QUANTUM CHAOS: LESSONS FROM DISORDERED METALS

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INTRODUCTION

The quantum description of systems which are chaotic in their classical limit is the subject of “Quantum Chaos”. A wide variety of physical systems fall into this category. Amongst those most commonly studied are the neutron resonances of complex atomic nuclei[1], Rydberg atoms in strong magnetic fields[2], and electrons in semiconducting nanostructures (known as “quantum dots”[3]). In contrast to integrable systems, eigenstates of quantum chaotic structures are characterized solely by their energy rather than by a set of quantum numbers. Therefore, a useful description of chaotic systems is a statistical one. It is the development of a framework in which the statistical properties of general quantum chaotic systems can be studied which forms the focus of these lectures.

The study of the quantum properties of systems which are chaotic in their classical limit has been conducted largely along two parallel lines. The first approach has been developed within the framework of a semi-classical approximation based on the Feynman path integral. This method, known in the literature as “Periodic orbit theory”, expresses the spectral quantum statistical properties of *clean chaotic* systems in terms of infinite sums over classical periodic trajectories — the “Gutzwiller Trace Formula”[4]. (Here the attribute ’clean’ means that systems without stochastic disorder are considered.) This theory has proved successful in describing the large scale structure of the spectrum, but its success in describing long-time universal behavior has been limited.

At the same time, a complementary approach based on methods of quantum field theory has been developed to study the average properties of an ensemble of *disordered* systems such as weakly disordered conductors[5]. As well as offering a description of the short-time diffusive dynamics, by surrendering information specific to each given system the field theoretic approach is capable of describing the longest time scales, where the quantum behavior displays a remarkable degree of universality.

Despite the fact that these two approaches account for the same qualitative behavior, it was only recently that a formal correspondence between them was established. The goal of these lectures will be to introduce and explore a third approach[6, 7] aimed at describing the quantum statistical properties of *individual* chaotic structures over a wide range of energy scales, encompassing both the universal and non-universal regimes.

This third approach, which relies on *energy averaging*, generates a field theory in which the effective action is associated with classical flow in phase space[7]. However, the basic ingredients of this approach are not individual periodic orbits but global modes of the time evolution operator of the underlying classical system. These results show that the statistical quantum properties of the system are intimately related to the *irreversible* classical chaotic dynamics or, more precisely, to the Perron-Frobenius modes, in which a classical probability density relaxes into the ergodic distribution. We will discuss how both random matrix theory, and the "diagonal approximation" of the periodic orbit theory emerge naturally as limits of the theory.

However, in arriving at the general field theoretic description we will review qualitative aspects of quantum coherence phenomena in disordered and mesoscopic structures. The lecture notes are organized as follows. In the section entitled "Spectral Statistics: A Brief History" we briefly review some of the milestones encountered in the development of a theory of quantum chaos. In the section "Coherence Effects in Disordered Conductors" we will investigate mesoscopic quantum coherence phenomena in weakly disordered metallic systems. Our emphasis will be placed on energy level correlations, weak localization phenomena, and universal mesoscopic fluctuations. This discussion will provide a useful platform on which to explore quantum coherence phenomena in non-random or ballistic chaotic structures. It will also reveal the pitfalls of a purely perturbative analysis. These qualitative ideas are put on a rigorous footing in the following section, "Supersymmetry Method". Using a field theoretic analysis, quantum statistical properties of weakly disordered conductors will be cast in the form of a functional field integral, involving an effective action which takes the form of a non-linear σ-model. Both perturbative and non-perturbative aspects of the theory, as well as the connection to random matrix theory, will be discussed. The field theoretic approach is not, however, restricted to random systems. In the section entitled "Quantum Chaos: The Ballistic σ-Model", we develop a general field theoretic approach to study quantum statistical properties of ballistic chaotic structures. Finally, in the closing section we summarize current understanding and offer an outlook on the outstanding problems.

SPECTRAL STATISTICS: A BRIEF HISTORY

Our discussion begins with a brief summary of the early progress in the field. The study of spectral correlations in chaotic quantum systems has a long history that dates back to the investigation of complex many-body quantum systems[9].

Wigner Surmise

Guided by the very complexity of the system, a statistical ansatz was introduced by Wigner to explain strong level correlations observed in resonance spectra of complex nuclei[10, 1]. Taking the matrix elements to be statistically uncorrelated Wigner obtained, within a two-level approximation, an estimate of the distribution of neighboring level spacings,

$$P(S) = \langle \delta\left(S - (E_{i+1} - E_i)\right)\rangle \sim \frac{\pi}{2}\frac{S}{\Delta} \exp\left[-\frac{\pi S^2}{4\Delta^2}\right], \tag{1}$$

where $\Delta = \langle E_{i+1} - E_i\rangle$ denotes the average level spacing. Surrendering information specific to each individual nucleus, Wigner was able to quantitatively account for the characteristic level repulsion observed in experiment. Forty years on, similar considerations are now given to the ground state properties of artificial atoms, or quantum dots[3] (with so far only limited success!).

Gor'kov-Eliashberg Theory

While successfully studied and applied to the physics of complex nuclei, it was not until almost twenty years later that similar ideas were applied to the study of the quantum properties of weakly disordered metallic grains. By identifying spectral correlations of the grains with those of random Hermitian matrix ensembles, Gor'kov and Eliashberg investigated the dielectric response of metallic grains embedded in an insulating matrix[8]. Although often neglected, this application anticipated applications of random matrix theory to the field of mesoscopic physics by two decades.

Bohigas-Giannoni-Schmidt Conjecture

Until 1981, the application of random matrix theory had been limited to the study of complex many-body or disordered quantum systems. An extensive numerical survey by Bohigas, Giannoni and Schmidt[11] lead them to the following conjecture:

- The spectral properties of quantum systems which are chaotic in their classical limit (even those with few degrees of freedom) are UNIVERSAL, independent of material properties of the system, and coincide with those of random matrix ensembles.

This conjecture presented a milestone in the study of quantum chaos. It provided a common thread which united a whole class of qualitatively different systems. So far, very few exceptions to this rule have been identified. And for those that have, it is usually simple to understand why they lie outside the present scheme. However, despite more than ten years of intense study, and the accumulation of a wealth of supporting data, this conjecture still awaits a rigorous mathematical proof.

Supersymmetry: Efetov's Non-linear σ-Model

Belonging to a statistical ensemble, average properties of disordered quantum systems are more susceptible to analytical investigation than individual systems. Providing the statistical average of an individual system (for example, over a range of energy levels) is statistically equivalent to average over an ensemble of similar systems, analytical results can be obtained systematically and rigorously.

Relying on such an *ergodicity hypothesis*, Efetov developed a field theoretic approach to the study of level correlations in weakly disordered metallic grains[5]. Building upon the seminal work of Wegner[12] and others[13, 14, 15, 16, 17, 18] on the phenomenon of Anderson localization, Efetov provided a rigorous mathematical foundation for the random matrix hypothesis introduced by Gor'kov and Eliashberg. However, perhaps more importantly, Efetov's approach provided a systematic way in which non-universal properties reflecting the intrinsic diffusive dynamics of the system could be studied.

Recent studies by Muzykantskii and Khmel'nitskii[6], and later by Andreev *et al.*[7] revealed that the supersymmetry approach introduced by Efetov is, in fact, not limited to random systems, but can find an extension to the wider class of chaotic structures. It is to this development that we turn towards the end of these lectures.

Random Matrix Theory

In the years that followed Wigner's statistical hypothesis, the general properties of random matrix ensembles was put on a firm footing by a number of mathematicians and physicists[19, 20]. In particular, a classification of three statistical (circular) ensembles was introduced by Dyson[21]. Focusing on the ensemble of Gaussian distributed random matrices,

$$P(H)dH \propto \exp\left[-\frac{N\beta}{\lambda^2}\mathrm{tr}H^2\right]dH, \tag{2}$$

three principle universality classes were identified according to whether the $N \times N$ matrix H is constrained to be real symmetric ($\beta = 1$, Orthogonal), complex Hermitian ($\beta = 2$, Unitary), or real quaternion ($\beta = 4$, Symplectic). Hamiltonians invariant under time-reversal belong to the orthogonal ensemble, while those which are not belong to the unitary ensemble. Time-reversal invariant systems with half-integer spin and broken rotational symmetry belong to the third symplectic ensemble.

In passing we note that more recent analyses, based on a general mathematical classification scheme for symmetric spaces, indicate that *ten* rather than just three fundamental symmetry classes exist [22]. By now all additional classes have been identified in physics (viz. in mesoscopic superconductor/normal metal systems and in models of random Dirac fermions).

Expressed in the basis of eigenstates $H = U^\dagger \Lambda U$, where Λ denotes the matrix of eigenvalues, the probability distribution takes the form

$$P(H)dH = J(\{E_i\})\exp\left[-\frac{\beta N}{\lambda^2}\sum_i^N E_i^2\right]\prod_l^N dE_l dU$$

where the invariant measure

$$J(\{E_i\}) = \prod_{i<j}|E_i - E_j|^\beta. \tag{3}$$

reveals the characteristic repulsion of the energy levels.

Spectral properties of the ensemble are characterized by correlations of the density of states (DoS),

$$\nu(E) = \mathrm{tr}\ \delta(E - H). \tag{4}$$

While the average DoS depends explicitly on the form of the probability distribution, statistical properties of the fluctuations $\delta\nu = \nu - \langle\nu\rangle$ are universal, independent of the support of the spectrum. For the Gaussian ensembles, the average DoS takes the form of a semi-circle,

$$\langle\nu(E)\rangle = \frac{N}{\pi\lambda}\left[1 - \left(\frac{E}{2\lambda}\right)^2\right]^{1/2}, \tag{5}$$

while the dimensionless two-point correlator of DoS fluctuations

$$R_2(\Omega) \equiv \Delta^2\langle\nu(E+\Omega/2)\nu(E-\Omega/2)\rangle - 1, \tag{6}$$

takes a universal form. For the unitary ($\beta = 2$) ensemble

$$R_2(\Omega) \stackrel{N\to\infty}{=} -\frac{\sin^2(\pi\Omega/\Delta)}{(\pi\Omega/\Delta)^2}. \tag{7}$$

General properties of all three random matrix ensembles are summarized by Mehta[19].

Although compelling in its simplicity, the conjecture of Bohigas, Giannoni and Schmidt begs the question: Which properties are universal and, perhaps more importantly, *which are not*? Empirically, the study of spectral statistics typically shows level correlations to be universal on energy scales comparable with the average energy level spacing Δ (see Fig. 1). However, universal correlations are typically observed to break down at some non-universal energy scale $E_c = \hbar/t_c$.

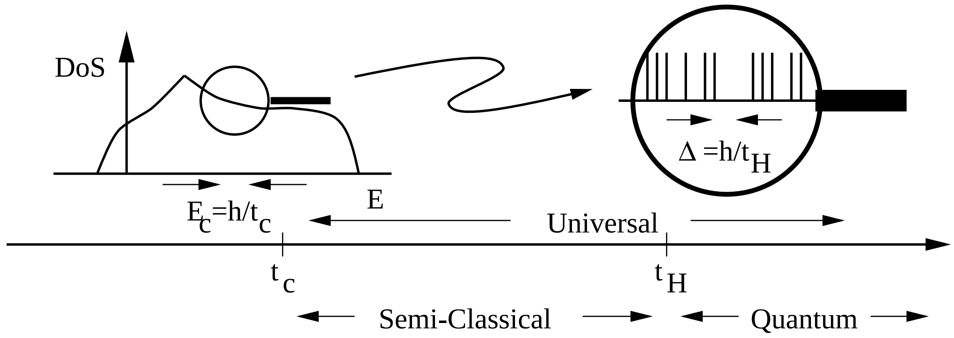

Figure 1. Time scales over which response functions such as $R_2(\Omega)$ are universal.

- What, in general, sets the range of universal correlations?
- How is chaos reflected in the statistics of the non-universal interval?

To address these questions and identify others we can gain useful insight by studying statistical properties of weakly disordered conductors.

COHERENCE EFFECTS IN DISORDERED CONDUCTORS

To guide our intuition in the technical analysis that follows we will begin with a survey of quantum coherence phenomena in weakly disordered conductors. To do so, we will focus on the quantum time evolution of a spreading wavepacket in a background of weakly scattering random impurities. We will see a history in which the particle explores several quite distinct regimes.

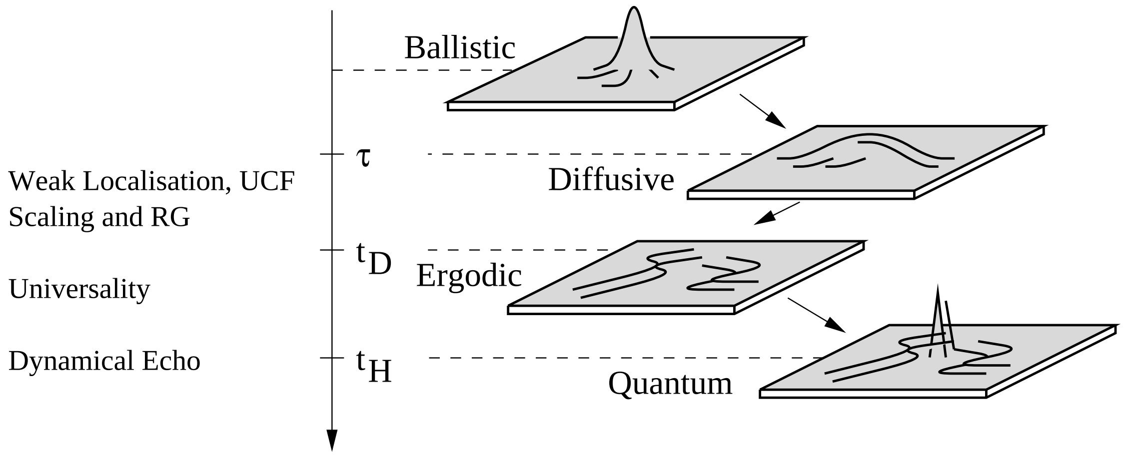

Figure 2. Times scales separating regimes of dynamics.

To be concrete, let us consider a particle confined to a d-dimensional region of size L, and subject to a random impurity potential. The corresponding Hamiltonian takes the form

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}, \tag{8}$$

where $\hat{\mathbf{p}} = -i\hbar\partial$ and, for simplicity, the random potential is taken to be Gaussian δ-correlated white-noise with zero average and a correlator given by

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle = \frac{\hbar}{2\pi\nu\tau}\delta^d\left(\mathbf{r}-\mathbf{r}'\right). \tag{9}$$

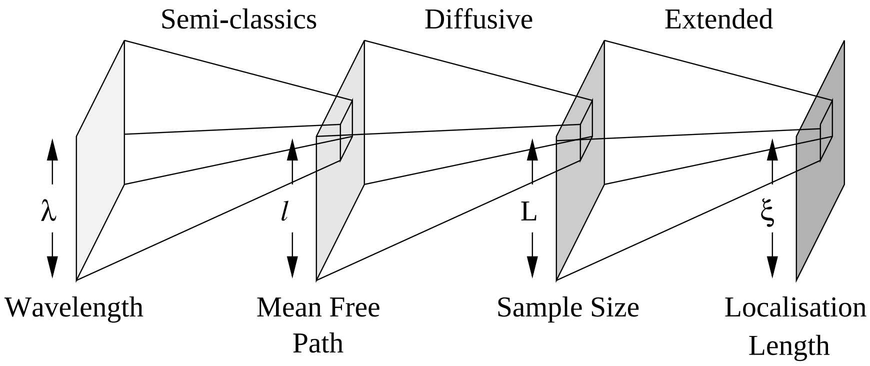

Figure 3. Length scales over which the properties of disordered conductors compare to general chaotic structures.

To stay within the regime in which properties of the disordered conductor bare comparison with those of general chaotic structures we will limit our discussion to length scales in which the dynamics is diffusive (see Fig. 3). Specifically, we will focus on the limit in which the wavelength of the particle λ_F is much smaller than the mean free path $\ell = v_F\tau$ (or, equivalently, $E_F\tau \gg \hbar$), wherein a semi-classical analysis applies. Moreover, we will consider the sample size L to be greatly in excess of the mean free path, while being much smaller than the localization radius ξ of any eigenstates which are localized.

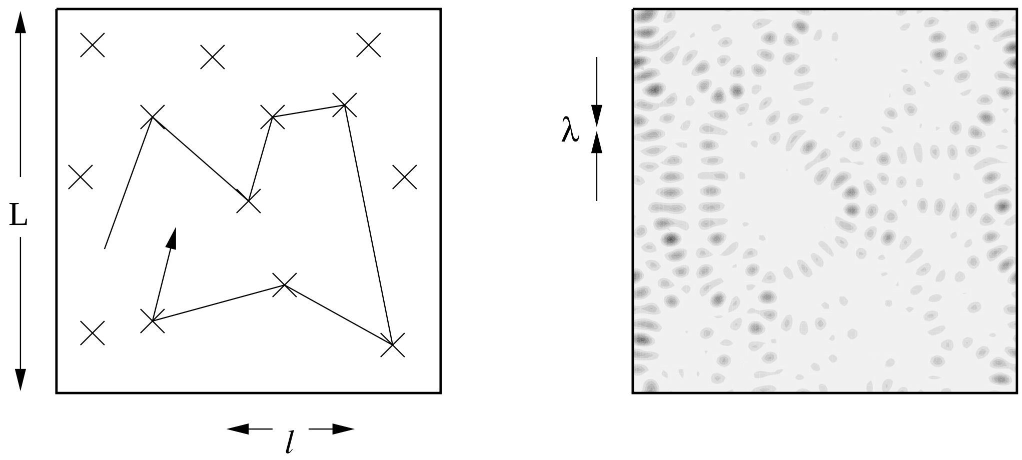

Figure 4. Typical wavefunction of a δ-correlated impurity potential.

A wavepacket, located in the vicinity of $\mathbf{r} = 0$ at a time $t = 0$, can be expressed at subsequent times as

$$\psi(\mathbf{r}, t) = \sum_\nu \psi^*_\nu(0)\psi_\nu(\mathbf{r})e^{-iE_\nu t/\hbar}, \tag{10}$$

where ψ_ν represents the stationary eigenstates of the time-independent Hamiltonian, and E_ν denotes the eigenenergy. Strictly speaking, the sum over eigenstates should be regarded as being truncated at some bandwidth E_B around an energy $E_F \gg E_B \gg \hbar/\tau$ so that the wavepacket is approximately localized in space and energy. The corresponding probability of finding the particle at a distance $\mathbf{r}$ after a time $t > 0$ can be expressed in terms of the propagator or (retarded) Green function as

$$\rho(\mathbf{r}, t) \equiv |\psi(\mathbf{r}, t)|^2 = |iG(0, \mathbf{r}; t)|^2. \tag{11}$$

On time scales in excess of the mean free scattering time τ the initial *ballistic* evolution of the wavepacket gives way to *diffusive* dynamics.

Diffusive Regime: $\tau < t < t_D = L^2/D$

Expressed as a Feynman path integral, the probability *amplitude* of propagation takes the form

$$G(0, \mathbf{r}; t) = \int_{\mathbf{r}(0)=0, \mathbf{r}(t)=\mathbf{r}} Dr \exp\left[-\frac{i}{\hbar}\int_0^t dt' \left(\frac{1}{2}m\dot{\mathbf{r}}^2 - V[\mathbf{r}(t')]\right)\right]. \tag{12}$$

Typically, as a result of the random impurity potential $V(\mathbf{r})$, trajectories arriving at $\mathbf{r}$ after a time t acquire different amplitudes and phases (see Fig. 5):

$$G(0, \mathbf{r}; t) = \sum_i \sqrt{\rho_i} e^{-i\varphi_i}, \tag{13}$$

where $\varphi_i = S_i/\hbar$ represents the total action associated with the trajectory i. Focusing on just two possible paths each having a probability amplitude $G_{1,2} = \sqrt{\rho_{1,2}} \exp[-i\varphi_{1,2}]$, the total probability density is given by

$$\rho \equiv |G_1 + G_2|^2 = \rho_1 + \rho_2 + 2\sqrt{\rho_1\rho_2} \cos(\varphi_1 - \varphi_2), \tag{14}$$

where the first two terms represent the sum of classical probabilities, and the last term describes the interference between the two trajectories.

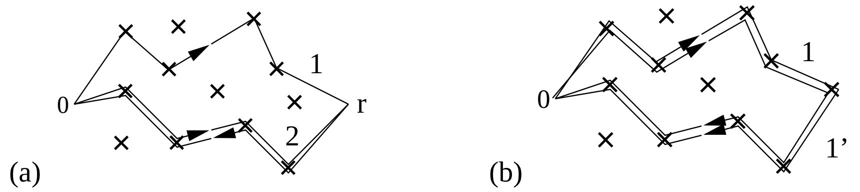

Figure 5. Schematic representation of different Feynman amplitudes contributing to the transmission probability.

Typically, different paths have lengths that differ substantially implying a statistical independence of φ_1 and φ_2 ($|\varphi_1 - \varphi_2| \gg 2\pi$). Therefore, the average over the random distribution of phases reproduces the classical result $\langle\rho\rangle = \rho_1 + \rho_2$. Applied to the Green function itself, this random phase cancelation implies a decay of the ensemble average on length scales comparable to the mean free path ℓ. Indeed, for a δ-correlated white noise impurity potential, it is a straightforward (and useful) exercise to show that, within diagrammatic perturbation theory,

$$\langle G(0, \mathbf{r}; t)\rangle = G_0(0, \mathbf{r}; t) e^{-t/\tau}, \tag{15}$$

where G_0 denotes the free particle Green function.

By contrast, the "diagonal" phase cancelation of each Feynman amplitude generates long-range correlations of the average probability density. In the leading approximation, this leads to a diffusive relaxation of the average probability density,

$$\langle\rho(\mathbf{r}, t)\rangle = \frac{1}{(2\pi Dt)^{d/2}} \exp\left[-\frac{\mathbf{r}^2}{2Dt}\right], \tag{16}$$

where $D = v^2\tau/d$ denotes the bare classical diffusion constant. Within diagrammatic perturbation theory this mechanism of density relaxation, known as a *diffuson mode*, can be expressed as a ladder series (see Fig. 6a),

$$\langle\rho(\mathbf{q}, \Omega)\rangle = \frac{1}{-i\Omega + \hbar D\mathbf{q}^2}. \tag{17}$$

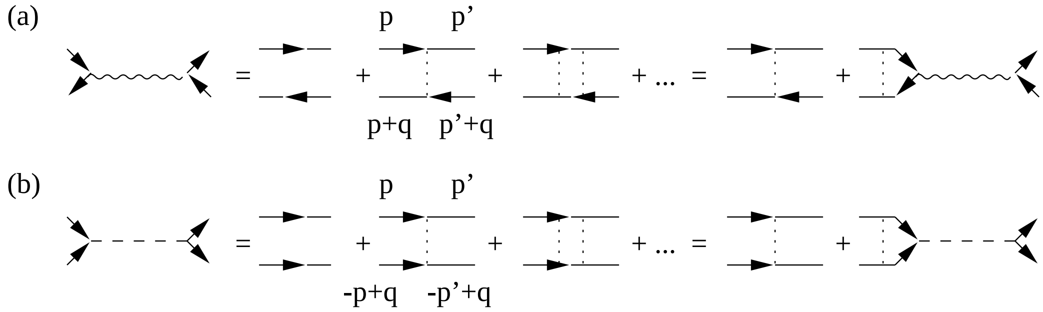

Figure 6. The diagrammatic representation of the dominant "diagonal" contribution to the probability amplitude arising from the interference of (a) overlapping trajectories (diffusons), and (b) the interference of paths with their time-reversed counterpart (Cooperons).

A second and quantum mechanism of density relaxation can be identified by studying the return probability. In the absence of an external magnetic field, the coherent superposition of a path (1) with its time-reversed counterpart (1′) leads to an enhancement of the average return probability (see Fig. 5b),

$$\rho = 2 \times 2\rho_1.$$

We will see below that this tendency to "weakly localize" the electron has important manifestations in quantum transport properties. Diagrammatically, contributions from the interference of time-reversed paths, known as *Cooperon* modes, correspond to a perturbative series involving maximally crossed impurity lines (see Fig. 6b). Together, the diffuson and Cooperon modes provide the elements from which a consistent theory of quantum coherence phenomena in disordered conductors can be assembled.

Fluctuations. While random phase cancelation removes mechanisms of quantum interference from the average probability density, such effects are manifest in *fluctuations*. Taking Eq. (14) for the probability density, its average fluctuation is shown to differ from the classical value by an interference term,

$$\left\langle \rho^2 \right\rangle = \left\langle \rho \right\rangle^2 + 2\rho_1\rho_2,$$

a signature of non-locality. This result, which relies on the non-vanishing average $\langle \cos^2(\varphi_1 - \varphi_2) \rangle = 1/2$, can be interpreted as an interaction involving both types of diffusion mode as shown in Fig. 7.

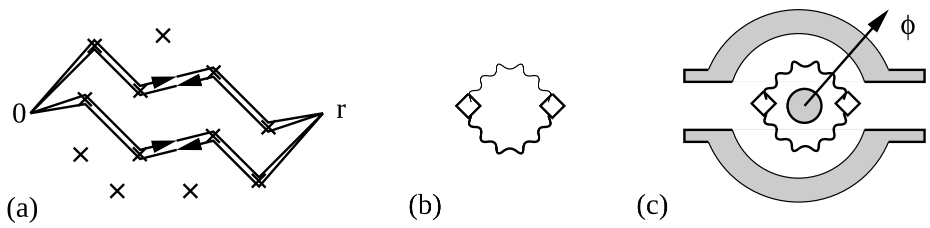

Figure 7. (a) Trajectories contributing to quantum interference corrections to fluctuations in the probability density together with (b) the diagrammatic interpretation. In the Aharonov-Bohm geometry (c) such contributions generate hc/e oscillations of the autocorrelation of conductance.

Aharonov-Bohm hc/e Oscillations. Both mechanisms of quantum interference shown in Fig. 7 are manifest in the transport properties of a metallic ring threaded by a magnetic flux, $\varphi = 2\pi\phi/\phi_0$, where $\phi_0 = hc/e$ denotes the flux quantum. In the

Aharonov-Bohm geometry (see Fig. 7) Feynman trajectories which pass anti-clockwise around the ring experience an additional phase shift relative to those that move clockwise. As a result, a modulation, periodic in hc/e, appears in the autocorrelator of the transmission probability, $\langle\rho(0)\rho(\varphi)\rangle$.

An analogous situation arises if a uniform magnetic field is applied to the disordered metal. In this case, an additional phase difference of $\delta\phi = \phi_1(H) - \phi_2(H)$ is induced between paths (1) and (2) in Fig. 5a. Thus, while leaving the fluctuation of the transmission probability unchanged, the autocorrelation, sensitive to the *relative* flux, decays on the scale of $H_c = \phi_0/A$, where A denotes the area of the sample:

$$\langle\rho(0)\rho(H)\rangle - \langle\rho\rangle^2 = \begin{cases} 2\rho_1\rho_2 & \phi \ll \phi_0, \\ 0 & \phi \sim \phi_0. \end{cases}$$

Magnetoconductance oscillations of this kind were observed experimentally by Webb *et al.*[23]

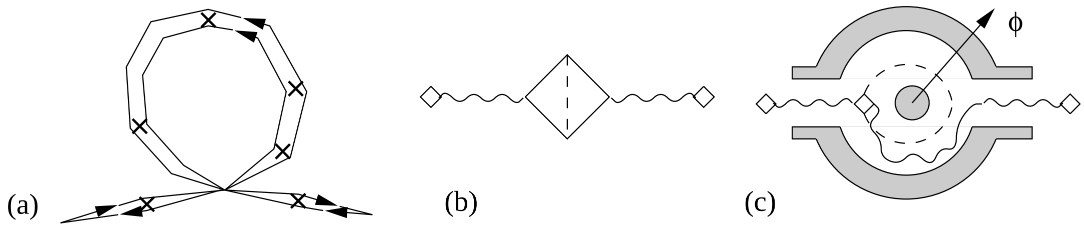

Figure 8. (a) Trajectories contributing to the quantum weak localization correction to the transmission probability. (b) Diagrammatic representation of the weak localization contribution. The junction between the two diffuson modes and the Cooperon is known as a "Hikami box". (c) Diagram responsible for $hc/2e$ oscillations of the conductance in the Aharonov-Bohm geometry.

Weak Localization. As discussed above, the interference of paths with their time-reversed counterpart generates an enhanced return probability. This effect leads to a *quantum renormalization* of the classical diffusion constant and the phenomenon of weak localization (for reviews see Refs. 24, 25, 26, 27, 28, 29). An estimate of the quantum correction to the diffusion constant can be obtained by calculating the change in transmission probability along a diffusing path due to redundant excursions (see Fig. 8a). Requiring that the particle return to a volume comparable to the wavelength, we obtain[24]

$$\frac{\delta D}{D} = \frac{\delta P}{P} \approx -\frac{v\lambda^2}{a^{3-d}}\int_\tau^{t_D} \frac{dt}{(Dt)^{d/2}} = -\frac{1}{\hbar\nu_d D}\frac{2}{2-d}\left(L^{2-d} - \ell^{2-d}\right), \tag{18}$$

where $P(t) = (2\pi Dt)^{d/2}$ represents the return probability, and the d-dimensional DoS $\nu_d = \nu a^{3-d}$ derives from the three dimensional DoS, $\nu = 1/\hbar\lambda^2 v$, with a denoting the transverse width. Applying the Einstein relation (or *Drude formula*),

$$\sigma = e^2\nu_d D, \tag{19}$$

this estimate of the weak localization correction (18) is in accord with the diagrammatic calculation showing that, in two dimensions[30],

$$\delta\sigma = -\frac{e^2}{\pi^2\hbar}\ln\left(\frac{L}{\ell}\right). \tag{20}$$

Anomalous Magnetoconductance. The sensitivity of the Cooperon mode to an external magnetic field has a remarkable effect on the conductance of a mesoscopic sample. While classical considerations would suggest an effect when $\Omega_c\tau \gtrsim 1$, where $\Omega_c = eB/mc$ represents the cyclotron frequency, an enhancement of the conductance is already observed at much lower field strengths, $B \sim \varphi_0/L^2$. This effect is easily understood as the phase breaking of the Cooperon mode by the external magnetic field. Its effect is easily estimated by including a factor to account for the typical flux enclosed by the random walk,

$$\frac{\delta D}{D} = \frac{\delta\sigma}{\sigma} \approx -\frac{\nu\lambda^2}{a^{3-d}} \int_\tau^{t_D} \frac{dt}{(Dt)^{d/2}} \left[1 - \cos\left(\frac{Dt^2}{L_\varphi}\right)\right],$$

where $L_\varphi = \sqrt{\varphi_0/B}$ denotes the phase coherence length. This leads to an anomalous magnetoconductance observed in experiment.

Aharonov-Bohm $hc/2e$ oscillations. The sensitivity of the weak localization correction to an external magnetic field also has a dramatic effect on the conductance of a ring in the Aharonov-Bohm geometry (see Fig. 8). The application of a magnetic flux imparts a relative phase into the time-reversed paths encircling the ring. As a result oscillations in the *non-averaged* conductance appear with a period of $hc/2e$. This effect, predicted by Altshuler and Aronov[31] was first observed by Sharvin and Sharvin[32].

Scaling Theory of Localization. In fact, Eq. (18) represents just the first term in a perturbative series which leads to a *quantum renormalization* of the diffusion constant D, and is in accord with a one-parameter scaling theory of localization*. Beginning with the pioneering work of Thouless[33], the scaling theory of localization as proposed by Abrahams, Anderson, Licciardello, and Ramakrishnan[34] represented a breakthrough in understanding. Its main elements are summarized below.

The bare conductance of a metallic sample of dimension L can be expressed through the Einstein relation as the ratio of characteristic energy scales,

$$G = \sigma L^{d-2} = \frac{e^2}{\hbar}\nu L^d \frac{\hbar D}{L^2} = \frac{e^2}{\hbar} g, \qquad g = \frac{E_c}{\Delta}. \tag{21}$$

According to the one-parameter scaling theory of localization, the variation of the dimensionless conductance with system size obeys the Gell-Mann Low equation

$$\frac{d \ln g}{d \ln L} = \beta(g). \tag{22}$$

In particular, it is independent of the microscopic properties of the sample, such as the bare conductance and ℓ/λ.

This result has profound consequences on the localization properties of disordered conductors. For an ohmic conductor the scaling function takes a constant value $\beta = d-2$, while deep in the insulating regime, where states are exponentially localized, $\beta(g) \approx \ln g$. A smooth interpolation between these limits suggests that below two-dimensions all states are localized, while above there is a critical conductance, g_*, above which states are extended. The unstable fixed point is associated with the Anderson localization transition.

*Although the leading quantum weak localization correction relies on the Cooperon channel, higher order corrections involving just diffuson modes can also operate.

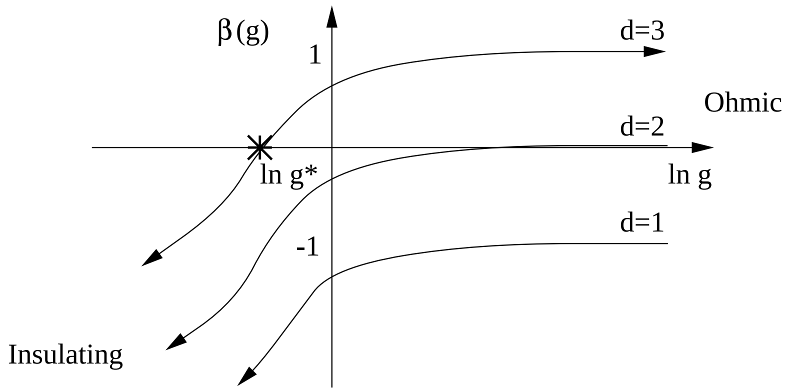

Figure 9. Scaling function of conductance, $\beta(g)$, in dimensions $d = 1$, 2 and 3. A localization transition is predicted in dimensions greater than two.

The situation in two-dimensions is more delicate. Localization properties depend sensitively on the asymptotic approach of the β function to the metallic limit. However, taking the first quantum correction from Eq. (20), we obtain $\beta(g) = -1/\pi^2 g$ which is consistent with localization of all states in two-dimensions. While the one-parameter scaling theory of localization has yet to find a rigorous mathematical basis, it nevertheless represents a milestone in phenomenology.

Level Statistics. Similar arguments can be applied to the two-point correlator of DoS fluctuations, R_2 (6). By making use of the identity

$$\nu(E) = \frac{1}{\pi}\mathrm{tr}\ \mathrm{Im}\ \hat{G}^-(E), \qquad \hat{G}^- = \frac{1}{E - \hat{H} - i0} \tag{23}$$

the fluctuation of the DoS, R_2, can be expressed as a correlation of Green functions. The dominant contribution arises from the interference of Feynman trajectories topologically equivalent to those shown in Fig. 5. In the language of diagrammatics, this contribution arises from the exchange of two diffusion ladders between two closed loops (see Fig. 10), and gives the celebrated result of Altshuler and Shklovskii[35]

$$R_2(\Omega) = \frac{1}{2\pi^2}\mathrm{Re}\sum_{\mathbf{q}} \frac{\Delta^2}{(-i\Omega + \hbar D\mathbf{q}^2)^2}. \tag{24}$$

In systems invariant under time-reversal, a second mechanism arising from the exchange of Cooperon ladders is accounted by a further factor of two.

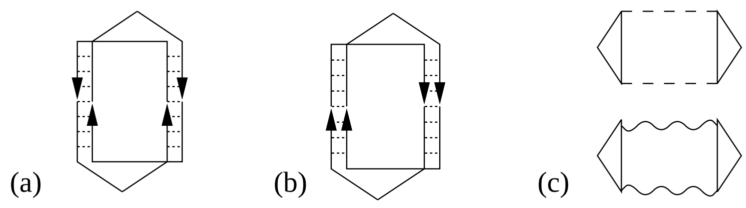

Figure 10. Diagrams showing the leading contribution to the two-point autocorrelator of DoS fluctuations.

Therefore, far from being universal, Eq. (24) shows that, at least on energy scales Ω in excess of the "Thouless energy" (or the inverse transport time),

$$E_c \equiv \frac{\hbar}{t_D} = \frac{\hbar D}{L^2}, \tag{25}$$

level statistics depend explicitly on material properties such as geometry and morphology. Yet, as we will see, Eq. (24) establishes an important connection between disordered and quantum chaotic structures as well as having important physical consequences.

Having strayed far, let us return to the spreading of the wavepacket. In the disordered conductor, we have seen that the dominant mechanism of density relaxation is diffusion. Mechanisms of quantum interference generate quantum corrections and give rise to strong mesoscopic, or sample-to-sample fluctuations. Quantum renormalization of the diffusion constant serves to weakly localize the electron. However, if the impurity potential is not strong enough to bring about complete localization, the wavepacket continues to spread. Beyond the typical transport time $t_D = L^2/D$ the wavepacket is distributed approximately uniformly throughout the sample and further evolution of the wavepacket is, in this sense, *ergodic.* Here the properties of the system become *universal*, independent of the individual features of the system.

Ergodic Regime: $t_D < t < t_H = \hbar/\Delta$

Universality of the response in the ergodic regime is manifest in both spectral and transport properties. Turning first to spectral properties of the system, we return to the perturbative analysis of the two-point correlator of DoS fluctuations, R_2 above.

Universal Level Statistics. Taking the energy $\Omega \ll E_c$, the expression for R_2 (24) can be approximated by the lowest spatial mode $\mathbf{q} = 0$,

$$R_2(\Omega) \simeq -\frac{1}{2\pi^2}\left(\frac{\Delta}{\Omega}\right)^2 . \tag{26}$$

In this limit fluctuations of the DoS become universal, independent of detailed properties of the system. Moreover, Eq. (26) is a manifestation of rigidity in the spectrum. Taking $N(\Omega)$ to be the number of levels in a band of width Ω, Eq. (26) shows fluctuations in N to increase only logarithmically with Ω,

$$\left\langle N^2(\Omega)\right\rangle_c \approx \frac{1}{\pi^2}\ln\langle N(\Omega)\rangle . \tag{27}$$

This compares with a random or Poissonian distribution of energy levels in which fluctuations of N grow as $\sqrt{\langle N\rangle}$.

Universal Conductance Fluctuations. Spectral rigidity (27) provided a simple physical understanding of the universality of conductance fluctuations observed in experiment[36]: It has been found that the dimensionless conductance $g = G\hbar/e^2$ exhibits fluctuations of $O(1)$ no matter how large the average conductance $\langle g\rangle$ (G: dimensionful conductance). To understand this phenomenon we note that Eq. (21) shows the dimensionless conductance to be equivalent to the number of levels inside a window of energy E_c. Thus, while the average conductance

$$\langle G\rangle = \frac{e^2}{\hbar}\langle N(E_c)\rangle = \frac{e^2}{\hbar}g, \tag{28}$$

is large in a good metal ($g \gg 1$), Eq. (27) implies that characteristic fluctuations are small,

$$\sqrt{\langle G^2\rangle_c} \sim O(e^2/\hbar), \tag{29}$$

and universal[37, 38] (independent of the average conductance).

Breakdown of Perturbation Theory. Eq. (26) illustrates a further point concerning the domain of validity of perturbation theory. At energy scales $\Omega \ll \Delta$, or equivalently at time scales $t \gg t_H = \hbar/\Delta$, response functions obtained with diagrammatic perturbation theory show unphysical IR divergences. Perturbation theory relies not only the semi-classical approximation, $\lambda/\ell \ll 1$, but also that (c.f. $\delta D/D$)

$$\left(\frac{\Delta}{E_c}\right)^{d/2} \left(\frac{\Delta}{\Omega}\right)^{(2-d)/2} \ll 1. \tag{30}$$

In the ergodic regime, where the effective dimensionality is zero, this condition is not met. From this we learn that, in general, one cannot expect "$\hbar \to 0$" to uniquely define what is meant by "semi-classics".

Quantum Regime: $t > t_H = \hbar/\Delta$

Within the quantum regime a separate approach must be sought that does not rely on the parameter (30) of the perturbation theory. Fortunately, at least for non-interacting electronic systems, such an approach exists. The supersymmetry method finds its origin in the pioneering work of Wegner[12]. Motivated by the scaling ideas which were being developed in parallel, Wegner introduced a description of weakly disordered conductors which was based on a field theory of nonlinear σ-model type. The approach allowed for the analysis of the perturbative sector of the theory, including a description of diffusion modes, higher order interference processes and a more rigorous formulation of scaling arguments.

Subsequent investigation provided a microscopic justification for the effective action proposed by Wegner. The crucial step came in a development by Efetov[39]. Employing an approach based on a supersymmetric formalism, Efetov was able to circumvent formal difficulties of the existing theory and thereby explore the non-perturbative regime. As a result, Efetov was able to explicitly confirm the conjecture employed by Gor'kov and Eliashberg: On time scales in excess of the typical transport or diffusion time, average spectral and transport properties of weakly disordered conductors coincide with those of random matrix ensembles. Since then, considerable effort has been directed towards the investigation of the universal limit of the theory. These include universal statistical properties of wavefunctions as well as spectra too numerous to report here.

So within the quantum regime, the dynamics of the spreading wavepacket are almost featureless, with the density relaxing into the uniform zero-mode configuration. The exception concerns the Heisenberg time. As a consequence of the rigidity of the spectrum, an approximately coherent superposition of the wavepacket generates an "echo" of the particle at $t_H = \hbar/\Delta$[40].

Ballistic Chaotic Structures

Does an analogous scenario describe the quantum time evolution of a wavepacket in, say, an irregular cavity ("quantum billiard") without impurities? In such systems there too exists some ergodic time after which properties of the system become universal. Our aim is to explore the dynamics at shorter time scales where the unstable nature of the classical dynamics is reflected in the quantum evolution:

- What plays the role of diffusion in describing the low lying relaxational degrees of freedom in general chaotic quantum systems?

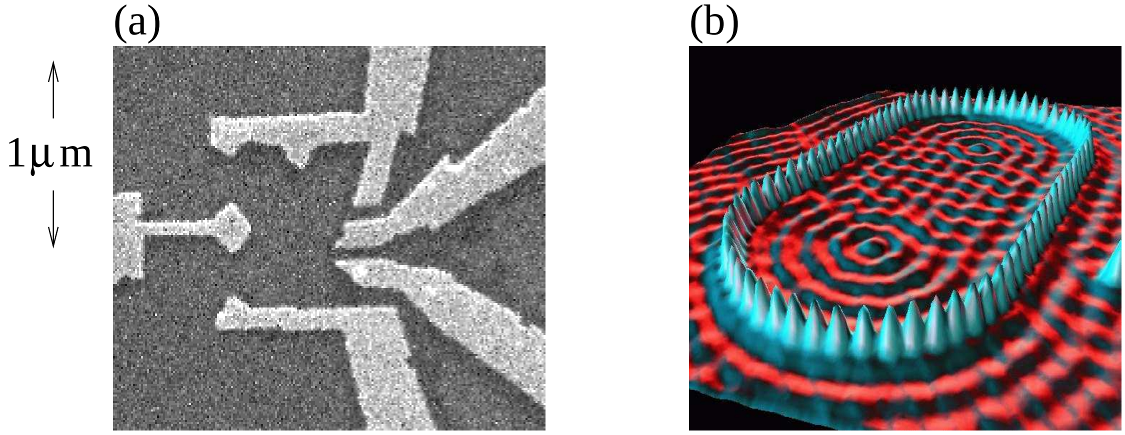

Figure 11. (a) Electron micrograph of a typical ballistic quantum dot taken from Ref. 3. Electrons are confined vertically to the ground state of a quantum well located at a GaAs/AlGaAs interface, and form a two-dimensional electron gas (2DEG). Metallic depletion gates confine electrons laterally. A mean free path and coherence length on the order of 10 microns ensures that the carriers are coherent and ballistic. (b) STM image taken from Ref. 41 of the surface electron density of a quantum stadium corral made from Iron atoms deposited on Copper (111).

- What is the analogue of *weak localization* and how are such quantum coherence effects manifest in experiment?
- What determines the domain of universality?

To address the first of these questions we begin with a semi-classical analysis based on the Feynman path integral.

Semi-classics and the Trace formula

The semiclassical analysis begins with Gutzwiller's trace formula, which expresses the dimensionless DoS as a sum over the classical periodic orbits[4, 42, 43, 44]

$$\frac{\nu(E)}{\nu} = 1 + \mathrm{Re}\frac{\Delta}{\pi\hbar}\sum_p T_p \sum_{r=1}^{\infty} \frac{e^{iS_p(E)r/\hbar - i\nu_p r}}{|\det(M_p^r - \mathbf{I})|^{1/2}}, \tag{31}$$

where p labels a primitive orbit with a period T_p, action $S_p(E)$, and Maslov phase ν_p. The sum over r accounts for repetitions of the classical periodic trajectory. Here M_p denotes the monodromy matrix associated with the linearized dynamics on the Poincarè section perpendicular to the orbit p.

Substituting Eq. (31) into Eq. (6), the two-point correlator of DoS fluctuations is associated with a double sum over periodic orbits. A random phase cancelation of long trajectories identifies the "diagonal contribution" as dominant. Taking this contribution alone, and expanding $S_p(E+\Omega) \simeq S_p(E) + T_p\Omega$, we obtain

$$R_2(\Omega) = \mathrm{Re}\frac{\Delta^2}{2\pi^2\hbar^2}\sum_p T_p^2 \sum_{r=1}^{\infty} \frac{e^{i\Omega T_p r/\hbar}}{|\det(M_p^r - \mathbf{I})|}. \tag{32}$$

The traditional way to deal with the above sum is to approximate it by an integral:

$$\sum_p \frac{f(T_p)}{|\det(M_p - \mathbf{I})|} \to \int_0^{\infty} \frac{dt}{t} f(t)$$

for any sufficiently smooth function $f(t)$. This approximation, known as the Hannay and Ozorio de Almeida sum rule[45], holds in the limit $t \to \infty$ where long periodic orbits which explore the whole energy shell uniformly are considered. In employing it for the calculation of $R_2(\Omega)$, the time t should be restricted to the regime where it is much larger than the shortest periodic orbits but still smaller than the Heisenberg time t_H. The result associated with it is therefore the universal one (26) which holds as long as $\Omega \gg \Delta$[43, 44]. Below we present a more careful treatment of the sum (32) that keeps the non-universal part of $R_2(\Omega)$.

To identify the physical operator to which the diagonal sum (32) corresponds we first consider the propagator of classical phase space density, the Liouville operator. Formally, the latter is defined by

$$\rho(\mathbf{x},t) = \hat{\mathcal{L}}^t \rho(\mathbf{x},0) = \int d\mathbf{x}' \delta\left(\mathbf{x} - \mathbf{u}(\mathbf{x}';t)\right) \rho(\mathbf{x}',0), \tag{33}$$

where $\mathbf{x}'$ and $\mathbf{x}$ are phase space vectors representing coordinates and momenta, and $\mathbf{u}(\mathbf{x};t)$ denotes the solution of the classical equations of motion of a particle starting at a position $\mathbf{x}$ after a time t.

From this definition, it is straightforward to obtain the following expression for the trace[46],

$$\mathrm{tr}\mathcal{L}^t = \sum_p T_p \sum_r \frac{\delta\left(t - rT_p\right)}{\left|\det(M_p^r - \mathbf{I})\right|}. \tag{34}$$

Comparing this expression to Eq. (32), and *neglecting repetitions*, i.e. the contribution of short periodic orbits that are traversed repeatedly, we obtain[47]

$$R_2(\Omega) = \frac{\Delta^2}{2\pi^2\hbar}\mathrm{Re}\frac{\partial}{\partial(i\Omega)}\int_0^\infty dt e^{i\Omega_+ t/\hbar}\mathrm{tr}\mathcal{L}^t = \frac{1}{2\pi^2}\mathrm{Re}\sum_\mu \frac{\Delta^2}{(-i\Omega_+ + \hbar\gamma_\mu)^2}, \tag{35}$$

where the second equality is based on $\mathrm{tr}\,\mathcal{L}^t = \sum_\mu \exp(t\gamma_\mu)$, $\{\gamma_\mu\}$ being the eigenvalues of (the logarithm of) the classical evolution operator. This result compares to Eq. (24) (with $\gamma_\mu \to D\mathbf{q}^2$), and identifies the modes of density relaxation as the eigenfunctions of the classical evolution operator.

But do the modes of the Liouville operator relax? What role is played by the repetitions neglected above? And how can we account for weak localization and corrections non-perturbative in Δ/Ω? These questions can not be answered with the framework of the diagonal approximation. Recent attempts to go beyond this approximation within the framework of periodic theory have met with some limited success[48]. However, these studies have, as yet, failed to identify weak localization corrections. Instead, we will employ a different approach based on a non-perturbative field theory. To introduce these ideas, we will first apply this approach to study quantum coherence phenomena in weakly disordered conductors.

SUPERSYMMETRY METHOD

Although the ideas presented in the previous section can be straightforwardly interpreted within the framework of a diagrammatic perturbation theory, the analysis of low energy, infrared phenomena relies on a non-perturbative description. The latter is provided by the above mentioned field theoretic approach introduced by Efetov. The purpose of the present section is to introduce this so-called supersymmetry method approach and demonstrate its application to the study of quantum coherence phenomena described in the previous section. The presentation will be by necessity concise. For a more comprehensive discussion the reader is referred to Ref. 5. Later we will show how an analogous approach can be developed for structures which are chaotic but not random.

Field Integral

The starting point of the field theoretic approach is the representation of the Green function as a functional field integral. Focusing on the advanced function, we have

$$G^-(\mathbf{r},\mathbf{r}';E) \equiv \left\langle \mathbf{r}' \left| \frac{1}{E-\hat{H}-i0} \right| \mathbf{r} \right\rangle \equiv \sum_\nu \frac{\psi_\nu(\mathbf{r}')^*\psi_\nu(\mathbf{r})}{E-E_\nu-i0} = \frac{i}{\mathcal{Z}}\int DSS(\mathbf{r})S^*(\mathbf{r}')e^{-\mathcal{L}[S]},$$

where the integration measure $DS = \Pi d(\mathrm{Re}S)d(\mathrm{Im}S)/(2\pi)$ is taken over all space points, $\mathcal{Z} = \int DS\exp(-\mathcal{L}[S])$ denotes the constant of normalization, and

$$\mathcal{L}[S] = i\int S^*\left(E-\hat{H}-i0\right)S.$$

Expressed in this form, the presence of the normalization factor leads to unphysical vacuum loops in the ensemble average. Accountancy of the latter can, at least in principle, be treated within the "replica formalism" in which one exploits the analytic continuation[49, 50]

$$\ln\mathcal{Z} = \lim_{n\to 0}\frac{\mathcal{Z}^n-1}{n}.$$

However, experience has shown considerable difficulty in correctly implementing this procedure in the non-perturbative sector of the theory[51]. Fortunately, at least for systems which are non-interacting, a second and more reliable approach can be developed.

The "supersymmetry method" exploits properties of Grassmann algebra to implement a scheme of "book-keeping". While a comprehensive introduction to superalgebra can be found in Ref. 52, essential properties of Grassmann variables are summarized in Ref. 5. Crucially, by exploiting the properties of Gaussian integrals over complex commuting or bosonic (B) variables S, and anticommuting or fermionic (F) variables χ,

$$\int e^{-S^*MS}d[S^*]d[S] = \frac{1}{\det M}, \qquad \int e^{-\bar{\chi}M\chi}d[\bar{\chi}]d[\chi] = \det M$$

the Green function can be expressed as a functional field integral in which the normalization, $\mathcal{Z}$, is by construction unity. Specifically, we can write

$$G^-(\mathbf{r},\mathbf{r}';E) = i\int D\psi S(\mathbf{r})S^*(\mathbf{r}')e^{-\mathcal{L}[\psi]}, \qquad \mathcal{L}[\psi] = i\int\psi^\dagger\left(E-\hat{H}-i0\right)\psi, \qquad (36)$$

where $\psi^T = (S,\chi)$ denotes a two-component superfield. The measure of the functional integral takes the form $D\psi = DS\,D\chi$, where $D\chi = \Pi\ 2d\chi^*d\chi$.

Finally, instead of representing each vertex separately, it is often more convenient to introduce a "source term" into the effective action from which arbitrary correlators can be constructed. Defining

$$\mathcal{Z}[J] = \int D\psi e^{-\mathcal{L}[\psi]}\exp\left[-i\int\left(J^\dagger\psi+\psi^\dagger J\right)\right],$$

the Green function can be expressed as a functional derivative over the vector superfields

$$G^-(\mathbf{r},\mathbf{r}';E) = -\frac{i}{2}\mathrm{tr}\left[\frac{\delta}{\delta J^\dagger(\mathbf{r})}\otimes\frac{\delta}{\delta J(\mathbf{r}')}\right]_{\mathrm{bf}}\mathcal{Z}\Big|_{J=0}.$$

By doubling the field space, this representation allows *two-point* correlators or response functions to be expressed in terms of the generating function,

$$\mathcal{Z} = \int D\psi e^{-\mathcal{L}[\psi]-\mathcal{L}_J[\psi]}, \tag{37}$$

$$\mathcal{L}[\psi] = i\int \bar{\psi}\left(E - \frac{\Omega_+}{2}\sigma_3^{\rm ar} - \hat{H}\right)\psi, \qquad \mathcal{L}_J[\psi] = i\int\left(\bar{J}\psi + \bar{\psi}J\right) \tag{38}$$

where $\Omega_+ \equiv \Omega + i0$ denotes the frequency source, and the field integral involves four-component superfields

$$\psi = \begin{pmatrix}\psi_- \\ \psi_+\end{pmatrix}_{\rm ar}, \qquad \psi_\mp = \begin{pmatrix}S_\mp \\ \chi_\mp\end{pmatrix}_{\rm bf}, \qquad \bar{\psi} = \psi^\dagger L, \qquad L = \begin{pmatrix}\sigma_3^{\rm ar} & 0 \\ 0 & \mathbf{I}^{\rm ar}\end{pmatrix}_{\rm bf}.$$

Pauli matrices† $\sigma_3^{\rm bf}$ and $\sigma_3^{\rm ar}$ respectively break the symmetry between boson/fermion and advanced/retarded degrees of freedom of Q. Convergence of $\mathcal{Z}$ determines the form of the metric L in the BB sector. While the choice in the fermionic sector seems arbitrary, a consideration of the saddle-point manifold of the σ-model determines the form above[53]. This implies a group structure in the bosonic sector which is non-compact, while the fermionic sector is compact.‡

Finally, our preliminary discussion in the previous section classified two modes of density relaxation in the disordered metal, diffusons and Cooperons. Both will be identified in the analysis that follows. Accordingly, anticipating the structure of the saddle-point action, it is convenient to manipulate the action into the form

$$\mathcal{L}[\Psi] = \frac{1}{2}\left(\mathcal{L}[\psi] + \mathcal{L}[\psi]^T\right)$$

where the dimension of the vector space is doubled to include complex or time-reversed components,

$$\Psi = \begin{pmatrix}\psi \\ \psi^*\end{pmatrix}.$$

The elements of the newly defined supervector Ψ are not independent, but fulfill the "time-reversal" symmetry relation

$$\Psi^\dagger = (C\Psi)^T, \qquad C = \begin{pmatrix}\sigma_1^{\rm tr} & 0 \\ 0 & i\sigma_2^{\rm tr}\end{pmatrix}_{\rm bf}.$$

Regarding ψ as analogous to the wavefunction, the transformation $\psi \to \psi^*$, $\hat{H} \to \hat{H}^T$ is the analogue of the quantum mechanical time-reversal operation. In fact, any discrete symmetry of the microscopic Hamiltonian doubles the number of low-lying modes of density relaxation. In each case, it is convenient to double the vector space[54].

Altogether, after doubling the space, we obtain the action

$$\mathcal{L}[\Psi] = \frac{i}{2}\int \bar{\Psi}\left(E - \frac{\Omega_+}{2}\sigma_3^{\rm ar} - \hat{H}\right)\Psi, \qquad \mathcal{L}_J[\Psi] = \frac{i}{2}\int\left(\bar{J}\Psi + \bar{\Psi}J\right)$$

---

†

$$\sigma_1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}.$$

‡Formally, the difference between the bosonic and fermionic sectors introduces a factor $\det(-1)$, taken over space points, into $\mathcal{Z}$. Since this factor is identically cancelled at a later stage we will, for convenience, incorporate it into a redefinition of the integration measure.

In this notation, advanced and retarded Green functions are obtained from the generating function by the operation,

$$\hat{G}^{\mp}(E \mp \Omega/2) = -\frac{i}{2}\mathrm{tr}\left[\frac{\delta}{\delta \bar{J}_{\mp}} \otimes \frac{\delta}{\delta J_{\mp}}\right]_{\mathrm{bf,tr}} \mathcal{Z}\Big|_{J=0}. \tag{39}$$

To proceed we will examine the Hamiltonian (8) describing a particle propagating in a weakly disordered impurity potential. Linear in the potential, the ensemble average over the Gaussian δ-correlated distribution (9), is straightforward.

$$\left\langle \exp\left[\frac{i}{2}\int \bar{\Psi} V \Psi\right]\right\rangle = \exp\left[-\int \frac{\hbar}{16\pi\nu\tau}\left(\bar{\Psi}\Psi\right)^2\right]. \tag{40}$$

The result is an effective quartic interaction of the fields. An expansion of the total action in the interaction recovers diagrammatic perturbation theory. However, our preliminary survey of two-dimensional systems below identifies the dominant contributions arising from mechanisms of quantum coherence as coming from IR or long-range fluctuations of the density. This suggests the validity of a mean-field decomposition of the action in which the important fluctuations are long-ranged.

The situation is comparable to the BCS mechanism of superconductivity. There the formation of the Cooper pair condensate is associated with the growth of the mean-field order parameter described by a saddle-point or gap equation. Massless Goldstone fluctuations of the order parameter around the saddle-point are described by a Ginzburg-Landau type action. In the present case we will find that the Goldstone modes associated with fluctuations of the mean field represent the diffusion modes.

In the absence of the symmetry breaking sources, Ω and J, the action is invariant under pseudounitary rotations of the fields in superspace,

$$\psi \longrightarrow U\psi, \qquad U^{\dagger}LU = L, \qquad U \in SU(2,2/4). \tag{41}$$

Hubbard-Stratonovich Decoupling

To establish a useful mean-field decoupling of the interaction requires the identification of the low-lying modes of the theory. A diagrammatic analysis of the theory identifies two relevant channels: the diffuson and Cooperon. Both are represented diagrammatically in Fig. 12.

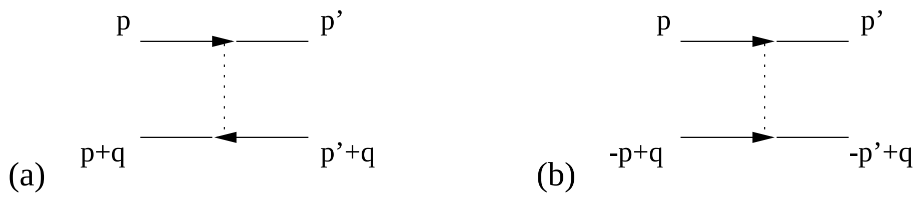

Figure 12. Diagrammatic representation of the terms in the Hubbard-Stratonovich decoupling that lead to (a) the diffusion mode, and (b) the Cooperon mode.

These processes are accounted for by contributions to the action from

$$\int\left(\bar{\psi}\psi\right)^2 \approx \sum_{\mathbf{p},\mathbf{p}'}\sum_{|\mathbf{q}|\ll\ell^{-1}}\left[\left(\bar{\psi}_{-\mathbf{p}'}\psi_{\mathbf{p}}\right)\left(\bar{\psi}_{-\mathbf{p}-\mathbf{q}}\psi_{\mathbf{p}'+\mathbf{q}}\right) + \left(\bar{\psi}_{-\mathbf{p}'}\psi_{\mathbf{p}}\right)\left(\bar{\psi}_{\mathbf{p}'-\mathbf{q}}\psi_{-\mathbf{p}+\mathbf{q}}\right)\right]$$

where the first and second terms generate the contributions from the diffuson and Cooperon respectively, and the summation over $\mathbf{q}$ is limited to low momentum transfer. Introducing the Hubbard-Stratonovich transformation,

$$\exp\left[\int \frac{\hbar}{16\pi\nu\tau}\mathrm{str}\left(\Psi\otimes\bar{\Psi}\right)^2\right] = \int DQ \exp\left[-\frac{\hbar}{4\tau}\int\left(\bar{\Psi}Q\Psi + \frac{\pi\nu}{2}\mathrm{str}Q^2\right)\right], \tag{42}$$

the contributions to the diffuson and Cooperon degrees of freedom are accounted for by *slow* fluctuations of the 8×8 supermatrix fields, Q.[§] Here the supertrace operation is defined by $\mathrm{str} M = \mathrm{tr} M_{FF} - \mathrm{tr} M_{BB}$. The supermatrix fields Q have an algebraic structure which reflects that of the dyadic product $\Psi \otimes \bar{\Psi}$,

$$Q = CLQ^T LC^T,$$

where the transpose of a supermatrix is defined by

$$M^T = \begin{pmatrix} M^T_{BB} & M^T_{FB} \\ -M^T_{BF} & M^T_{FF} \end{pmatrix}.$$

Altogether we obtain the following expression for the average

$$\langle \mathcal{Z} \rangle = \int D\psi DQ e^{-\mathcal{L}_J} \exp\left[-\int \left(\frac{\pi\hbar\nu}{8\tau} \mathrm{str} Q^2 + \frac{i}{2} \bar{\Psi} \hat{\mathcal{G}}^{-1} \Psi \right) \right],$$

where

$$\hat{\mathcal{G}}^{-1} = E - \frac{\Omega_+}{2}\sigma_3^{\mathrm{ar}} - \frac{\hat{p}^2}{2m} - \frac{i\hbar}{2\tau} Q, \tag{43}$$

defines the supermatrix Green function. Integrating over the fields Ψ, we obtain

$$\langle \mathcal{Z} \rangle = \int DQ \exp\left[-\int \mathrm{str} \left(\frac{\pi\hbar\nu}{8\tau} Q^2 - \frac{1}{2} \ln \hat{\mathcal{G}}^{-1} \right) + \int\int \frac{i}{2} \bar{J} \hat{\mathcal{G}} J \right].$$

A legitimate mean-field decomposition justifies treating the action within a saddle-point approximation. We will proceed by identifying a manifold of degeneracy associated with the saddle-point. Fluctuations around this degenerate manifold will generate an effective action for the diffusion modes.

Saddle-Point Equation

A variation of the action with respect to Q generates the saddle-point equation

$$Q_{\mathrm{sp}}(\mathbf{r}) = -\frac{i}{\pi\nu} \mathcal{G}(\mathbf{r}, \mathbf{r}). \tag{44}$$

From this equation it is possible to interpret the saddle-point solution of Q as representing the self-consistent Born approximation to the self-energy. Taking the symmetry breaking sources Ω and J to be vanishingly small, the ansatz that the saddle-point solution Q_{sp} is constant, independent of position, and takes non-zero matrix elements only on the diagonal, implies

$$\mathcal{G}(\mathbf{r}, \mathbf{r}) = \int d\mathbf{p} \left[E - \frac{\mathbf{p}^2}{2m} - \frac{i\hbar}{2\tau} Q_{\mathrm{sp}} \right]^{-1} = \nu \int d\xi \left[E - \xi - \frac{i\hbar}{2\tau} Q_{\mathrm{sp}} \right]^{-1} = i\pi\nu \mathrm{sgn}(Q_{\mathrm{sp}}).$$

From this we deduce that the elements of Q_{sp} take values of ± 1. However, signs cannot be assigned arbitrarily and analyticity of the Green function in each sector demands

[§] Note that as written, the Hubbard-Stratonovich transformation is not exact. If *all* degrees of freedom of Q (fast and slow) are taken into account, the decoupling (42) involves an overcounting by a factor of 2. This is because the saddle-point corresponding to the Cooperonic sector ($[Q, \sigma_3^{\mathrm{tr}}]$) can be found in fast fluctuations of the diffuson section, and *vice versa*

$Q_{\rm sp} = \sigma_3^{\rm ar}$. Moreover, the invariance of the action under pseudounitary rotations implies a degenerate manifold of solutions

$$Q_{\rm sp} = T^{-1}\sigma_3^{\rm ar}T, \qquad T = LU, \qquad Q_{\rm sp}^2 = \mathbf{I}. \tag{45}$$

Dividing out rotations that leave $\sigma_3^{\rm ar}$ invariant, the degeneracy of the manifold is specified by the factor space $SU(2,2/4)/SU(2/2)\otimes SU(2/2)$.

Substituting Eq. (45) into Eq. (43), we obtain the following expression for the supermatrix Green function at the saddle-point:

$$\mathcal{G}_{\rm sp}(\mathbf{r},\mathbf{r}') = {\rm Re}\left\langle G^-(\mathbf{r},\mathbf{r}')\right\rangle + i\pi\nu f_d(|\mathbf{r}-\mathbf{r}'|)Q_{\rm sp}, \tag{46}$$

where the average Green function, or "Friedel function"

$$f_d(r) \equiv \frac{{\rm Im}\,\langle G^-(\mathbf{r},0)\rangle}{{\rm Im}\,\langle G^-(0,0)\rangle} = \Gamma(d/2)\left(\frac{2}{k_F r}\right)^{d/2-1} J_{d/2-1}(k_F r)e^{-r/2\ell}, \tag{47}$$

decays on a scale comparable to the mean free path. Note that a non-zero contribution to ${\rm Re}\langle G^-\rangle$ can be accommodated by a shift of the chemical potential (and can therefore be set to zero).

Non-linear σ-model

The degeneracy of the non-linear manifold implies the existence of Goldstone modes which can be obtained by expansion of the action near the extremum. To classify the fluctuations let us consider small deviations $\delta Q(\mathbf{r}) = Q(\mathbf{r}) - Q_{\rm sp}$. Neglecting symmetry breaking sources, an expansion of the effective action to quadratic order yields

$$\langle \mathcal{Z}\rangle = \int DQ e^{-S^{(2)}[Q]}$$

with

$$S^{(2)}[Q] = -\int d\mathbf{r}d\mathbf{r}'\frac{\hbar^2}{16\tau^2}{\rm str}\left[\mathcal{G}_{\rm sp}(\mathbf{r},\mathbf{r}')\delta Q(\mathbf{r}')\mathcal{G}_{\rm sp}(\mathbf{r}',\mathbf{r})\delta Q(\mathbf{r}) - \frac{2\pi\nu\tau}{\hbar}\delta^d(\mathbf{r}-\mathbf{r}')\delta Q(\mathbf{r})\delta Q(\mathbf{r}')\right]$$

Depending on whether or not they change the eigenvalues of the supermatrix Q, the modes of fluctuation can be classified as longitudinal, δQ_l, or transverse, δQ_t,

$$[Q,\delta Q_l]_- = 0, \qquad [Q,\delta Q_t]_+ = 0.$$

Expressed in terms of these modes, the first term separates into two contributions, $S_l + S_t$, where

$$S_{\substack{l\\t}}[\delta Q] = \pm\left(\frac{\pi\hbar\nu}{4\tau}\right)^2\int d\mathbf{r}d\mathbf{r}' f_d^2(|\mathbf{r}-\mathbf{r}'|){\rm str}\left[\delta Q_{\substack{l\\t}}(\mathbf{r})\delta Q_{\substack{l\\t}}(\mathbf{r}')\right].$$

Since the Q-matrices vary slowly over the length scale ℓ at which the average Green function decays, this facilitates a gradient expansion. Using the identity

$$\int d\mathbf{r} f_d^2(|\mathbf{r}|) = \frac{2\tau}{\pi\hbar\nu},$$

we find that, in the leading approximation, the action of the transverse fluctuations vanishes, while the action for the longitudinal fluctuations takes the form

$$S_l^{(2)} = \int \frac{\pi\hbar\nu}{4\tau}{\rm str}\left[\delta Q_l\right]^2. \tag{48}$$

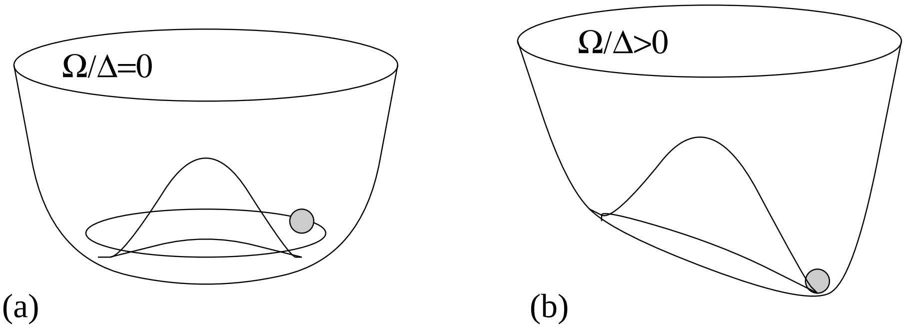

Figure 13. Schematic diagram of the structure of the saddle-point manifold (a) in the absence, and (b) in the presence of the symmetry breaking source Ω. For $\Omega/\Delta = 0$, fluctuations of the supermatrix Q separate into massive longitudinal modes, and massless transverse Goldstone modes. The latter become massive in the presence of symmetry breaking source. An expansion in the vicinity of the saddle-point $\sigma_3^{\rm ar}$ recovers diagrammatic perturbation theory.

Thus, in the limit $\Delta\tau \gg 1$, longitudinal fluctuations generate only a small contribution to the effective action. (In the Gaussian approximation, their contribution vanishes due to supersymmetry.)

Transverse fluctuations, on the other hand, provide a non-vanishing contribution to the effective action only at second order of the gradient expansion. Using the identity

$$\int d\mathbf{r} f_d^2(|\mathbf{r}|) r_\alpha r_\beta = \frac{4\tau^2 D}{\pi\nu\hbar}\delta_{\alpha\beta},$$

where $D = v^2\tau/d$, we obtain the effective action

$$S^{(2)} = \frac{\pi\nu}{8}\int \hbar D \mathrm{str}\,(\partial Q)^2\,,$$

where $Q(\mathbf{r}) = T^{-1}(\mathbf{r})\sigma_3^{\rm ar}T(\mathbf{r})$. Finally, taking $\Omega\tau \ll 1$, an expansion of the effective action to leading order in the symmetry breaking sources Ω and J yields

$$\langle Z\rangle = \left\langle e^{-S_J[Q]}\right\rangle, \qquad \langle\cdots\rangle_Q = \int DQ\cdots e^{-S[Q]}, \tag{49}$$

where

$$S[Q] = \frac{\pi\nu}{8}\int \mathrm{str}\left[\hbar D\,(\partial Q)^2 + 2i\Omega_+\sigma_3^{\rm ar}Q\right]. \tag{50}$$

In the same approximation, the leading contribution to the source term takes the form

$$S_J[Q] = \frac{\pi\nu}{2}\int d\mathbf{R}\; d\mathbf{r}\; f_d(|\mathbf{r}|)\; \bar{J}(\mathbf{R}+\mathbf{r}/2)Q(\mathbf{R})J(\mathbf{R}-\mathbf{r}/2). \tag{51}$$

We have thus succeeded in expressing the response of weakly disordered metallic grains in the form of a functional supersymmetric non-linear σ-model (50). Spontaneous symmetry breaking induced by the external source Ω leads to the existence of Goldstone modes associated with the diffusion modes. The action (50) can be compared to the Landau-Ginzburg free energy of a superconductor or a ferromagnet.

Magnetic Field

An extension of the present theory to include an external magnetic field introduces a magnetic vector potential into the action.

$$\mathcal{L} = \frac{i}{2}\int \bar\Psi\left[E - \frac{\Omega_+}{2}\sigma_3^{\rm ar} - \frac{1}{2m}\left(\hat p - \frac{e}{c}\sigma_3^{\rm tr}\mathbf{A}\right)^2 - V\right]\Psi.$$

The magnetic field breaks time-reversal symmetry and couples to the symmetry breaking matrix $\sigma_3^{\rm tr}$. Treating the magnetic field as a weak perturbation, and proceeding as before, it is straightforward to show that the effective action takes a diffusive form involving a covariant derivative,

$$S[Q]=\frac{\pi\nu}{8}\int {\rm str}\left[\hbar D\left(\tilde{\partial}Q\right)^2+2i\Omega_+\sigma_3^{\rm ar}Q\right],\qquad \tilde{\partial}=\partial-\frac{ie}{\hbar c}\mathbf{A}\left[\sigma_3^{\rm tr},\ \right]. \tag{52}$$

From this result one can identify the degrees of freedom of Q corresponding to the diffuson and Cooperon modes. Those degrees of freedom which commute with $\sigma_3^{\rm tr}$ remain unperturbed by the weak magnetic field and represent the diffuson modes. The remaining degrees of freedom, which correspond to the Cooperon modes, acquire a mass and are frozen out by the magnetic field. This structure compares to the phenomenological analysis made in the previous section.

Correlators

Another virtue of the supersymmetry approach is that, within this representation, statistical correlators of wavefunctions as well as spectra can be obtained with the same efficiency. As an example, employing Eqs. (39) and (51), the two-point correlator of DoS fluctuations is given by

$$R_2(\Omega)=\frac{1}{64}\left\langle\left[\int\frac{d\mathbf{r}}{L^d}{\rm str}\left(\sigma_3^{\rm ar}\otimes\sigma_3^{\rm bf}Q(\mathbf{r})\right)\right]^2\right\rangle_Q-1, \tag{53}$$

while the density response function is given by

$$G^-(0,\mathbf{r})G^+(\mathbf{r},0)=\frac{\pi^2\nu^2}{8}\left\langle{\rm str}\left[\sigma_3^{\rm bf}(\mathbf{I}^{\rm ar}-\sigma_3^{\rm ar})Q(0)\sigma_3^{\rm bf}(\mathbf{I}^{\rm ar}+\sigma_3^{\rm ar})Q(\mathbf{r})\right]\right\rangle_Q$$

Perturbation Theory

To help digest the form of the effective action we will focus on several applications identified in the previous section. Specifically, we will examine the interpolation of level statistics from the perturbative to the non-perturbative limit, and examine the role of weak localization and scaling. We begin by describing a perturbative expansion of the action in the vicinity of the global saddle-point $\sigma_3^{\rm ar}$. For this purpose a number of convenient parametrizations in terms of the generators of the coset exist. Those most commonly used include

$$Q=\begin{cases} e^{-W}\sigma_3^{\rm ar}e^{W} & \text{exponential,}\\ W+\sigma_3^{\rm ar}(\mathbf{I}^{\rm ar}-W^2)^{1/2}, & \text{square root,}\\ (\mathbf{I}^{\rm ar}+W)^{-1}\sigma_3^{\rm ar}(\mathbf{I}^{\rm ar}+W) & \text{rational,}\end{cases} \tag{54}$$

where $[W,\sigma_3^{\rm ar}]_+=0$.¶

Level Statistics. To apply the perturbation theory, we turn first to the question of level statistics. Employing the exponential parametrization, to leading order in W, the effective action takes the form

$$S_0[W]=\frac{1}{8L^d}\int {\rm str}W\hat{\Pi}^{-1}W,\qquad \hat{\Pi}^{-1}=-\left(\frac{\pi}{\Delta}\right)\left[\hbar D\partial^2+i\Omega_+\right], \tag{55}$$

¶While, order by order, the perturbative expansion is independent of the parametrization, a formal identification with diagrammatic perturbation theory is achieved most easily in the square root parametrization. In this case, each diagram finds its counterpart in the perturbative expansion.

while an expansion of the vertices (53), gives

$$R_2(\Omega) = \frac{1}{256}\left\langle \left[\int \mathrm{str}\left(\sigma_3^{\mathrm{bf}} W^2\right)\right]^2 \right\rangle_W, \qquad \langle\cdots\rangle_W = \int DW \cdots e^{-S_0[W]}.$$

The average over fields W can be determined by applying Wick's theorem and employing the corresponding contraction rules. Focusing on unitary and orthogonal symmetry, for general P and R, the contraction rules take the form

$$\begin{aligned}
&\langle \mathrm{str}\,[W(\mathbf{r})PW(\mathbf{r}')R]\rangle_W \\
&\quad = \Pi(\mathbf{r},\mathbf{r}')\left[\mathrm{str}P\mathrm{str}R - \mathrm{str}P\sigma_3^{\mathrm{ar}}\mathrm{str}R\sigma_3^{\mathrm{ar}} + (2-\beta)\mathrm{str}\left(P\sigma_3^{\mathrm{ar}}\widetilde{R}\sigma_3^{\mathrm{ar}} - P\widetilde{R}\right)\right], \\
&\langle \mathrm{str}\,[W(\mathbf{r})P]\,\mathrm{str}\,[W(\mathbf{r}')R]\rangle_W \\
&\quad = \Pi(\mathbf{r},\mathbf{r}')\mathrm{str}\left[PR - P\sigma_3^{\mathrm{ar}}R\sigma_3^{\mathrm{ar}} + (2-\beta)\left(P\sigma_3^{\mathrm{ar}}\widetilde{R}\sigma_3^{\mathrm{ar}} - P\widetilde{R}\right)\right],
\end{aligned}$$

where, as usual, $\beta = 1, 2$ for orthogonal and unitary symmetry respectively, and $\widetilde{R} = C^T R^T C$ reflects the operation of charge conjugation. Applying these identities, we recover the Altshuler-Shklovskii correction (24)

$$R_2(\Omega) = \frac{1}{\beta}\mathrm{tr}\hat{\Pi}^2, \tag{56}$$

previously found within diagrammatic perturbation theory. This contribution is associated with the exchange of two diffuson or Cooperon ladders between two closed loops (see Fig. 14a). Performing the trace we find that, for $\Omega \gg E_c = \hbar D/L^2$, $R_2(\Omega) \sim g^{-d/2}(\Delta/\Omega)^{2-d/2}$. In the zero-dimensional limit this result recovers the perturbative random matrix limit. Interestingly, in two-dimensions, the numerical prefactor identically vanishes, and a non-trivial dependence of R_2 on Ω emerges from higher order weak localization corrections[55].

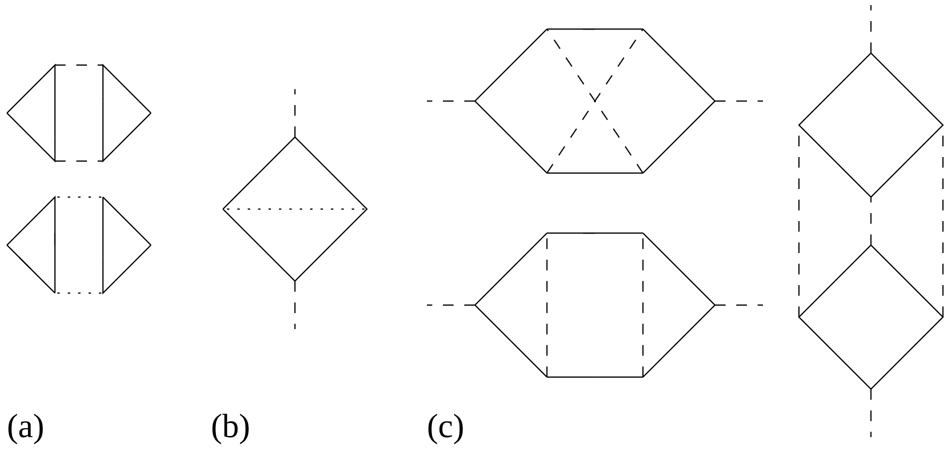

Figure 14. Diagrams contributing to (a) $R_2(\Omega)$ and (b) the weak localization correction to the diffusion constant at one loop. Diagrams contributing to weak localization corrections involving just diffusons are shown in (c) at two loop order.

Scaling and Renormalization Group. As a second application, we consider scaling and the renormalization of the diffusion constant. To do so, we employ a perturbative renormalization group separating slow from fast ($|\mathbf{q}| \gg \Lambda$) degrees of freedom. Employing the parametrization $Q = T_<^{-1}Q_>T_<$, where $T_<$ includes slow degrees of freedom and $Q_> = T_>^1\sigma_3^{\mathrm{ar}}T_>$ involves the fast, we obtain

$$S[Q] = \frac{\pi\nu}{8}\int \mathrm{str}\left[\hbar D\left((\partial Q_>)^2 + 2[Q_>,\partial Q_>]\cdot\mathbf{\Phi}_< + [Q_>,\mathbf{\Phi}_<]^2\right) + 2i\Omega_+ U_< Q_>\right],$$

where $\mathbf{\Phi}_< = T_<^{-1}\partial T_<$ and $U_< = T_< \sigma_3^{\rm ar} T_<^{\rm ar}$.

A perturbative expansion of $Q_>$ in the generators of the coset yields the series $S = S_< + S_>^{(0)} + S_>^{(1)} + \cdots$ where

$$
\begin{aligned}
S_< &= \frac{\pi\nu}{8}\int {\rm str}\left[\hbar D\left(\partial Q_<\right)^2 + 2i\Omega_+ Q_<\right], \\
S_>^{(0)} &= -\frac{\pi\nu}{8}\int \hbar D {\rm str}\left(\partial W_>\right)^2, \\
S_>^{(1)} &= \frac{\pi\nu}{8}\int \hbar D {\rm str}\left[8[W_>, \partial W_>]\mathbf{\Phi}_< + 2\left(\mathbf{\Phi}_< \sigma_3^{\rm ar} W_>\right)^2 + 4\left(\mathbf{\Phi}_< \sigma_3^{\rm ar}\right)^2 W_>^2\right] \\
&\qquad + \frac{\pi\nu}{8}\int 2i\Omega_+ {\rm str}\left[U_< \sigma_3^{\rm ar}(W_> + W_>^2/2)\right].
\end{aligned}
$$

At one loop, the renormalised action takes the simple form

$$S[Q] = S_< + \left\langle S_>^{(1)}\right\rangle_{W_>}.$$

Applying the contraction rules only one term survives and we obtain

$$\left\langle S_>^{(1)}\right\rangle_{W_>} = \frac{\pi\nu}{8}\int \hbar D \times 2(2-\beta){\rm str}\left[\mathbf{\Phi}_< \sigma_3^{\rm ar} C^T \mathbf{\Phi}^T C \sigma_3^{\rm ar} - \mathbf{\Phi}_< C^T \mathbf{\Phi}^T C\right].$$

Using the fact that $\mathbf{\Phi}_<$ is off-diagonal in ar-space, $C^T(\mathbf{\Phi}_<^{\rm ar})^T C = \mathbf{\Phi}_<^{\rm ra}$, and the identity ${\rm str}[\mathbf{\Phi}_<, \sigma_3^{\rm ar}]^2 = {\rm str}(\partial Q_<)^2$, we recover the usual diffusive non-linear σ-model with

$$D \to D\left[1-(2-\beta)\Pi(0,0)\right] = D\left[1-(2-\beta)\frac{\Delta}{\pi}\int_\Lambda \frac{d\mathbf{q}}{(2\pi)^d}\frac{1}{\hbar D\mathbf{q}^2 + i\Omega_+}\right].$$

This result recovers the standard weak localization correction to the diffusion constant at one loop (see Fig. 14b). It relies on the existence of the Cooperonic mode and is therefore destroyed by the application of a magnetic field ($\beta = 2$). Weak localization corrections arising from the interaction of diffuson modes do exist but only at two loop order (Fig. 14c) and higher.

This completes our brief survey of perturbation theory. We remark that, as well as recovering known results from diagrammatic perturbation theory, the non-linear σ-model affords the possibility of seeking other non-perturbative stationary points of the effective action[56]. Of particular interest are frequency scales comparable to the average level spacing. A stationary phase saddle-point of the σ-model can be identified at $Q = \sigma_3^{\rm ar} \otimes \sigma_3^{\rm bf}$. A surprising connection between the perturbation theory in the vicinity of this saddle-point and the conventional saddle-point was recently stressed by Andreev and Altshuler[56].

Zero-Mode

Having examined the short-time perturbative dynamics within the framework of diagrammatic perturbation theory, we now switch attention to consider the long-time non-perturbative limit. In particular, taking $\Omega \ll E_c = \hbar D/L^2$, contributions to the effective action involving spatial fluctuations of Q are strongly suppressed. In this limit, the contribution at leading order in $1/g$ arises from the zero-mode. In parentheses we note that this argument has to be applied with some care: Indeed, each individual non-zero mode configuration is small in $1/g$. However, it may still happen – and does so in systems with localization – that the *cumulative* contribution of all these modes still overpowers the zero-mode. In the simplest case, the zero-mode corresponds to $Q = Q_0$ independent of coordinates, and the effective action takes the universal form

$$S[Q_0] = i\frac{\pi\Omega_+}{4\Delta}{\rm str}\left(\sigma_3^{\rm ar} Q_0\right), \tag{57}$$

independent of dimensionality, diffusion constant, or geometry. In fact, as will be shown below, Eq. (57) is an exact representation of statistical correlators of random matrix ensembles.

Reducing the functional integral to a definite integral allows statistical correlators to be evaluated directly. However, zero-mode integrations require an explicit parametrization of the saddle-point manifold. For this purpose several parametrizations exist in the literature. Each has its own merit and the optimum choice of parametrization depends on the precise nature of the effective action and source. As an application of the zero-mode integration, we will apply the effective action (57) to the two-point correlator of DoS fluctuations for unitary ensembles (i.e. in the absence of time-reversal symmetry). To do so, we will employ the parametrization introduced by Efetov in Ref. 5.

Employing the result of Eq. (53), in the zero-dimensional limit, the two-point correlator of DoS takes the form

$$R_2(\Omega) = \frac{1}{64}\mathrm{Re}\left\langle \left[\mathrm{str}\left(\sigma_3^{\mathrm{ar}} \otimes \sigma_3^{\mathrm{bf}} Q_0\right)\right]^2 \right\rangle_{Q_0} - 1.$$

Separating the components into eigenvalues and rotations, the zero-dimensional Q-matrix takes the form

$$Q_0 = UH\bar{U},$$

where matrices H and U have the block structure

$$H = \begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix}_{\mathrm{ar}}, \qquad U = \begin{pmatrix} u_1 u_2 & 0 \\ 0 & v \end{pmatrix}_{\mathrm{ar}}.$$

These matrices are further separated into components

$$\begin{aligned}
\hat{\theta} &= \begin{pmatrix} i\theta_1 & 0 \\ 0 & \theta \end{pmatrix}_{\mathrm{bf}} \otimes \mathbf{I}^{\mathrm{tr}}, \qquad \theta_1 > 0, \qquad 0 < \theta < \pi, \\
u_1 &= \exp\begin{bmatrix} 0 & -2\bar{\eta} \\ 2\eta & 0 \end{bmatrix}_{\mathrm{bf}}, \qquad \eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & -\eta_1^* \end{pmatrix}_{\mathrm{tr}}, \qquad u_2 = \begin{pmatrix} e^{i\chi\sigma_3^{\mathrm{tr}}} & 0 \\ 0 & e^{i\phi\sigma_3^{\mathrm{tr}}} \end{pmatrix}_{\mathrm{bf}}, \\
v &= \exp\begin{bmatrix} 0 & -2i\bar{\kappa} \\ 2i\kappa & 0 \end{bmatrix}_{\mathrm{bf}}, \qquad \kappa = \begin{pmatrix} \kappa_1 & 0 \\ 0 & -\kappa_1^* \end{pmatrix}_{\mathrm{tr}},
\end{aligned}$$

where $\bar{u}_i \equiv u_i^\dagger$, $\bar{v} \equiv \sigma_3^{\mathrm{bf}} v^\dagger \sigma_3^{\mathrm{bf}}$. Here the coordinates θ_1 and θ represent the eigenvalues of the non-compact bosonic and compact fermionic sector respectively.

Employing this parametrization we find

$$\begin{aligned}
\mathrm{str}\left[\sigma_3^{\mathrm{ar}} Q_0\right] &= 4(\lambda - \lambda_1), \\
\mathrm{str}\left[\sigma_3^{\mathrm{fb}} \otimes \sigma_3^{\mathrm{ar}} Q\right] &= 4(\lambda_1 + \lambda) - 16\left(\lambda_1 - \lambda\right)\left[\eta_1^*\eta_1 - \kappa_1^*\kappa_1\right],
\end{aligned}$$

where $\lambda = \cos\theta$, $\lambda_1 = \cosh\theta_1$, $\mu = (1-\lambda^2)^{1/2}$, and $\mu_1 = (\lambda_1^2 - 1)^{1/2}$. Finally, taking into account the invariant measure,

$$dQ_0 = \frac{1}{128\pi^2}\frac{\mu\mu_1}{(\lambda_1 - \lambda)^2} d\theta d\theta_1 d\phi d\chi d\eta_1^* d\eta_1 d\kappa_1^* d\kappa_1,$$

we obtain

$$R_2(\Omega) = \frac{1}{2}\mathrm{Re}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda \exp\left[i\Omega_+(\lambda_1 - \lambda)\right]. \tag{58}$$

Explicit integration over the eigenvalues λ, λ_1 recovers the random matrix result (7).

In conclusion, we have demonstrated that, by exploiting the ensemble average, an effective field theory can be developed which is capable of describing the response of disordered conductors from time scales short as compared to the typical transport or diffusion time, to times in excess of the Heisenberg time where properties become universal. As an appendix to this section, we will employ the same supersymmetry approach to show that the zero-dimensional σ-model is equivalent to the supermatrix representation of correlators of random matrix ensembles.

Random Matrix Theory

As indicated above, the supersymmetry approach provides a convenient way of representing statistical correlators of random matrix ensembles. In the following, we will employ this approach to obtain an expression for the generating function of two-point correlators, $\langle \mathcal{Z} \rangle$, in the form of a zero dimensional integral over Q-matrices.

Expressed as a definite integral over $8 \times N$-component supervectors, Ψ, the generating function of the matrix Green function

$$G^{\pm} = \frac{1}{E - H \pm i0}, \qquad H = H_s + i\alpha H_a,$$

of an arbitrary complex Hermitian matrix takes the form

$$\mathcal{Z} = \int d[\Psi] e^{-\mathcal{L}[\Psi] - \mathcal{L}_J[\Psi]},$$

where

$$\mathcal{L}[\Psi] = \frac{i}{2}\bar{\Psi}\left[E - \frac{\Omega_+}{2}\sigma_3^{\mathrm{ar}} - \hat{H}_s - i\alpha\sigma_3^{\mathrm{tr}}\hat{H}_a\right]\Psi, \qquad \mathcal{L}_J = i\left(\bar{J}\Psi + \bar{\Psi}J\right)$$

The complex $N \times N$ Hamiltonian is comprised of symmetric (H_s) and antisymmetric (H_a) matrix elements drawn at random from a Gaussian ensemble (2). For generality, a parameter α allows a smooth interpolation from the Gaussian orthogonal ($\alpha = 0$) to the unitary ($\alpha = 1$) ensemble.

Taking our notation from Ref. 57, the ensemble average over the Gaussian distribution generates an interaction of the supervectors,

$$\langle Z \rangle = \int d[\Psi] e^{-\mathcal{L}_0[\Psi] - \mathcal{L}_I[\Psi] - \mathcal{L}_J},$$

$$L_0[\Psi] = \frac{i}{2}\bar{\Psi}\left[E - \frac{\Omega_+}{2}\sigma_3^{\mathrm{ar}}\right]\Psi, \qquad L_I[\Psi] = \frac{\lambda^2}{4N}\mathrm{str}\left(A^2 + \alpha^2(A\sigma_3^{\mathrm{tr}})^2\right),$$

where λ denotes the variance of the random distribution, and the 8×8 supermatrix takes the form $A = i\Psi \otimes \bar{\Psi}$. Note that, without the introduction of the composite vector Ψ, the interaction of the fields generated by the ensemble average could not be expressed in terms of the single dyadic product A.

In order to decouple the interaction by means of a Hubbard-Stratonovich transformation, it is convenient to introduce the matrix

$$\widetilde{A} = \frac{a_+ + a_-}{2}A + \frac{a_+ - a_-}{2}\sigma_3^{\mathrm{tr}}A\sigma_3^{\mathrm{tr}}, \qquad a_{\pm} = (1 \pm \alpha^2)^{1/2},$$

such that $\mathcal{L}_I = (\lambda^2/4N)\mathrm{str}\widetilde{A}^2$. As a result we obtain

$$\langle Z \rangle = \int d[\Psi]d[Q]\exp\left[-\frac{N}{4}\mathrm{str}Q^2 - \frac{\lambda}{2}\bar{\Psi}\tilde{Q}\Psi - \mathcal{L}_0[\Psi] - \mathcal{L}_J[\Psi]\right],$$

where

$$\tilde{Q} = \frac{a_+ + a_-}{2} Q + \frac{a_+ - a_-}{2} \sigma_3^{\rm tr} Q \sigma_3^{\rm tr}.$$

Performing the Gaussian integral over Ψ we obtain

$$\langle Z \rangle = \int d[Q] \exp\left[-\frac{N}{4} {\rm str}\left(Q^2 - 2 \ln \mathcal{G}^{-1} \right) + \frac{i}{2} \bar{J} \mathcal{G} J \right],$$

where the supermatrix Green function takes the form

$$\mathcal{G}^{-1} = E - \frac{\Omega_+}{2} \sigma_3^{\rm ar} - i\lambda \tilde{Q}.$$

To proceed, we first establish the mean-field, or saddle-point of the action to leading order in N. Taking the frequency Ω/Δ, and symmetry breaking source α to be parametrically smaller than N, the saddle-point equation takes the form

$$Q_{\rm sp} = -i\lambda \frac{1}{E - i\lambda Q_{\rm sp}}.$$

Analyticity then requires the solution

$$Q_{\rm sp} = -\frac{i}{2\lambda} E + \left(1 - \frac{E^2}{4\lambda^2} \right)^{1/2} \sigma_3^{\rm ar}.$$

From this result we recover the semi-circular average DoS (5),

$$\langle \nu(E) \rangle = \frac{1}{8} \frac{N}{\pi\lambda} {\rm Re} \left\langle {\rm str}\left(\sigma_3^{\rm bf} \otimes \sigma^{\rm ar} Q_{\rm sp} \right) \right\rangle_{Q_{\rm sp}} = \frac{N}{\pi\lambda} \left(1 - \frac{E^2}{4\lambda^2} \right)^{1/2}.$$

Taking the energy at the center of the band, $E = 0$, and expanding to leading order in N, we obtain the effective zero-dimensional non-linear σ-model

$$\langle Z \rangle = \int d[\Psi] d[Q] e^{-S_0[Q] - S_J[Q]}, \tag{59}$$

with the effective action

$$S_0[Q] = i \frac{\pi\Omega_+}{4\Delta} {\rm str}\left(\sigma_3^{\rm ar} Q \right) - \frac{N\alpha^2}{8} {\rm str}\left[Q, \sigma_3^{\rm tr} \right]^2. \tag{60}$$

In particular, for $\alpha = 0$, we recover the zero-dimensional action (57) obtained previously. Time-reversal symmetry breaking implied by the application of H_a leads to a freezing out of the Cooperon degrees of freedom for $\alpha \sim \sqrt{N}$.

We have seen how the application of ensemble averaging allowed quantum statistical properties of weakly disordered metals to be cast in the form of a functional field integral involving the low-lying modes of density relaxation. Does an analogous theory exist for systems which are irregular but not random? The key to the construction of an effective field theory of clean chaotic structures relies on the recognition that the spectrum itself provides a statistical ensemble. By averaging over an interval of energy it is possible to establish a "ballistic" non-linear σ-model within the same framework. It is to the derivation of this effective action that we now turn.

The derivation of the diffusive non-linear σ-model relied explicitly on the impurity average over realizations of the random potential. Can an analogous approach be developed for systems which, while not belonging to a statistical ensemble, can be considered as quasi-classical? The first suggestion that such a description was possible came in an insightful work by Muzykantskii and Khmel'nitskii[6]. Guided by the quasi-classical Boltzmann description of density relaxation in the ballistic regime of weakly disordered conductors, they proposed a generalization of the diffusive action. Although still phenomenological, this "ballistic action" established a crucial bridge between the semi-classical description of level statistics offered by the Trace formula, and the universal random matrix correlations.

In a subsequent development by Andreev *et al.*[7] a formal justification of the phenomenological approach was obtained. The crucial step involved the realization that a statistical ensemble is provided by the spectrum itself. By averaging over a range of energy levels, the ballistic σ-model, proposed by Muzykantskii and Khmel'nitskii, was constructed explicitly. Whether, when properly regularised, correlations converge to well defined distributions is a matter of ongoing research in quantum chaos (and not just within the restricted subset of field theoretical approaches). The question behind is whether energy averaging in a individual system does actually suffice to define smooth distributions or whether extra averaging over external parameters needs to be introduced. To the best of our knowledge a conclusive answer is still outstanding.

In its first application, the statistical average over energy was applied directly to the Green function $\hat{G} = (E - \hat{H})^{-1}$. However, as a matter of convenience, to obtain the ballistic model here we will exploit a more recent development by Zirnbauer[58]. This approach requires the introduction of the generalized Green function

$$\hat{G}^{+}(\varphi) = \frac{1}{1 - e^{i\varphi_{+}}\hat{U}^{\dagger}}, \qquad \hat{G}^{-} = (\hat{G}^{+})^{\dagger},$$

where $\varphi^{+} = \varphi + i0$, and the Hamiltonian is expressed through the unitary time evolution operator $\hat{U} = \exp[i\hat{H}t/\hbar]$. With this definition, the DoS

$$\nu(\varphi) = \frac{1}{\pi}\mathrm{tr}\left[\mathrm{Re}\,\hat{G}^{-}(\varphi) - \frac{1}{2}\right], \tag{61}$$

is periodic on the interval $[0, 2\pi]$. Since, when projected onto the unitary evolution operator $\hat{U}$, the semi-infinite spectrum of the Hamiltonian generates a multiple covering of the unit circle, caution is required in deducing spectral statistics from eigenphase statistics. However, if the number of levels inside one wind of the phase is large, and the statistical correlations decay quickly, the statistics of each interval become separable.

Hamiltonian

To keep our discussion concrete, we will focus on the particular case of a Hamiltonian describing a particle confined to a two-dimensional closed system of size L and subject either to a slowly varying inhomogeneous magnetic field $\mathbf{B}(\mathbf{r}) = \partial_{\mathbf{r}} \times \mathbf{A}(\mathbf{r})$,

$$\hat{H} = \frac{1}{2m}\left(\hat{p} - \frac{e}{c}\mathbf{A}\right)^{2}, \tag{62}$$

or, to make contact with our previous analysis, to a slowly varying random scalar potential (8). The correlation length ξ of the inhomogeneous magnetic field is chosen to be large such that $\xi \gg v/\max(\omega_0, \omega_{\rm rms})$ where ω_0 and $\omega_{\rm rms}$ represent respectively the average and standard deviation of the random cyclotron frequency,

$$\omega_c(\mathbf{r}) = \frac{eB(\mathbf{r})}{mv},$$

and v denotes the velocity at the Fermi energy E. Finally, in both cases, the strength of the random potential is adjusted such that the dynamics can be treated quasi-classically, (i.e. $\lambda \ll \ell$ where ℓ denotes the scattering mean free path of the potential). To keep our discussion simple, in contrast to the previous analysis, we will limit our discussion to the case in which time-reversal symmetry is fully broken. In practice, this confines attention to time scales $t \gg L_c/v$, where $L_c = (eB_{\rm rms}/hc)^{1/2}$ represents the typical magnetic length.

Although, in the present case, it is possible to define a statistical ensemble involving the random scalar or vector potential, we will suppose a single realization of each and implement the procedure of phase averaging described above.

Field Integral

Taking our notation from above, statistical two-point correlators of advanced and retarded Green functions can be obtained from the generating function

$$\mathcal{Z}(J) = \int D\psi e^{-\mathcal{L}[\psi]-\mathcal{L}_J[\psi]}, \tag{63}$$
$$\mathcal{L}[\psi] = \mathrm{tr}\left[\bar\psi\left(\mathbf{I} - e^{-i\sigma_3^{\rm ar}(\bar\varphi - \sigma_3^{\rm ar}\varphi^+/2)}\begin{pmatrix}\hat U & 0\\ 0 & \hat U^\dagger\end{pmatrix}_{\rm ar}\right)\psi\right], \qquad \mathcal{L}_J[\psi] = \mathrm{tr}\left(\bar\psi J + \bar J\psi\right),$$

where the trace is taken over the Hilbert space. Here the structure and symmetry properties of the supervector fields mirror those introduced previously. In this case ψ only has four components, in ar and bf-blocks. The corresponding Green functions are obtained from the generating function by

$$\hat G^{\mp} = \frac{1}{2}\mathrm{tr}\left[\frac{\delta}{\delta \bar J_{\mp}} \otimes \frac{\delta}{\delta J_{\mp}}\right]_{\rm bf} \mathcal{Z}\Big|_{J=0}.$$

Phase Averaging

To obtain the statistical average over the center of mass $\bar\varphi$, we will exploit an ingenious and general integral identity introduced by Zirnbauer[58]. For a unitary subgroup $U(N)$ of $GL(N)$, there holds the identity

$$\int_{\Theta(N)} d\Theta \exp\left(i\left[\bar\psi_+^\alpha \Theta\psi_+^\alpha + \bar\psi_-^\alpha\Theta^\dagger\psi_-^\alpha\right]\right) = \int d\mu(z,\bar z)\exp\left[\bar\psi_+^\alpha z^{\alpha\beta}\psi_-^\beta + \bar\psi_-^\alpha \bar z^{\alpha\beta}\psi_+^\beta\right] \tag{64}$$

where $d\Theta$ represents the Haar measure of $U(N)$, and

$$d\mu(z,\bar z) = d(z,\bar z)\mathrm{sdet}\left(\mathbf{I} - \bar z z\right)^{-N}$$

denotes the integration measure with $d(z,\bar z)$ defining the flat Berezin measure on the symmetric space of supermatrices $U(1,1/2)/[U(1/1)\times U(1/1)]$. The matrices z and $\bar z = z^\dagger\sigma_3^{\rm bf}$ are precisely those parametrising the non-linear supermatrix manifold $Q = T^{-1}\sigma_3^{\rm ar}T$, where

$$T = \begin{pmatrix}\mathbf{I} & z\\ \bar z & \mathbf{I}\end{pmatrix}, \qquad T^{-1} = \begin{pmatrix}\mathbf{I} & -z\\ -\bar z & \mathbf{I}\end{pmatrix}\begin{pmatrix}(\mathbf{I}-z\bar z)^{-1} & 0\\ 0 & (\mathbf{I}-\bar z z)^{-1}\end{pmatrix}. \tag{65}$$

Eq. (64) can be thought of as a Hubbard-Stratonovich transformation decoupling the interaction generated amongst the fields by the matrices Θ. Remarkably, in contrast to Gaussian averaging, the Hubbard-Stratonovich fields already belong to the non-linear manifold, apparently circumventing the need for separation of massive from massless degrees of freedom.

In the present case this result may be applied directly: $\alpha = (a, \sigma)$, where $a = 1, \ldots, n$ denotes the Hilbert space indices, $\sigma = \mathrm{b/f}$, and the average over the $U(1)$ phase $\bar{\varphi}$ corresponds to $N = 1$. Averaging the generating function $\mathcal{Z}$, and applying Eq. (64), we obtain

$$\begin{aligned}
\langle \mathcal{Z} \rangle_{\bar{\varphi}} &\equiv \frac{1}{2\pi} \int_0^{2\pi} d\bar{\varphi} \mathcal{Z} = \int d\mu(z, \bar{z}) \int D\psi e^{-\mathcal{L}' - \mathcal{L}_J}, \\
\mathcal{L}'[\psi] &= \mathrm{tr} \bar{\psi} \mathcal{G}^{-1} \psi, \qquad \mathcal{G}^{-1} = \begin{bmatrix} \mathbf{I} & -e^{i\varphi_+/2} \hat{U} z \\ -e^{i\varphi_+/2} \hat{U}^\dagger \bar{z} & \mathbf{I} \end{bmatrix}_{\mathrm{ar}}.
\end{aligned}$$

Notice that z and $\bar{z}$, and hence Q, are non-local in the Hilbert space indices. Integrating over the superfields ψ we then obtain

$$\langle \mathcal{Z} \rangle = \int d\mu(z, \bar{z}) \mathrm{sdet} \mathcal{G}^{-1} e^{-S_J}, \qquad S_J = -\mathrm{tr} \bar{J} \mathcal{G} J. \tag{66}$$

By making use of the parametrization in Eq. (65), from which $dQ = d(z, \bar{z})$, as well as the identity

$$\sigma_3^{\mathrm{ar}} T + T \sigma_3^{\mathrm{ar}} = 2\sigma_3^{\mathrm{ar}}, \qquad T^{-1} = \frac{1}{2}(Q\sigma_3^{\mathrm{ar}} + \mathbf{I}),$$

Eq. (66) can be expressed in the coordinate free or invariant form,

$$\langle \mathcal{Z} \rangle = \int dQ e^{-S - S_J} \tag{67}$$

$$\begin{aligned}
S &= -\mathrm{str} \ln \left[\frac{1}{2} \begin{pmatrix} e^{-i\varphi_+/2} \hat{U}^\dagger - \mathbf{I} & 0 \\ 0 & e^{-i\varphi_+/2} \hat{U} - \mathbf{I} \end{pmatrix} (Q\sigma_3^{\mathrm{ar}} + \mathbf{I}) + \mathbf{I} \right] \\
&= -\mathrm{str} \ln \left[\mathbf{I} - \tanh \left(\frac{i}{4} \varphi_+ \sigma_3^{\mathrm{ar}} + \frac{1}{2} \ln \hat{U} \right) Q \right].
\end{aligned} \tag{68}$$

This result, which is *exact*, is also quite general. The statistical average properties of a general unitary operator $\hat{U}$ are expressed as an integral over a non-linear manifold of supermatrices, Q. In particular, the same procedure can be equally applied to a quantum chaotic map, $\hat{U}$[59]. Applied directly to the unitary (orthogonal or symplectic) matrix ensemble, Eq. (68) correctly recovers the statistical correlations of Dyson's circular ensemble[58]. In the following section we will apply this action to study the quantum statistical properties of the Hamiltonian (62). By implementing a semi-classical approximation we will identify the low-lying modes of density relaxation and thereby obtain a ballistic analogue of the diffusive σ-model.

Semi-classics

Applying Eq. (68) to the Hamiltonian (62), and taking $\varphi_+ = \Omega_+ t/\hbar$, we obtain the effective action

$$S = -\mathrm{str} \ln \left(\mathbf{I} - i \tan \left[\left(\frac{\Omega_+}{2} \sigma_3^{\mathrm{ar}} + \hat{H} \right) \frac{t}{2\hbar} \right] Q \right). \tag{69}$$

Until now the calculation has involved only formal manipulations of the average generating function. To continue we have to apply approximations, i.e. we need to identify contributions to the field integral whose action is small. The physical significance of these 'soft modes' will become clear below. At any rate, however, a word of caution is due at this point: It has turned out that the σ-model (69) as it stands is *ill-defined* [60]. The point is that, besides the soft modes we are going to discuss below, a host of unphysical zero-mode configurations to the field integral exist. The presence of these modes is intimately related to the above question whether or not energy (or phase) averaging actually suffices to give smooth correlation functions. The question if and how these extra modes can be eliminated via suitably designed averaging procedures is a matter of current research. We repeat that this problem is not exclusive to the field theoretical approach taken here but does affect statistical approaches to chaos in general. Here we take a naïve approach, i.e. we will assume that the unphysical modes have been regularized away such that only the below discussed contributions – which are insensitive to extra averaging procedures[60] – survive. At present it can not be safely excluded that the picture discussed here will have to undergo qualitative changes.

Our starting point will be to assume the legitimacy of the semi-classical expansion $\delta\hat{H} = \hat{H} - H(\mathbf{x})$, where $H \equiv W(\hat{H})$ denotes the Wigner transform of the quantum Hamiltonian:

$$\begin{aligned} W(\hat{H}) \equiv H(\mathbf{x}) &= \int d\mathbf{r} e^{-i\mathbf{r}\cdot\mathbf{P}/\hbar} H(\mathbf{R}+\mathbf{r}/2, \mathbf{R}-\mathbf{r}/2), \\ &= \int \frac{d\mathbf{p}}{h^d} e^{i\mathbf{R}\cdot\mathbf{p}/\hbar} H(\mathbf{P}+\mathbf{p}/2, \mathbf{P}-\mathbf{p}/2), \end{aligned} \tag{70}$$

$$H(\mathbf{r}_1, \mathbf{r}_2) = \left\langle \mathbf{r}_1 | \hat{H} | \mathbf{r}_2 \right\rangle, \qquad H(\mathbf{p}_1, \mathbf{p}_2) = \left\langle \mathbf{p}_1 | \hat{H} | \mathbf{p}_2 \right\rangle. \tag{71}$$

The phase space coordinate $\mathbf{x} \equiv (\mathbf{R}, \mathbf{P})$ at which we take the Wigner transform will be fixed later such that $\delta\hat{H}$ is small. Translational invariance of the generating function in energy ensures that an expansion in $\delta\hat{H}$ does not involve factors of the absolute energy $H(\mathbf{x})$. In this notation, an expansion leads to the approximation $S \simeq S_0 + S_R$, where

$$\begin{aligned} S_0 &= i\frac{t}{2\hbar}\mathrm{str}\left[\frac{\Omega_+}{2}\sigma_3^{\mathrm{ar}} Q + \delta\hat{H} Q\right], \qquad S_R = -\frac{t^2}{8\hbar^2}\mathrm{str}\left[\delta\hat{H} Q\right]^2 \\ S_J &= -\frac{1}{2}\int d\mathbf{x} \left[\bar{J} Y^{-1} \left(\sigma_3^{\mathrm{ar}} Q + \mathbf{I}\right) Y J\right], \end{aligned} \tag{72}$$

where $Y = \mathrm{diag}(e^{-iH(\mathbf{x})t/2\hbar}, e^{+iH(\mathbf{x})t/2\hbar})$

In the following, those contributions to the Hamiltonian which vary smoothly on a scale comparable to the mean free path, ℓ, will be included in S_0 and treated within a semi-classical approximation. For reasons which will become apparent, quantum corrections to the classical expansion are included through contributions to S_R.

To facilitate the semi-classical expansion it is convenient to switch to the Wigner representation. Treating each term separately, let us consider the first order term in t. Since, in the Wigner representation, the trace of a product of operators involves the product of the Wigner transforms, we find

$$\begin{aligned} \mathrm{str}\left(\delta\hat{H} Q\right) &= \mathrm{str}\left(T^{-1}\sigma_3^{\mathrm{ar}}[T, \delta\hat{H}]\right) = \int d\mathbf{x}\,\mathrm{str}\left(W(T^{-1})\sigma_3^{\mathrm{ar}} W([T, \delta\hat{H}])\right) \\ &= -i\hbar \int d\mathbf{x}\,\mathrm{str}\left(T^{-1}(\mathbf{x})\sigma_3^{\mathrm{ar}}\{H(\mathbf{x}), T(\mathbf{x})\}\right), \end{aligned}$$

where $\{H, \ \}$ represents the classical Poisson bracket

$$\{O_1, O_2\} = \partial_{\mathbf{q}} \cdot O_1 \partial_{\mathbf{p}} O_2 - \partial_{\mathbf{p}} O_1 \cdot \partial_{\mathbf{q}} O_2,$$

and $d\mathbf{x} \equiv d\mathbf{r}d\mathbf{p}/h^d$. Here, for simplicity, we assume that H commutes with Q in the superindices. As a result, we obtain

$$S_0 = i\frac{t}{4\hbar}\int d\mathbf{x}\mathrm{str}\left[\Omega_+\sigma_3^{\mathrm{ar}}Q - 2i\hbar T^{-1}\sigma_3^{\mathrm{ar}}\left\{H,T\right\}\right]. \tag{73}$$

In the semi-classical approximation ($\hbar \to 0$), the contribution from this leading order term dominates the short-time dynamics.

Mode Locking

In its present form the effective action (73) is ill-defined. The propagator generates the classical flow of phase space density, and thereby admits arbitrary fluctuations transverse to the constant energy shell. This circumstance admits to the existence of a continuum of zero modes. Anticipating (from our previous analysis) the relevance of a single zero mode, it is necessary to seek a mechanism of mode-locking.

The mechanism by which fluctuations of Q transverse to the constant energy shell become frozen out remains an important and outstanding question. Recent, and as yet preliminary, investigations by Zirnbauer[60] suggest a resolution in which the ambiguity in the quantization of an individual classical Hamiltonian can be used to identify a statistical ensemble (cf. the above remarks on 'extra-averaging'). In the semi-classical approximation, ensemble averaging generates a mass which strongly suppresses transverse fluctuations of Q.

To proceed, we will adopt a more circumspect approach and apply the condition of mode-locking as an ansatz to be checked self-consistently. We therefore take the degrees of freedom transverse to the constant energy shell to be locked, which fixes the normalization,

$$\int d\mathbf{x} = \frac{h}{t\Delta}, \tag{74}$$

where the integration runs over the energy shell. Taken together, Eqs. (73) and (74) determine the ballistic σ-model

$$\langle \mathcal{Z} \rangle = \int DQe^{-S_0-S_J} \tag{75}$$

with the in-shell action

$$S_0 = i\frac{\pi}{2\Delta}\int \mathrm{str}\left[\Omega_+\sigma_3^{\mathrm{ar}}Q - 2i\hbar T^{-1}\sigma_3^{\mathrm{ar}}\left\{H,T\right\}\right], \tag{76}$$

where the supermatrix fields $Q = T^{-1}\sigma_3^{\mathrm{ar}}T$ depend on the $2d-1$ phase space coordinates parametrising the constant energy shell, $\mathbf{x}_\parallel = (\mathbf{r},\mathbf{p})_{2d-1}$. With this definition the integration measure is normalized such that $\int \equiv \int d\mathbf{x}_\parallel = 1$.

Perturbation Theory

To make sense of the functional integral (76) we must identify the low lying modes of the action. To do so, we adopt a perturbative expansion of the effective action around the zero mode. Employing the exponential parametrization (54), we obtain

$$S[W] = -\frac{\pi}{\Delta}\int \mathrm{str}\left[W\hat{\Pi}^{-1}W\right] + O(W^4), \qquad \hat{\Pi}^{-1} = \hbar\sigma_3^{\mathrm{ar}}\{H,\ \} - i\Omega_+. \tag{77}$$

It is, therefore, tempting to associate the low energy degrees of freedom with eigenmodes of the unitary (*reversible*) evolution operator $e^{-\{H,\ \}t}$. However this identification is incorrect.

As with any functional integral there is a need to define an appropriate regularization. For example the functional integral (75) may be understood as the limit $a \to 0$ of a product of definite integrations over a discretized space, where a denotes the discretization cell size. This admits to functions $T(\vec{x}_{\|})$ which are *smooth* and square integrable. In seeking such a basis, the eigenfunctions of the classical evolution operator seem to be the natural choice. However, the intricate nature of chaotic classical evolution cause these eigenfunctions to lie outside the Hilbert space: The chaotic dynamics of probability densities involves contraction along stable manifolds, together with stretching along unstable ones (see Fig. 15). Thus, in the course of time, an initially non-uniform distribution turns into a singular function on the unstable manifold, which in turn covers the whole energy shell densely. Therefore the eigenfunctions of $\hat{\Pi}$ are not square integrable and their contribution to the functional integral cannot be directly recovered by the discretization procedure involved in evaluating the functional integral.

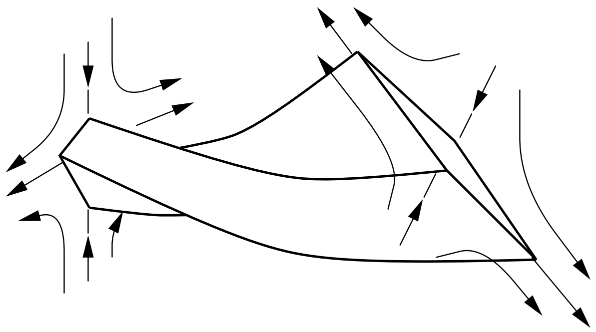

Figure 15. Schematic diagram showing the tendency for a hyperbolic system to stretch exponentially rapidly along the unstable direction and contract along the stable direction.

Thus, in choosing a convenient basis one has to take account of the regularization. Its primary effect is to truncate the contraction along the stable manifold and thereby render the classical evolution *irreversible*. It is the eigenfunctions of this regularized classical evolution operator that serve as a suitable basis for the quantum mechanical correlator. Remarkably, as the strength of the regulator is taken to zero, the spectrum, $\{\gamma_n\}$, of the resulting operator, known in the literature as the Perron-Frobenius operator, reflects intrinsic irreversible properties of the *purely classical dynamics*[61, 62, 63, 64, 65].

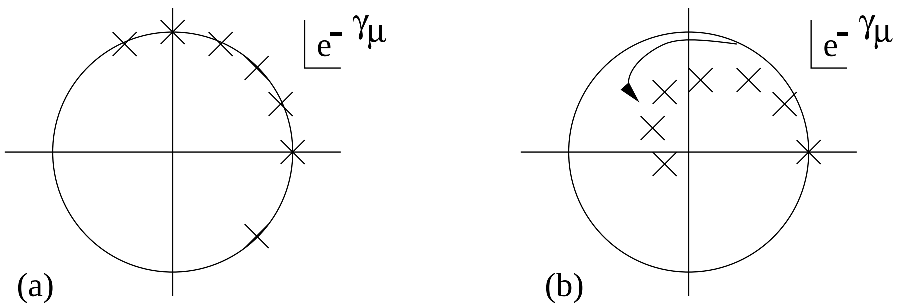

Figure 16. (a) Formal representation of the eigenvalues of the reversible evolution operator which lie on the unit circle. In non-integrable systems, the eigenvalues form an infinitely degenerate and dense set. (b) Poles or resonances of the coarse-grained, regularised Perron-Frobenius operator. In systems which are uniformly hyperbolic, all resonances lie within the unit circle (apart from the zero mode), and correspond to modes of density relaxation.

In ergodic systems, the leading eigenvalue $\gamma_0 = 0$ is non-degenerate, and manifests the conservation of probability density. Thus any initial density relaxes, in the course of time, to the state associated with γ_0. If, in addition, this relaxation is exponential in time, then the Perron-Frobenius spectrum has a gap associated with the slowest decay rate. Thus, the first non-zero eigenvalue has the property $\gamma_1' \equiv \mathrm{Re}(\gamma_1) > 0$. This gap sets the ergodic time scale, $t_c = 1/\gamma_1'$ over which the classical dynamics relaxes to equilibrium. In the case of disordered metallic grains it coincides with the transport time, t_D, while in ballistic systems or quantum billiards it is of order of the flight time across the system.

In the limit $\Omega \gg \hbar\gamma_1'$ the perturbative expansion (77) is justified, and R_2 takes the form of Eq. (56) but where the diffusion operator is replaced by the Perron-Frobenius operator. This result coincides with that obtained within the "diagonal approximation" of periodic orbit theory[47] thereby establishing a direct correspondence of the field theoretic and semi-classical approaches. In the limit $\Omega \ll \hbar\gamma_1'$ the dominant contribution to the semi-classical action (76) arises from the ergodic classical distribution, the zero-mode $\{H, T\} = 0$. Taking this contribution alone, the action coincides with the *universal* action (57) from which one can deduce that correlations coincide with those of random matrix ensembles.

Although much attention has been paid to the spectral properties of the Perron-Frobenius operator, little is known about the resonance spectrum in dynamical systems. Uniformly hyperbolic or axiom A systems, such as billiards with constant negative curvature, are characterized by exponential decay of classical correlation functions and the resonance spectrum of the Perron-Frobenius operator has a gap. Irregular boundary scattering in two-dimensional billiards on flat surfaces is characterized by isolated resonances, associated with macroscopic modes, together with gapless resonances (of low spectral weight) associated with a degeneracy of weakly unstable periodic orbits.

Although the resonance spectra of the Perron-Frobenius operator are well defined, states associated with the resonances are singular. This has important implications on the role of quantum weak localization corrections to the perturbative result.

Weak Localization

Having identified the degrees of freedom associated with modes of classical phase space density relaxation we now turn our attention to weak localization corrections. To do so, we digress from our consideration of a system in a magnetic field and temporarily restore T-invariance (for the significance of T-invariance, cf. the above discussion on (one-loop) weak localization). As with the disordered conductor, such a generalization implies an action involving Q-matrices of twice the rank,

$$S[Q] = i\frac{\pi}{4\Delta}\int \mathrm{str}\left[\Omega_+\sigma_3^{\mathrm{ar}}Q - 2i\hbar T^{-1}\sigma_3^{\mathrm{ar}}\{H, T\}\right], \tag{78}$$

where, under the Wigner transform, the symmetry properties of the generators under charge conjugation induce the constraint

$$T^\dagger(\mathbf{x}) = CT^T(\bar{\mathbf{x}})C^T,$$

where $\bar{\mathbf{x}} = (-\mathbf{p}, \mathbf{q})$.

Previously, we have seen that the principle mechanism of weak localization arises from the enhancement of return probability due to the interference of Feynman paths with their time-reversed counterpart. At a perturbative level, this mechanism is described by a one-loop contribution to the perturbation theory. Applying the perturbative expansion described previously, to one loop order, the effective action takes the form

$$S[W] = -\frac{\pi}{2\Delta}\int \mathrm{str}\left[W\left(1 - W^2\right)\hat{\Pi}^{-1}W\right] + O(W^6). \tag{79}$$

Following Aleiner and Larkin[66], from a perturbative expansion, we obtain the weak localization correction to the (retarded component of the) propagator

$$2\pi\nu\delta\Pi(1,2) = \Pi(1,\bar{2})\Pi(\bar{2},2) + \Pi(1,\bar{1})\Pi(\bar{1},2) + \int d3\Pi(1,3)\Pi(\bar{3},2)(2i\Omega - \hbar\{H,\ \})_3\Pi(3,\bar{3}), \tag{80}$$

where we have employed the short-hand notation $1 = (\mathbf{p}_1, \mathbf{q}_1)_\parallel$ etc.[‖] Here the first two terms represent vertex corrections while the third term accounts for the standard weak localization correction.

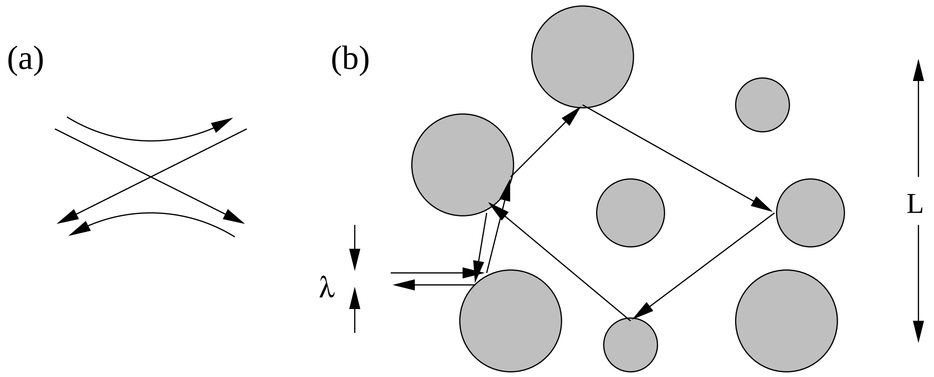

Figure 17. Schematic diagram showing (a) the junction corresponding to the Hikami box in ballistic structures, and (b) the insertion of the Hikami box into a configuration of trajectories in the Lorentz gas.

The mechanism of weak localization is reminiscent of that encountered in disordered systems. In particular, for the weak localization correction (80) to operate a particle must, at some point, return with opposite momentum (see Fig. (17)). At the level of classical trajectories such an event is ruled out (save for normal reflection from a boundary). We might, however, hope that weak localization corrections are reinstated by the regularization scheme described previously. However, in contrast to the perturbative contribution to the two-point correlator of the DoS, the quantum correction (80) relies explicitly on the eigenfunctions of the Perron-Frobenius operator. These eigenfunctions are strongly sensitive to the regularization scheme, and become singular as the regulator is taken to zero.

We are, therefore, faced with a dilemma. The singular contribution to the quantum correction is ill-defined. To circumvent these difficulties, Aleiner and Larkin[66] introduced a coarse graining of the space to stabilize the theory. Operationally, this involved the introduction of a weak but finite *quantum* impurity potential. This regularising scheme, which serves to mimic the quantum resolution, introduced a new quantum time scale into the action. Taking the resolution to be set by the Fermi wavelength λ, the time scale on which weak localization corrections can operate is set by the time scale for two trajectories which are initially at a separation λ to become separated by some macroscopic scale, L. With a typical Lyapunov exponent of the system, Γ, this time scale, known as the "Ehrenfest time", is given by

$$t_E = \Gamma^{-1}\ln\left(\frac{L}{\lambda}\right). \tag{81}$$

[‖]Notice that, as a consequence of particle conservation, Eq. (80) generates a correction only to the non-zero modes of the Perron-Frobenius operator (i.e. $\int d1\delta\Pi(1,2) = 0$).

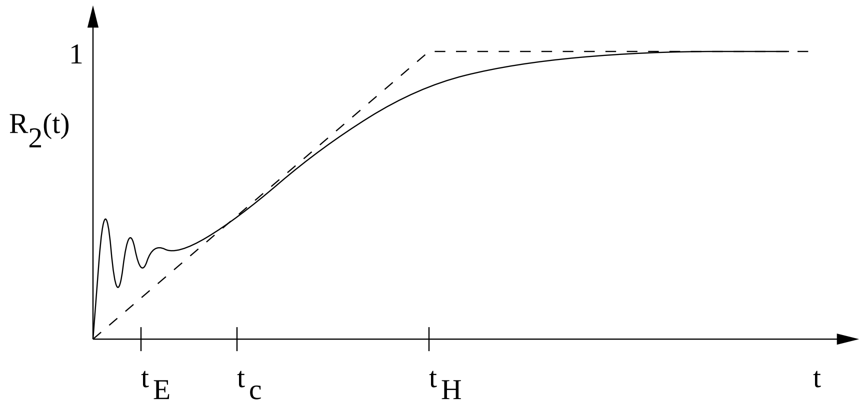

Figure 18. Schematic diagram showing the dynamic structure factor of the two-point correlator of DoS fluctuations.

To assess the implications that these conclusions have on level statistics we can consider the dynamic structure of the two-point correlator of DoS fluctuations (see Fig. 18). According to the analysis above it is possible to identify three relevant time scales, the Ehrenfest time t_E, the lowest non-zero eigenvalue of the Perron-Frobenius operator $t_c = 1/\gamma_1'$, and the Heisenberg time $t_H = \hbar/\Delta$. On time scales shorter than t_E, the results above show that weak localization corrections can not operate and the "diagonal approximation" (together with the random matrix zero mode contribution) is exact. On time scales in excess of t_E, weak localization corrections emerge. However, whether or not the dominant contributions to these corrections arise solely from properties of the classical relaxation modes, or are sensitive to the artificial coarse-graining of the system, depends on the relative magnitude of the quantum (t_E) and classical (t_c) times scales. Indeed, the $\hbar$ dependence of t_E suggests that, in the limit $\hbar \to 0$, all weak localization corrections outside the ergodic regime vanish.

Faced with the scenarios outlined above, the following conclusion can be drawn. If the weak localization correction is controlled by the classical dynamics of the Perron-Frobenius operator, the particular mechanism of coarse-graining is irrelevant. It serves merely to regularise singularities in the classical phase space density and thereby open the channel to weak localization. If, however, the weak localization corrections depend explicitly on the coarse-graining, consistency requires a careful consideration of the leading quantum processes contributing corrections to the classical dynamics. Taking the second point of view, in the next section we examine the leading quantum corrections to the classical action for the Hamiltonian with a smoothly varying inhomogeneous scalar or vector potential.

Quantum Irreversibility

The manner in which classical irreversibility enters the action depends sensitively on the dominant mechanism of quantum scattering or "diffraction". Operationally, this requires the identification and exploitation of slow and fast degrees of relaxation in the system. To illustrate the general approach, let us consider the case of the inhomogeneous magnetic and scalar fields.

As indicated above, irreversibility enters through a mechanism of quantum scattering and can appear only at second order (or higher) in the Hamiltonian. We therefore focus on the second order term in the expansion, S_R (72). Expressed in terms of the

Wigner transform,

$$S_R = -\frac{t^2}{8\hbar^2}\int d\mathbf{X}d\mathbf{X}' O(\mathbf{X},\mathbf{X}')\mathrm{str}\left[Q(\mathbf{X})Q(\mathbf{X}')\right], \tag{82}$$

where

$$\begin{aligned} O(\mathbf{X},\mathbf{X}') &= \int d\mathbf{r}d\mathbf{r}' e^{i(\mathbf{r}\cdot\mathbf{P}+\mathbf{r}'\cdot\mathbf{P}')/\hbar}\left\langle \mathbf{R}-\mathbf{r}/2|\hat{H}|\mathbf{R}'+\mathbf{r}'/2\right\rangle\left\langle \mathbf{R}'-\mathbf{r}'/2|\hat{H}|\mathbf{R}+\mathbf{r}/2\right\rangle, \quad (83) \\ &= \int \frac{d\mathbf{p}d\mathbf{p}'}{h^{2d}} e^{-i(\mathbf{p}\cdot\mathbf{R}+\mathbf{p}'\cdot\mathbf{R}')/\hbar}\left\langle \mathbf{P}-\mathbf{p}/2|\hat{H}|\mathbf{P}'+\mathbf{p}'/2\right\rangle\left\langle \mathbf{P}'-\mathbf{p}'/2|\hat{H}|\mathbf{P}+\mathbf{p}/2\right\rangle. \end{aligned}$$

To employ this representation, we will apply Eq. (83) to specific examples.

Scalar Potential. Applied to the scattering from a scalar potential, Eq. (83) takes the form

$$O(\mathbf{X},\mathbf{X}') = L^d\delta^d(\mathbf{R}-\mathbf{R}')\int\frac{d\mathbf{r}}{L^d}e^{i\mathbf{r}\cdot(\mathbf{P}-\mathbf{P}')/\hbar}V(\mathbf{R}-\mathbf{r}/2)V(\mathbf{R}+\mathbf{r}/2). \tag{84}$$

This expression is further simplified by applying the ansatz that the low-lying modes of density relaxation are governed by field configurations of Q that vary slowly in space. The validity of this approximation can be checked self-consistently. In this case, the kernel above can be replaced by the coarse grained average $\langle V(\mathbf{R}-\mathbf{r}/2)V(\mathbf{R}+\mathbf{r}/2)\rangle$. With this assumption S_R takes the form of a collision integral with

$$O(\mathbf{X},\mathbf{X}') \simeq L^d\delta^d(\mathbf{R}-\mathbf{R}')|\widetilde{V}(\mathbf{P}-\mathbf{P}')|^2, \tag{85}$$

where the Fourier transform $\widetilde{V}$ represents the Born scattering amplitude.

Taking the matrices Q to be locked over a scale $\hbar/t$ perpendicular to the constant energy shell, and assuming that, in this range, the matrix element $\widetilde{V}$ depends only on the angle of scattering, we obtain

$$S_R = -\frac{\pi^2}{2\Delta^2}\int\frac{d\mathbf{R}}{L^d}\frac{d\mathbf{n}d\mathbf{n}'}{S_d^2}|\widetilde{V}(\mathbf{n}-\mathbf{n}')|^2\mathrm{str}\left[Q(\mathbf{R},\mathbf{n})Q(\mathbf{R},\mathbf{n}')\right]. \tag{86}$$

As the simplest example, taking the potential to be δ-correlated, $|\widetilde{V}(\mathbf{n}-\mathbf{n}')|^2 = \hbar\Delta/2\pi\tau$, the physical regulator takes the form

$$S_R = -\frac{\pi\hbar}{4\Delta\tau}\int\frac{d\mathbf{R}}{L^d}\frac{d\mathbf{n}d\mathbf{n}'}{S_d^2}\mathrm{str}\left[Q(\mathbf{R},\mathbf{n})Q(\mathbf{R},\mathbf{n}')\right], \tag{87}$$

where $\mathbf{n}=\mathbf{p}/p_F$. If, on the other hand, the potential is taken to be a smooth, slowly varying function, the effective action takes the form

$$S_R = \frac{\pi\hbar}{4d\Delta\tau_{\mathrm{tr}}}\int\frac{d\mathbf{R}}{L^d}\frac{d\mathbf{n}}{S_d}\mathrm{str}\left(\partial_{\mathbf{n}}Q(\mathbf{R},\mathbf{n})\right)^2, \tag{88}$$

where

$$\int\frac{d\mathbf{n}}{S_d}|\widetilde{V}(\mathbf{n}-\mathbf{n}')|^2(\mathbf{n}-\mathbf{n}')_\alpha(\mathbf{n}-\mathbf{n}')_\beta = \frac{\delta_{\alpha\beta}}{\pi d}\frac{\hbar\Delta}{\tau_{\mathrm{tr}}} \tag{89}$$

defines the transport mean free time. Interpreting the kinetic component of the action as the left-hand side of a Boltzmann operator, Eqs. (87) and (88) serve as a collision integral introducing irreversibility into the effective action**. Taking $\hbar/\Delta\tau \gg 1$, a moment expansion of the fields can be performed. As a result, the effective action takes the form of the diffusive action. Before showing explicitly how the diffusive action is recovered, we turn to the inhomogeneous magnetic field.

**A construction of similar type has recently been performed by Khmel'nitskii and Muzykantskii[67].

Inhomogeneous Magnetic Field. For a smoothly varying inhomogeneous vector potential, we can employ the same approximation as above. Assuming the low-lying modes to be those associated with field configurations of Q that vary slowly in space, we find

$$O(\mathbf{X},\mathbf{X}') \simeq 2^d L^d \delta^d(\mathbf{R}-\mathbf{R}') \int \frac{d\mathbf{p}}{L^d h^d} e^{-2i\mathbf{p}\cdot\mathbf{R}/\hbar} \times \left\langle \mathbf{P}-\mathbf{p}/2|\hat{H}|\mathbf{P}'+\mathbf{p}/2\right\rangle \left\langle \mathbf{P}'-\mathbf{p}/2|\hat{H}|\mathbf{P}+\mathbf{p}/2\right\rangle$$

Further integration over $\mathbf{R}$ generates the approximation

$$O(\mathbf{X},\mathbf{X}') \simeq L^d \delta^d(\mathbf{R}-\mathbf{R}')|\widetilde{H}(\mathbf{P}-\mathbf{P}')|^2, \tag{90}$$

where $\widetilde{H}(\mathbf{P}-\mathbf{P}') = \left\langle \mathbf{P}|\hat{H}|\mathbf{P}'\right\rangle$ again represents the general Born scattering amplitude. Again, taking the Born amplitude to depend only on the angle of scattering, S_R takes the form of Eq. (86).

Applying this approximation to the inhomogeneous magnetic field in the two-dimensional geometry, and taking the Born scattering amplitude as short-ranged, we obtain the effective action

$$\begin{aligned} S_0[Q] &= i\frac{\pi}{2\Delta}\int \frac{d\mathbf{R}}{L^2}\frac{d\theta}{2\pi}\mathrm{str}\left[\Omega_+\sigma_3^{\mathrm{ar}}Q + 2i\hbar T^{-1}\sigma_3^{\mathrm{ar}}\left(v\hat{\mathbf{n}}\cdot\partial_{\mathbf{R}}T - \frac{\omega_c v}{2c}\partial_\theta T\right)\right], \\ S_R[Q] &= \frac{\pi\hbar}{8\Delta\tau_{\mathrm{tr}}}\int \frac{d\mathbf{R}}{L^2}\frac{d\theta}{2\pi}\mathrm{str}\left(\partial_\theta Q\right)^2, \end{aligned}$$

where $\hat{\mathbf{n}} = (\cos\theta, \sin\theta)$ and

$$\frac{\hbar\Delta}{2\pi\tau_{\mathrm{tr}}} \sim \omega_{\mathrm{rms}}^2 \int \frac{d\theta}{2\pi}(\theta/2)^2\cot^2(\theta/2).$$

identifies the transport mean free time. As with the scalar potential, taking $\hbar/\Delta\tau_{\mathrm{tr}} \gg 1$, a moment expansion of the fields is justified. Employing the parametrization,

$$T(\mathbf{X}_\parallel) = \exp\left[i\hat{\mathbf{n}}\cdot\mathbf{J}(\mathbf{R})\right]\widetilde{T}(\mathbf{R}),$$

where $[\mathbf{J},\sigma_3^{\mathrm{ar}}]_+ = 0$, and expanding the effective action to quadratic order in the generators $\mathbf{J}$ we obtain

$$\begin{aligned} S_0[\widetilde{Q}] &= i\frac{\pi}{2\Delta}\int \frac{d\mathbf{R}}{L^2}\mathrm{str}\left[\Omega_+\sigma_3^{\mathrm{ar}}\widetilde{Q} - \hbar\left(v\left[(\partial_{\mathbf{R}}\widetilde{T})\widetilde{T}^{-1},\sigma_3^{\mathrm{ar}}\right]\cdot\mathbf{J} + i\omega_c\sigma_3^{\mathrm{ar}}\epsilon^{ij}J_iJ_j\right)\right], \\ S_R[\widetilde{Q}] &= \frac{\pi\hbar}{2\Delta\tau_{\mathrm{tr}}}\int\frac{d\mathbf{R}}{L^2}\mathrm{str}\,\hat{\mathbf{J}}^2 \end{aligned}$$

where $\widetilde{Q} = \widetilde{T}^{-1}\sigma_3^{\mathrm{ar}}\widetilde{T}$. Integrating over the degrees of freedom $\mathbf{J}$ we finally obtain the effective action

$$\begin{aligned} S_0[\widetilde{Q}] &= \frac{\pi}{4\Delta}\int\frac{d\mathbf{R}}{L^2}\mathrm{str}\left[\hbar\widetilde{D}_{xx}\partial_{\mathbf{R}}\widetilde{Q}\cdot\begin{pmatrix}\mathbf{I} & -\beta\widetilde{Q}\\ \beta\widetilde{Q} & \mathbf{I}\end{pmatrix}\cdot\partial_{\mathbf{R}}\widetilde{Q} + 2i\Omega_+\sigma_3^{\mathrm{ar}}\widetilde{Q}\right], \\ &= \frac{\pi}{4\Delta}\int\frac{d\mathbf{R}}{L^2}\mathrm{str}\left[\hbar\widetilde{D}_{xx}(\partial_{\mathbf{R}}\widetilde{Q})^2 + \hbar\widetilde{D}_{xy}\widetilde{Q}\epsilon^{ij}\partial_{R_i}\widetilde{Q}\partial_{R_j}\widetilde{Q} + 2i\Omega_+\sigma_3^{\mathrm{ar}}\widetilde{Q}\right], \end{aligned} \tag{91}$$

where

$$\beta = \omega_c\tau_{\mathrm{tr}}, \qquad \widetilde{D}_{xx} = \frac{1}{1+\beta^2}\frac{v^2\tau_{\mathrm{tr}}}{2}, \qquad \widetilde{D}_{xy} = \beta\widetilde{D}_{xx}. \tag{92}$$

Taking the magnetic field to be homogeneous, the effective action recovers the diffusive non-linear σ-model (91) with an additional contribution involving the "topological term" introduced by Pruisken[68]. For an inhomogeneous field configuration, modes of density relaxation are described by the diffusion operator

$$\hat{\Pi} = -\left(\frac{\pi}{\Delta}\right)\left[\hbar\partial_{\mathbf{R}}\cdot(\widetilde{D}_{xx}\partial_{\mathbf{R}}) + \hbar\sigma_3^{\mathrm{ar}}\epsilon^{ij}\partial_{R_i}\left(\widetilde{D}_{xy}\partial_{R_j}\right) + i\Omega_+\right]. \tag{93}$$

This operator, which has a structure similar to the passive scalar operator, describes *advected* diffusion[69], and its long-time properties coincide with the conventional diffusion operator[70, 71]. The short-time properties of the non-hermitian operator depend explicitly on the nature of the random field distribution.

This concludes our preliminary investigation of the properties of the quantum ballistic action. While, from a pedagogical point of view, the ballistic action (76) gives considerable insight into generic properties of the quantum chaotic systems, operationally, the generalized action is complex. It seems likely that, from a practical point of view, this approach will be of use only when there is some separation of fast and slow degrees of freedom already on the level of the action. For the inhomogeneous scalar or vector potential, the fast relaxation of momentum degrees of freedom allowed the ballistic action to be reduced to an effective diffusive action involving only spatial degrees of freedom. On can envisage other scenarios such as the rough billiard or quantum kicked rotor where the spatial degrees of freedom relax faster than the momentum. In such cases, one can expect a quantitative analysis of the quantum dynamics.

DISCUSSION

To conclude, we have discussed mechanisms of quantum coherence in disordered and chaotic structures. For this general class of systems, a quasi-classical approximation allows quantum statistical properties to be expressed in the form of a functional integral involving a supersymmetric non-linear σ-model. The merit of the field theoretic approach lies in its ability to account within a single framework for the shortest time scales where the behavior is well described by the Trace formula, to the longest times where the quantum properties become universal. The formal connection between the field theoretic approach and that developed to study random statistical ensembles reveals a "dictionary" in which coherence phenomena associated with the physics of disordered conductors, such as weak localization, find a counterpart in the more general class of quantum chaotic systems.

Moreover, the field theoretic approach provides a solid platform on which other mechanisms of quantum coherence can be addressed. In particular, our analysis has focussed on aspects of *non-interacting* quantum systems. Manifestations of strong electron interaction play a crucial role in defining properties of disordered electronic structures[72]. The interplay of mesoscopic quantum interference phenomena with phase coherence effects such as the proximity effect in superconducting-normal metal structures are only just beginning to be explored.

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