Tale 3
Supersymmetric Quantum Mechanics

1. let us begin from well-known example of a linear oscillator, which has the Hamiltonian:

$$\hat{H}\psi = E\psi, \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 x^2}{2} \quad [\hat{p}, \hat{x}] = -i\hbar \tag{1}$$

Or, in dimensionless variables:

$$P = \sqrt{\frac{1}{m\hbar\omega}}p, \qquad q = \sqrt{\frac{m\omega}{\hbar}}x; \tag{2}$$

$$H = \hbar\omega\tilde{H}, \quad \tilde{H} = \frac{1}{2}\left(P^2 + q^2\right), \quad [\hat{P}, \hat{q}] = -i. \tag{3}$$

The Hamiltonians of Eqs (1) and (3) lead to a differential equation of the second order, which, however, could be reduced to that of the first order by introduction the creation and annihilation operators $\hat{a}^+$ and $\hat{a}$:

$$\hat{a} = \frac{\hat{q} - i\hat{P}}{\sqrt{2}}, \quad \hat{a}^+ = \frac{q + iP}{\sqrt{2}}, \quad [\hat{a}^+, \hat{a}] = 1, \tag{4}$$

$$\tilde{H} = \frac{1}{2}\left(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+\right) = \hat{a}^+\hat{a} + \frac{1}{2}. \tag{5}$$

This remarkable substitution allows to find all energy levels. Indeed,

$$|0>: \quad \hat{a}|0>= 0 \to E_0 = \frac{1}{2},$$

then

$$|1>: \quad |1>= \hat{a}^{+}|0> \rightarrow E_1 = \frac{3}{2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|n>: \quad |n>= (\hat{a}^{+})^n|0> \rightarrow E_n = n + \frac{1}{2}$$

The condition $\hat{a}|0>= 0$ is the first order equation

$$\left(q + \frac{d}{dx}\right)\psi_0(q) = 0, \qquad \psi_0(q) \propto \exp\left[-\frac{q^2}{2}\right]. \tag{6}$$

The wave functions of the excited states could be found by the action of the a^{+}-operator on $\psi_0(q)$.

One more example: Dirac's electron in two dimensions with the spin 1/2 and Lande-factor $g = 2$. Its Hamiltonian has the form

$$H = \frac{\left(p - \frac{e}{c}A\right)^2}{2m} - g\mu H\sigma^z = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 - \frac{e\hbar}{mc}H\sigma^z. \tag{7}$$

This Hamiltonian[1] can be written as

$$H = \frac{1}{2}\hbar\omega(a^{+}a + aa^{+} + f^{+}f - ff^{+})$$

where

$$[a, a^{+}] = a^{+}a - aa^{+} = 1; \qquad \left\{f^{+}, f\right\} = f^{+}f + ff^{+} = 1$$

The operators $\hat{a}$ and $\hat{a}^{+}$ are Bose-like, while the operators $\hat{f}$ and $\hat{f}^{+}$ are Fermi-like. So,

$$H = \hbar\omega(a^{+}a + \frac{1}{2} + f^{+}f - \frac{1}{2}) = \hbar\omega(a^{+}a + f^{+}f) = \hbar\omega(n_B + n_F).$$

[1]We omitted an infinite degeneracy in values of momenta p, which does not enter, finally, into Hamiltonian

Therefore, the energy levels $E(n_B, n_F)$ are doubly degenerate, if the numbers of bosons and fermions both do not vanish. If one introduces the vector notation

$$\Phi = \begin{pmatrix} a \\ f \end{pmatrix}, \qquad \Phi^+ = (a^+ \quad f^+)$$

then

$$\Phi\Phi^+ = \begin{pmatrix} aa^+ & af^+ \\ fa^+ & ff^+ \end{pmatrix}, \qquad H = \Phi^+\Phi.$$

The vector Φ may be transformed by the unitary transformation $\Phi' = \hat{U}\Phi$. It gives $\Phi'^+ = \Phi^+\hat{U}^+$ for the conjugate vector Φ^+, and $\Phi'^+\Phi' = \Phi^+\hat{U}^+\hat{U}\Phi = \Phi^+\Phi$. The unitary matrix $\hat{U}$ has a substructure

$$U = \begin{pmatrix} U_{aa} & U_{af} \\ U_{fa} & U_{ff} \end{pmatrix}$$

It is convenient now to introduce the operators transforming fermions into bosons and vice versa

$$\hat{Q} \propto a^+ f \qquad \hat{Q}^+ \propto af^+$$

$$\hat{Q}|n_B, n_F> \propto |n_B{+}1, n_F{-}1> \quad \hat{Q}^+|n_B, n_F> \propto |n_B{-}1, n_F{+}1>$$

The operators $\hat{Q}$ and $\hat{Q}^+$ are nilpotent, i.e. $Q^2 = (Q^+)^2 = 0$. It is also convenient to introduce the Hermitian operators

$$\hat{Q}_1 = \hat{Q}^+ + \hat{Q}, \quad \hat{Q}_2 = -i(\hat{Q}^+ - \hat{Q})$$

Direct calculation gives

$$\{Q_1, Q_2\} = Q_1Q_2 + Q_2Q_1 = 0,$$

and

$$Q_1^2 = Q_2^2 = \{Q^+, Q\}.$$

If the Hamiltonian has the form $H = \{Q^+, Q\}$, then the system this Hamiltonian describes is supersymmetrically invariant, i.e $[H, Q] = [H, Q^+] = [H, Q_{1,2}] = 0$. It is interesting to look at the superalgebra of the operators

$$\{Q_i, Q_j\} = 2\delta_{ij}H; \quad [H, Q_i] = 0 \quad i, j = 1, 2$$

A mathematician would say that this is the Lie Superalgebra or the Gradiate Lie algebra, but it sounds more like a boring science rather, than a fairy tale. In the fairy tale, Q-operators are square roots of the Hamiltonian H. Therefore, if

$$Q_1\psi_1 = q\psi_1,$$

then

$$H\psi_1 = q^2\psi_1$$

and if

$$\psi_2 = Q_2\psi_1,$$

then

$$Q_1\psi_2 = Q_1Q_2\psi_1 = -Q_2Q_1\psi_1 = -q\psi_1$$

i.e. ψ_2 is an eigenfunction of $\hat{Q}_1$ with the eigenvalue equal to $-q$. On the other hand, since $[H, Q] = 0$,

$$H\psi_2 = HQ_2\psi_1 = Q_2H\psi_1 = Q_2q^2\psi_1 = q^2\psi_2$$

and, therefore, ψ_2 is an eigenfunction of $\hat{H}$ with the same eigenvalue q^2. Therefore, if $q \neq 0$, then all levels of H are two-fold degenerate (and correspond to superpartners).

One more step forward. Consider the operators $\hat{Q}_{1,2}, \hat{Q}, \hat{Q}^+$ acting on spinors ($n_F = 0$ and $n_F = 1$). In this basis $f^+ = \sigma^+$

and $f = \sigma^-$, while $Q^+ = Bf^+$ and $Q = B^+f$ where $\hat{B}$ is an arbitrary boson operator. Introducing

$$B^+ = B_1 + iB_2 \qquad B = B_1 - iB_2$$

one has

$$Q_1 = Q^+ + Q = (B_1 - iB_2)\frac{\sigma_1 + i\sigma_2}{2} + (B_1 + iB_2)\frac{\sigma_1 - i\sigma_2}{2} = B_1\sigma_1 + B_2\sigma_2$$

In the same way,

$$Q_2 = -i(Q^+ - Q) = B_1\sigma_2 - B_2\sigma_1$$

and

$$H = Q_1^2 = Q_2^2 = \frac{\{B, B^+\} + [B, B^+]\sigma_3}{2}$$

The particular case $\hat{B} = \hat{b}$ corresponds to the case of Dirac's electron in magnetic field, considered above. The representation we have found gives hope of splitting more complicated Hamiltonians, than we considered before. The main point of this splitting is to reduce the order of the Hamiltonian with respect to derivatives. Consider, for instance,

$$B = \frac{ip + W(x)}{\sqrt{2}}, \quad B = \frac{-ip + W(x)}{\sqrt{2}},$$

$$\{B, B^+\} = p^2 + W^2, \quad [B, B^+] = W'.$$

Then,

$$H = \frac{p^2 + W^2 + W'\sigma_3}{2},$$

or, in the matrix form,

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p^2 + W^2 + W' & 0 \\ 0 & p^2 + W^2 - W' \end{pmatrix} = \begin{pmatrix} BB^+ & 0 \\ 0 & B^+B \end{pmatrix}$$

Thus, we have found a general form of supersymmetric quantum mechanics. If now we want to solve an equation, say, of the form

$$H_-\psi = \frac{p^2 + W^2 - W'}{2}\psi = E\psi \tag{8}$$

we must recall that it has a superpartner

$$H_+\psi = \frac{p^2 + W^2 + W'}{2}\psi = E\psi \tag{9}$$

These two operators $H_\pm$ have the same eigenvalues E_n, except for the eigenvalue $E_0 = 0$, which only the operator H_- has. This very special eigenfunction ψ_0 is solution of the first order equation

$$B\psi_0 = 0.$$

Since $\hat{H}_- = \hat{B}^+\hat{B}$, the function ψ_0 is the eigenfunction of H_- with the eigenvalue $E_0 = 0$. Let the function ψ_1 be an eigenfunction of H_+ with the eigenvalue E_1. Then $\hat{B}^+\psi_1$ is an eigenfunction of H_- with the same eigenvalue. This helps us to solve the Schroedinger equation for a particular potential.

2. Let us choose

$$W(\alpha, x) = \alpha \tanh x, \tag{10}$$

then

$$H_-(\alpha) = \frac{1}{2}(p^2 + \alpha^2 - \frac{\alpha(\alpha+1)}{\cosh^2 x}) \tag{11}$$

$$H_+(\alpha) = \frac{1}{2}(p^2 + \alpha^2 - \frac{\alpha(\alpha-1)}{\cosh^2 x}) \tag{12}$$

Now we can see that

$$H_+(\alpha) = H_-(\alpha_1) - \frac{\alpha_1^2}{2} + \frac{\alpha^2}{2}$$

for $\alpha_1 = \alpha - 1$. Thus, in the potential well shown in the Fig 1 there is a level with $E = 0$. If $a > 0$, then

$$E_1 = \frac{\alpha^2 - \alpha_1^2}{2} = \frac{\alpha^2 - (\alpha - 1)^2}{2} = \frac{2\alpha - 1}{2}$$

$$E_2 = \frac{\alpha^2 - \alpha_1^2 + \alpha_1^2 - \alpha_2^2}{2} = \frac{\alpha^2 - (\alpha - 2)^2}{2} = \frac{4\alpha - 4}{2}$$

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$$E_n = \frac{(\alpha^2 - \alpha_1^2) + (\alpha_{n-1}^2 - \alpha_n^2)}{2} = \frac{\alpha^2 - (\alpha - n)^2}{2} = \frac{2n\alpha - n^2}{2}$$

It is obvious now that the exact solvability of the Schroedinger equation with the potential

$$U(x) = -\frac{a(a+1)}{\cosh^2 x}$$

is not accidental fact. It is connected with the hidden supersymmetry of the Hamiltonian, which has the eigenvalues

$$E_n = -\frac{(a-n)^2}{2}$$

for all n, satisfying the condition $0 \leq n \leq [a+1]$ [2]. It turns out to be convenient, in order to take the supersymmetry into account explicitly, to consider not only one Schroedinger equation

$$H_- \psi = E\psi,$$

but a couple of them, adding the superpartner equation. The matrix Hamiltonian

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

[2] Notation $[x]$ stays for integer par of x

has the supersymmetry, which helps to solve the equation.

3. One may remember, from the tale about solitons, that the solutions of the Schroedinger equation with the potential

$$U(x) = -\frac{1}{\cosh^2 x}$$

correspond to zero reflection coefficient at all positive energies. This quite strange statement may be supported by arguments, based on supersymmetry. Indeed, let $a = 1$ and the superpartners have the form

$$H_- = B^+B = \frac{p^2}{2} - \frac{1}{\cosh^2 x} + \frac{1}{2}, \tag{13}$$

$$H_+ = BB^+ = \frac{p^2}{2} + \frac{1}{2} \tag{14}$$

Therefore, the Hamiltonian H_- with the considered potential is supersymmetrically equivalent to the free motion Hamiltonian H_+. The Hamiltonian H_- has a zero level, and the eigenfunction, which corresponds to this level, obeys the equation

$$B\psi_0^{(-)} = (\frac{\partial}{\partial x} + \tanh x)\psi_0^{(-)} = 0$$

which has the solution $\psi_0^{(-)} = \cosh^{-1} x$. On the other hand, the ground state wave function of the Hamiltonian H_+ is just a constant and all the rest are running waves $\psi_k^{(+)} = exp^{ikx}$ The operator B^+ transforms the eigenfunctions of the Hamiltonian H^+ into the eigenfunctions of the Hamiltonian H_- with the same energy. Therefore,

$$\psi_k^{(-)} = B^+e^{ikx} = (-\frac{\partial}{\partial x} + \tanh x)e^{ikx} = (-ik + \tanh x)e^{ikx}$$

Thus, we have constructed the eigenfunction of continuous spectrum, valid at any real values of x and containing no reflected wave. Therefore the reflection coefficient is zero due to the supersymmetric equivalence of the Hamiltonian considered to that with no potential at all and of the eigenfunctions of its continuous spectrum to simple running waves. It is interesting that the wave function

$$\psi_{k=0}^{(-)} = \tanh x$$

corresponds to the bottom of continuous spectrum in the soliton-like potential.