \sqrt{\frac{1}{S}}\exp[i\mathbf{kr}], \quad v_0 = 1 \tag{7}$$

(where S is the total area of the network) with eigenvalues

$$\lambda(\mathbf{k}) = 2 - \cos k_x - \cos k_y \tag{8}$$

So, the Green's function is equal to the following integral over the Brillouin zone (BZ) $-\pi < k_{x,y} < \pi$:

$$G(\mathbf{r},\mathbf{r}') = \int_{BZ}(d\mathbf{k})\frac{\exp[i\mathbf{k}\ (\mathbf{r}-\mathbf{r}')]}{2-\cos k_x - \cos k_y}. \tag{9}$$

As a result, $R_{\rm eff}$ can be expressed through a double integral:

$$R_{\rm eff}(|\mathbf{r}|) = R_0 \int_{-\pi}^{\pi}\frac{dk_x}{2\pi}\int_{-\pi}^{\pi}\frac{dk_y}{2\pi}\ \frac{1-\cos k_x x \cos k_y y}{2-\cos k_x - \cos k_y}. \tag{10}$$

In the case $x = 1, y = 0$ the Eq (10) gives

$$R_{\rm eff} = \frac{R_0}{2}.$$

For $x = 1, y = 1$ (diagonal), the integral in Eq (10) is equal to

$$R_{\rm eff}(1,1) = \int_{-\pi}^{\pi}\frac{dk_x}{2\pi}\int_{-\pi}^{\pi}\frac{dk_y}{2\pi}\ \frac{1-\cos k_x \cos k_y}{2-\cos k_x - \cos k_y} = \frac{2R_0}{\pi}. \tag{11}$$